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Unconditionally Stable Numerical Scheme to Study the Dynamics of Hepatitis B Disease

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Abstract. Hepatitis B is a contagious liver infection caused by hepatitis B virus and is a worldwide public health problem. According to WHO about 350 million people are suffering from this chronic disease. Therefore millions of peoples are at stack of cirrhosis, hepatocellular carcinoma, and cancer of liver or even liver failure. Hepatitis B still has necessary complex issues that must be addressed effectively. In this research paper, we have developed NSFD scheme for HBV model. The simulations show that our proposed NSFD scheme gives stable, unconditional, positive and converging results at all step sizes as compared to the traditional RK-4 method and Euler method which diverge at large step sizes and produce negative and unstable results with large oscillations.Later we have analyzed the local asymptotic stability of this proposed NSFD scheme using Linearized Stability Theorem and Schur-Cohn Stability Criteria.

AMS (MOS) Subject Classification Codes: 92-XX; 92-BXX; 92B05
Key Words: Hepatitis B Virus, Horizontal Transmission, Vertical Transmission, Stability Analysis, NSFD Scheme, RK-4 Method.

1. Introduction

Hepatitis B Virus is a life threatening liver disease and a pandemic health problem [1, 4] especially in developing countries [3]. This liver disease is caused by Hepatitis B Virus [1]. Nearly 360 million people are suffering from this contagious disease and about 350-400 million are HBV carriers worldwide [2, 3, 6]. Among them more than 240-400 million are HBV chronic liver infected and 150-170 are HCV chronic infected [1,4, 5, 7, 8]. Acute or

chronic hepatitis B causes 780,000 deaths annually [4]. WHO estimates that more than two billions of people are HBV infected and 360 millions of people are chronically infected [2, 10] and 15% - 40% chronically infected peoples develop serious complications like Cirrhosis, Hepatocellular Carcinoma (HCC) and cancer of liver that causes millions of deaths annually [2, 5, 9].

Mathematical models have become prominent tool for good decision making to analyze the control and treatment of infectious diseases [11] because they help to study underlying dynamics and quantitative behavior of viral infections like HBV from certain population[33]. G. F. Medley et al. presented a historical model [12] of HBV endemicity, dynamics and control where they extended the HBV model of natural history. They demonstrated that infection remains despite reproduction number is less than unity($R_0 < 1$). They plotted the proportion of population (seropositive) against R_0 and observed backward bifurcation [14]. After that [13, 14, 38] and many other authors extended model [12] by incorporating different compartments and parameters like horizontal and vertical transmissions, external controls and feedback parameters. Later [15, 36, 37] also extended models [13] and [38] by utilizing latent period and migration effect respectively to develop their own HBV models. So their Models also indirectly based on [12]. Another extension of model [12] was developed by [1] by incorporating the vaccination and treatment impacts and this model [1] is our model of interest.

Non-Standard Finite Difference Scheme (NSFD) is a discrete representation of a system of DE's [25]. This scheme possesses consistent, convergent and stable solutions [28, 29]. This scheme possesses some properties of continuous model for extreme values for given set of parametric values, which generally do not satisfied by Euler and RK-4 methods. Moreover in some cases frequency of oscillations is not conserved at large step sizes. Therefore more accurate numerical schemes in form of discretization of continuous model is required [39].Historically different researchers [30, 31, 32] have worked on NSFD scheme dealing with such issues. All these authors developed NSFD schemes for different classes of dynamical systems. Marcus and Mickens [23] developed positive numerical methods for the model of photoconductivity of semiconductors. Piyanwong et al. in [40] developed positive NSFD scheme for SIR epidemic model andM. Y. Ongun and I. Turhanin [34] constructed and analyzed a nonstandard numerical scheme for a mathematical model describing the HIV infection of CD4+ T cells. S. Riaz et al. [41] proposed unconditional NSFD scheme for quadratic Riccati differential equation. Some authors have also numerically dealt with such models taking into consideration the impact of fluid flow [42, 43, 44, 45, 46]. Here we are going to develop NSFD scheme for HBV model [2]. In this paper we proposed an unconditional stable NSFD scheme for the mathematical model of hepatitis B virus under vaccination and treatment impacts. Numerical simulations show a great potential towards stability and convergence of NSFD scheme with both smaller and larger step sizes where other numerical schemes such as RK-4 and Euler methodsdo not agree with it at larger step sizes. These schemes do not remain convergent as we increase the step size. The main use of our proposed NSFD scheme for HBV is very effective in presenting certain qualitative properties of continuous model. We will show that the discrete model constructedusing NSFD scheme is unconditionally guarantee the positivity of solution of the corresponding original system of ODE's. Numerical simulations show that when we go on increasing step

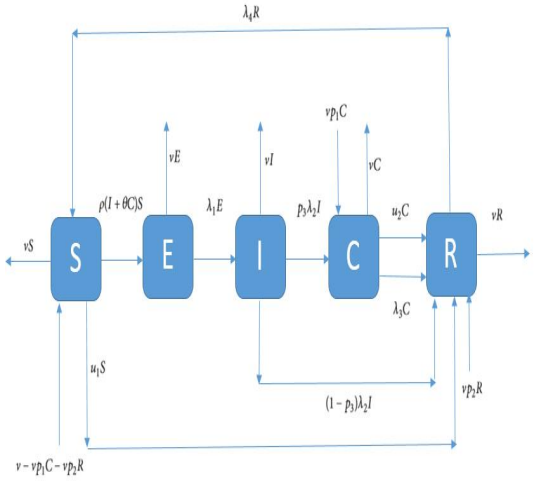

FIGURE 1. Schematic representation of HBV Model

sizes the traditional numerical schemes RK-4 and Euler method diverges but NSFD always gives converging results. Thus NSFD is better than other two traditional schemes.

2. MATERIAL AND METHODS

2.1. **Mathematical Model.** We consider HBV disease model [1] under vaccination and treatments effects. This model has been demonstrated by system (2.1-2.5) of coupled ODE's and its schematic representation has been given in Figure 1.

$$\dot{S(t)} = \nu - \nu p_1 C - \nu p_2 R - \rho(I + \theta C)S - \nu S - u_1 S + \lambda_4 R \tag{2. 1}$$

$$\dot{E(t)} = \rho(I + \theta C)S - (\nu + \lambda_1)E \tag{2. 2}$$

$$\dot{I(t)} = \lambda_1 E - (\nu + \lambda_2)I \tag{2. 3}$$

$$\dot{C(t)} = \nu p_1 C + p_3\lambda_2 I - (\nu + \lambda_3)C - u_2 C \tag{2. 4}$$

$$\dot{R(t)} = \nu p_2 R + (1 - p_3)\lambda_2 I + \lambda_3 C - \nu R - \lambda_4 R + u_1 S + u_2 C \tag{2. 5}$$

With initial conditions $S(0); E(0); I(0); C(0); R(0) \geq 0$[15]

Where all the parameters are defined in Table1. For simplicity and convenience normalizing the population size to 1, then S, E, I, C, R are fractions of susceptible, exposed,

infected, chronic carriers and recovered population. Therefore $S+E+I+C+R=1$ holds [1, 13, 15]. Thus fifth equation can be omitted and substituting $R=1-S-E-I-C$ in first equations yields the following reduced model

$$\dot{S(t)} = \nu - \nu p_1 C - \rho(I+\theta C)S - \nu S - u_1 S + (\lambda_4 - \nu p_2)(1-S-E-I-C) \quad (2.\ 6)$$

$$\dot{E(t)} = \rho(I+\theta C)S - (\nu+\lambda_1)E \quad (2.\ 7)$$

$$\dot{I(t)} = \lambda_1 E - (\nu+\lambda_2)I \quad (2.\ 8)$$

$$\dot{C(t)} = \nu p_1 C + p_3\lambda_2 I - (\nu+\lambda_3)C - u_2 C \quad (2.\ 9)$$

Taking stable population such that per capita birth and death rate ν are equal. Disease induced death rate is neglected here and λ_1 is the rate of transforming exposed individuals to infections individuals. The rate of acute infection is termed as λ_2, and λ_3 represents spontaneous recovery rate from carrier to recovered class. θ Represents infectiousness of carriers w.r.t. acute infections. p_3 is acute infection proportion where individual become HBV carriers and remaining move towards immunity state. $\rho(I+\theta C)S$, represents disease propagation phase of horizontal transmission and is formulated using mass action term in which ρ is contact rate. Vertical transmission is formulated as $\nu p_1 C$ and $\nu - \nu p_1 C - \nu p_2 R$ is birth flux into susceptible class.

2.2. **Construction of Non-Standard Finite Difference (NSFD) Scheme for HBV Model.** In order to develop NSFD scheme for model (2.6-2.9) we start with following rules [22-29].

Rule 1. Order of discrete derivatives and derivatives appearing in the DEq's should be same. Higher order of discrete derivative results into numerical instabilities.

Rule 2. Discrete representation must possess nontrivial denominator.

Rule 3. Non-linear terms are replaced by nonlocal discrete representation.

Rule 4. Special conditions holding for differential equations and their solutions must hold for difference equations and its solutions.Mickens in [25] has discussed these rules in details. We have briefly narrated the basic rules here for objective of constructing NSFD scheme for our model of interest. In [26] J. Sunday et al. have included the fifth rule

Rule 5. Solutions offinite-difference equations and differential equations should not be exactly same.

Further we replace the first derivative of x(t) by discrete representation [25, 27] given by

$$\frac{dx(t)}{dt} \approx \frac{x_{i+1} - f(h)x_i}{\Phi(h)} \quad (2.\ 10)$$

Where f and Φ are functions having step size $h=\Delta t$ and $t_i = ih$and $x(t) \to x_i$ where (f,Φ) are such that $f(h) = 1+O(h)$and $\Phi(h) = h + O(h^2)$. Conventionally, we take $f(h)=1$ and $\Phi(h)=h$(to make huge calculations simple). By utilizing the above mentioned techniques, rules and [16-24], we propose following NSFD scheme for system (2.6-2.9)

$$\frac{S^{n+1}-S^n}{\Phi(h)} = \nu - \nu p_1 C^n - \rho(I^n + \theta C^n)S^{n+1} - \nu S^{n+1} - u_1 S^{n+1}$$
$$+(\lambda_4 - \nu p_2)(1 - S^n - E^n - I^n - C^n) - \lambda_4 S^{n+1} + \nu p_2 S^n \quad (2.\ 11)$$

$$\frac{E^{n+1}-E^n}{\Phi(h)} = \rho(I^n + \theta C^n)S^{n+1} - (\nu+\lambda_1)E^{n+1} \tag{2. 12}$$

$$\frac{I^{n+1}-I^n}{\Phi(h)} = \lambda_1 E^n - (\nu+\lambda_2)I^{n+1} \tag{2. 13}$$

$$\frac{C^{n+1}-C^n}{\Phi(h)} = \nu p_1 C^n + \lambda_2 I^n - (\nu+\lambda_3)C^{n+1} - u_2 C^{n+1} \tag{2. 14}$$

Where $\Phi(h) = \frac{1-e^{-hM}}{\rho}$ and $\rho = max(\nu,\nu)$ which guarantee the condition of positivity [19]. Note that this scheme satisfies $\dot{S}+\dot{E}+\dot{I}+\dot{C}+\dot{R} = 0$ (i.e. total population equals to constant)when $n \to \infty$ then $h \to 0$. Rearranging explicit formulation of system (2.11-2.14) we get

$$S^{n+1} = \frac{S^n + \Phi(h)(\nu - \nu p_1 C^n + (\lambda_4 - \nu p_2)(1 - E^n - I^n - C^n) + \nu p_2 S^n)}{1 + \Phi(\rho(I^n + \theta C^n) + u_1 + \nu + \lambda_4)} \tag{2. 15}$$

$$E^{n+1} = \frac{E^n + \Phi(h)\rho(I^n + \theta C^n)S^{n+1}}{1 + \Phi(h)(\nu+\lambda_1)} \tag{2. 16}$$

$$I^{n+1} = \frac{I^n + \Phi(h)\lambda_1 E^n}{1 + \Phi(h)(\nu+\lambda_2)} \tag{2. 17}$$

$$C^{n+1} = \frac{C^n + \Phi(h)(\nu p_1 C^n + p_3 \lambda_2 I^n)}{1 + \Phi(h)(\nu+\lambda_3+u_2)} \tag{2. 18}$$

3. Simulations of Matheamtical Model

In this sectionwe have simulated HBV model using Euler Method, RK-4 Method and NSFD scheme. In all simulations we have used parametric values from Table. 1. In these simulations we have taken the horizontal axis as the time (in years) and vertical axis as Susceptible, Exposed, Acute Infected and Carriers populations such that total population do not exceed unity. But in simulations demonstrated in Figures(2-8) we have only simulated the susceptible population versus time under the impact of both vaccination and treatment.

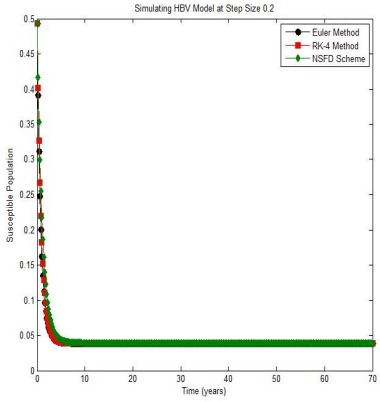

FIGURE 2. Behavior of susceptible population at step size 0.2

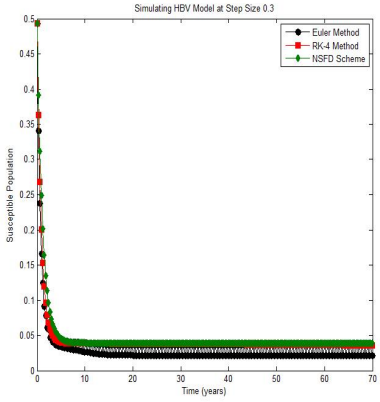

FIGURE 3. Behavior of susceptible population at step size 0.3

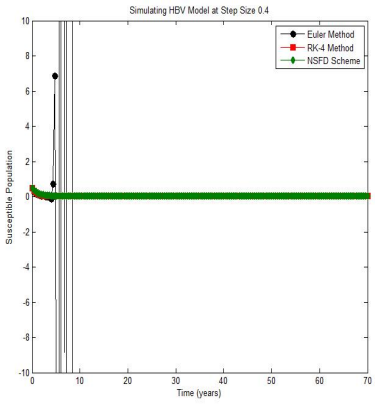

FIGURE 4. Behavior of susceptible population at step size 0.4

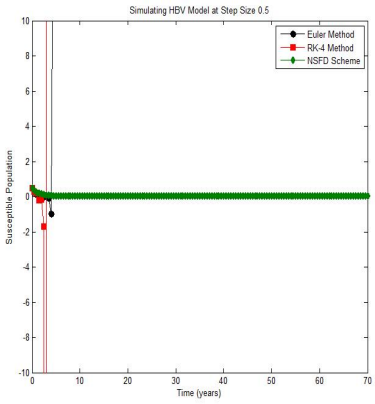

FIGURE 5. Behavior of susceptible population at step size 0.5

Table 1: Parametric values used in our simulations			
Parameter	Description	Values	Sources
ν	Birth and death rates	0.0121	[1]
ρ	Transmission rate	10(0.8-20.49)	[1]
θ	Carrier Infectiousness w.r.t Acute Infections	0.5	[1]
λ_1	The rate at which exposed become acute	6 per year	[1]
λ_2	Rate of acute infection	4 per year	[1]
λ_3	The rate at which carrier become recovered	0.025 per year	[1]
λ_4	Loss of recovery rate	0.03	[1]
p_1	Infected newborns probability	0.11	[1]
p_2	Immune newborns probability	0.1	[1]
p_3	Proportion at which Acute Infection becomes Carriers	0.05	[1]
S_0	Susceptible individuals	0.493	[1]
E_0	Exposed individuals	0.0035	[1]
I_0	Acute infection individuals	0.0035	[1]
C_0	Chronic HBV carriers	0.007	[1]

We are interested to investigate the behaviour of HBV model populations at both small and large step sizes. We have simulated first the behavior of susceptible population using Euler method, RK-4 method and NSFD Scheme at step size 0.2 for the case u1=1, u2=1. Simulations depicted in Figure.2 show that susceptible remain convergent for three numerical schemes at small step sizes. Then we increased the step size to 0.3 for the same case. At this step size Euler method starts to oscillate and RK-4 and NSFD Scheme remain convergent as have been demonstrated in Figure.3. Similarly at step size 0.4 Euler method diverges with large oscillation but RK-4 method and NSFD Scheme remain convergent. Although Euler method diverges with large oscillation but we have taken the limits of vertical axis from -10 to 10 in order to show the impact of three methods simultaneously. At step size 0.5, RK-4 method also diverges but NSFD Scheme remain convergent as have been demonstrated in Figure. 5. At Step sizes 0.6 and 0.7 we get almost similar results that both Euler and RK-4 method diverges but NSFD Scheme converges. At all large step size NSFD Scheme remain convergent for example at step size 5 both RK-4 and Euler method diverges but NSFD again remain convergent. Thus NSFD gives stable, positive, unconditional and converging results in all cases for all step sizes.Bhavior of susceptible population for case when vaccination and treatment applied at different step sizes has been tabulated in Table.2. Similarly other populations exposed, infected and carrier behaves almost similar for different step sizes as in case of susceptible population.

Table 2: Response of Susceptible Population for NSFD Scheme, RK-4 and Euler Method			
Step Size	NSFD Scheme	RK-4 Method	Euler Method
0.01	Converges	Converges	Converges
0.05	Converges	Converges	Converges
0.09	Converges	Converges	Converges
0.1	Converges	Converges	Converges
0.2	Converges	Converges	Converges
0.3	Converges	Converges	Converges
0.4	Converges	Converges	Diverges
0.5	Converges	Diverges	Diverges
0.6	Converges	Diverges	Diverges
0.8	Converges	Diverges	Diverges
1	Converges	Diverges	Diverges
10	Converges	Diverges	Diverges
10	Converges	Diverges	Diverges

4. Stability Analysis of Non-Standard Finite Difference Scheme

For stability of discrete NSFD Scheme (2.15-2.18) we use the following theorems which confirms the local asymptotic stability of this scheme.

4.1. **Linearized Stability Theorem.** Let $\widehat{t}$ be an equilibrium point of the difference equation $t_{n+1} = F(t_n, t_{n-1}, , t_{n-k}), n = 0, 1,$ where function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\widehat{t}$. Then the following statements are true.

- If all the roots of characteristic polynomial have absolute value less than one, then the equilibrium point $\widehat{t}$ is locally asymptotically stable.
- If at least one root of the characteristic polynomial has absolute value greater than one, then the equilibrium point $\widehat{t}$ is unstable[34].

Now we are interested to develop jacobian matrix for system of equations (2.15-2.18). For our convenience we replace $\Phi(h)$ by h in order to make calculations as simple and brief as possible and let

$$F = \frac{\nu - \nu p_1 C + (\lambda_4 - \nu p_2)(1 - E - I - C) + \nu p_2 S)}{1 + h(\rho(I + \theta C)S + u_1 + \nu + \lambda_4)} \tag{4. 19}$$

$$G = \frac{E + h(\rho(I + \theta C)S}{1 + h(\nu + \lambda_1)} \tag{4. 20}$$

$$H = \frac{I + h\lambda_1 E}{1 + h(\nu + \lambda_2)} \tag{4. 21}$$

$$K = \frac{C + h(\nu p_1 C + p_3 \lambda_2)I}{1 + h(\nu + \lambda_3 + u_2)} \tag{4. 22}$$

Constructing Jacobian matrix for the system of equations (4.19-4.22)

$$J_{NSFD} = \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \\ E_5 & E_6 & E_7 & E_8 \\ 0 & E_9 & E_{10} & 0 \\ 0 & 0 & E_{11} & E_{12} \end{bmatrix} \tag{4. 23}$$

Where $E_1 = \frac{1+\nu p_2 h}{1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4)}$
$E_2 = \frac{-h(\lambda_4-\nu p_2)}{1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4)}$
$E_3 = \frac{(1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4))(-h(\lambda_4-\nu p_2))-(S+h(\nu-\nu p_1 C+(\lambda_4-\nu p_2)(1-E-I-C)+\nu p_2 S)h\nu}{(1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4))^2}$
$E_4 = \frac{(1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4))(-\nu p_1 h-(\lambda_4-\nu p_2))}{(1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4))^2}$
$\frac{-(S+h(\nu-\nu p_1 C+(\lambda_4-\nu p_2)(1-E-I-C)+\nu p_2 S)h\theta}{(1+h(\rho(I+\theta C)+u_1+\nu+\lambda_4))^2}$
$E_5 = \frac{h(\rho(I+\theta C)}{1+h(\nu+\lambda_1)}$
$E_6 = \frac{1}{1+h(\nu+\lambda_1)}$
$E_7 = \frac{h\rho S}{1+h(\nu+\lambda_1)}$
$E_8 = \frac{h\rho S\theta}{1+h(\nu+\lambda_1)}$
$E_9 = \frac{h\lambda_1}{1+h(\nu+\lambda_2)}$
$E_{10} = \frac{1}{1+h(\nu+\lambda_2)}$
$E_{11} = \frac{hp_3\lambda_2}{1+h(\nu+\lambda_3+u_2)}$
$E_8 = \frac{1+h\nu p_3}{1+h(\nu+\lambda_3+u_2)}$

Firstly we investigate the local asymptotic stability at disease free equilibria. So we need dfe which can be calculated as follows. Put $\dot{S} = \dot{E} = \dot{I} = \dot{C} = 0$ in system (2.6-2.9) and put $E = I = C = 0$ (for disease free case) then the given system of equations yields disease free equilibria $E_{DFE} = (S_{dfe}, E_{dfe}, I_{dfe}, C_{dfe}) = (S_{dfe}, 0, 0, 0)$Where $S_{dfe} = \frac{\nu+(\lambda_4-\nu p_2)}{\nu+u_1+\lambda_4-\nu p_2}$ i.e. $E_{dfe} = (S_{dfe}, 0, 0, 0) = (\frac{\nu+(\lambda_4-\nu p_2)}{\nu+u_1+\lambda_4-\nu p_2}, 0, 0, 0)$. In the absence of vaccination, $u_1 = 0; S_{dfe} = \frac{\nu+(\lambda_4-\nu p_2)}{\nu+\lambda_4-\nu p_2} = 1$. Therefore $E_{dfe} = (1, 0, 0, 0)$

Now at $E_{dfe} = (1, 0, 0, 0)$ the above jacobian matrix becomes

$$J_{NSFD} = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ 0 & F_5 & F_6 & F_7 \\ 0 & F_8 & F_9 & 0 \\ 0 & 0 & F_{10} & F_{11} \end{bmatrix}$$

Where $F_1 = \frac{1+\nu p_2 h}{1+h(u_1+\nu+\lambda_4)}$
$F_2 = \frac{-h(\lambda_4-\nu p_2)}{1+h(u_1+\nu+\lambda_4)}$
$F_3 = \frac{(1+h(u_1+\nu+\lambda_4))(-h(\lambda_4-\nu p_2))-(1+h(\nu+(\lambda_4-\nu p_2)(\nu p_2 h\nu)}{(1+h(u_1+\nu+\lambda_4))^2}$
$F_4 = \frac{(1+h(u_1+\nu+\lambda_4))(-\nu p_1 h-(\lambda_4-\nu p_2))-(1+h(\nu+(\lambda_4-\nu p_2)\nu p_2 h\theta}{(1+h(u_1+\nu+\lambda_4))^2}$
$F_5 = \frac{1}{1+h(\nu+\lambda_1)}$
$F_6 = \frac{h\rho}{1+h(\nu+\lambda_1)}$

$F_7 = \frac{h\rho\theta}{1+h(\nu+\lambda_1)}$
$F_8 = \frac{h\lambda_1}{1+h(\nu+\lambda_2)}$
$F_9 = \frac{1}{1+h(\nu+\lambda_2)}$
$F_{10} = \frac{hp_3\lambda_2}{1+h(\nu+\lambda_3+u_2)}$
$F_{11} = \frac{1+h\nu p_3}{1+h(\nu+\lambda_3+u_2)}$

Applying row operations on above matrix we get the following matrix

$$J_{NSFD} = \begin{bmatrix} G_1 & G_2 & G_3 & G_4 \\ 0 & G_5 & G_6 & G_7 \\ 0 & 0 & G_8 & G_9 \\ 0 & 0 & 0 & G_{10} \end{bmatrix}$$

Where $G_1 = \frac{1+\nu p_2 h}{1+h(u_1+\nu+\lambda_4)}$
$G_2 = \frac{-h(\lambda_4-\nu p_2)}{1+h(u_1+\nu+\lambda_4)}$
$G_3 = F_3$
$G_4 = F_4$
$G_5 = F_5$
$G_6 = F_6$
$G_7 = F_7$
$G_8 = \frac{1}{1+h(\nu+\lambda_2)} + \frac{-h\lambda_1 h\rho}{1+h(\nu+\lambda_2)}$
$G_9 = \frac{-h\lambda_1 h\rho\theta}{1+h(\nu+\lambda_2)}$
$G_{10} = \frac{1+h\nu p_3}{1+h(\nu+\lambda_3+u_2)} + (\frac{-h\nu p_3\lambda_2}{1+h(\nu+\lambda_3+u_2)})(\frac{-h\lambda_1 h\theta\rho}{1-h\lambda_1 h\rho})$

Characteristics Equation of above Jacobian is

$$|J_{NSFD} - \eta I| = 0$$

$$\begin{vmatrix} G_1-\eta & G_2 & G_3 & G_4 \\ 0 & G_5-\eta & G_6 & G_7 \\ 0 & 0 & G_8-\eta & G_9 \\ 0 & 0 & 0 & G_{10}-\eta \end{vmatrix} = 0$$

This is upper triangular matrix whose Eigen values are given by

$\eta_1 = G_1$
$\eta_2 = G_5 = F_5$
$\eta_3 = G_8$
$\eta_4 = G_{10}$

At $h = 0.01$ and $u_1 = 1, u_2 = 1$
$\eta_1 = 0.9832 < 1, \eta_2 = 0.9433 < 1, \eta_3 = 0.9454 < 1, \eta_4 = 0.9345 < 1,$

At $h = 0.01$ and $u_1 = 0, u_2 = 0$
$\eta_1 = 0.9929 < 1, \eta_2 = 0.9433 < 1, \eta_3 = 0.9554 < 1, \eta_4 = 0.9433 < 1,$

At $h = 0.05$ and $u_1 = 1, u_2 = 1$
$\eta_1 = 0.9212 < 1, \eta_2 = 0.7689 < 1, \eta_3 = 0.6829 < 1, \eta_4 = 0.7413 < 1,$

At $h = 0.05$ and $u_1 = 0, u_2 = 0$
$\eta_1 = 0.9657 < 1, \eta_2 = 0.7689 < 1, \eta_3 = 0.6829 < 1, \eta_4 = 0.7698 < 1,$

Therefore, Spectral Radius=Maximum absolute Eigen value =$\eta_1 = |0.9929| < 1$. Similarlyat large step sizes h and using different values of u_1 and u_2 all eigen values remain less than unity. Therefore all the Eigen values of Characteristic equation lie in the unit circle thus by linearized stability theorem our developed NSFD scheme unconditionally possesses local asymptotic stability at DFE.

4.2. **Schur-Cohn Stability Criteria.** For the characteristics polynomial $p(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$ carrying solutions $\lambda_j : j = 1, 2, 3$ of equation $p(\lambda) = 0$ satisfy $|\eta_j| < 1$ if following conditions are satisfied [34]

1. p(1)=1+A$_1 + A_2 + A_3 > 0$
2.$(-1)^3 p(-1) = 1 - A_1 + A_2 - A_3 > 0$
3.$1 - (A_3)^2 > |A_2 - A_1A_3|$

The characteristics equation od Jacobian (4.23) is given by

$$\begin{vmatrix} \eta - F_1 & F_2 & F_3 & F_4 \\ 0 & \eta - F_5 & F_6 & F_7 \\ 0 & F_8 & \eta - F_9 & 0 \\ 0 & 0 & F_{10} & \eta - F_{11} \end{vmatrix} = 0$$

Expanding with respect to first column

$$(F_1 - \eta) \begin{vmatrix} \eta - F_5 & F_6 & F_7 \\ F_8 & \eta - F_9 & 0 \\ 0 & F_{10} & \eta - F_{11} \end{vmatrix} = 0$$

Which implies that

$$(\eta - F_1)(\eta^3 - (F_{11} + F_5 - F_9)\eta^2 + ((F_9F_{11} + F_5(F_{11} + F_9) - (F_8F_6))\eta - ((F_5F_9F_{11}) + (F_8F_6F_{11}) + (F_8F_7F_{10})) = 0 \quad (4.\ 24)$$

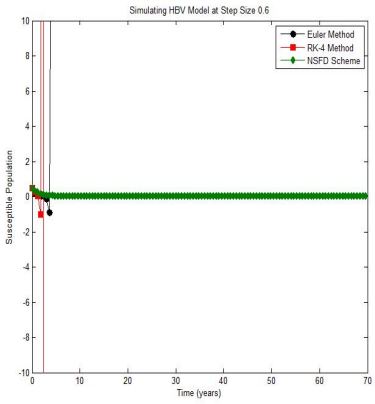

FIGURE 6. Behavior of susceptible population at step size 0.6

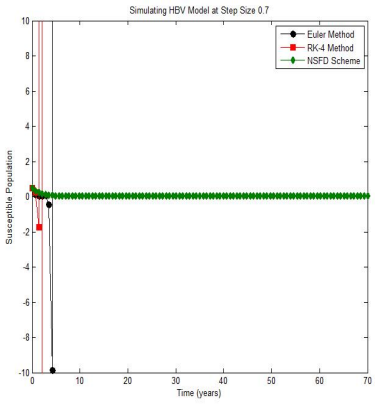

FIGURE 7. Behavior of susceptible population at step size 0.7

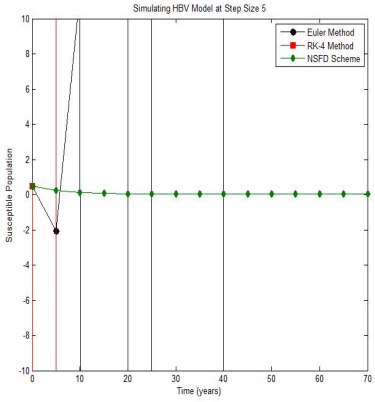

FIGURE 8. Behavior of susceptible population at step size 5

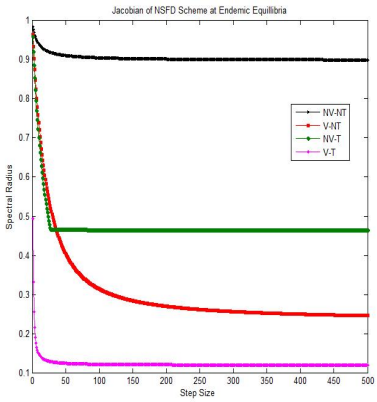

FIGURE 9. Spectral Radius vs Step Size

Above expression can be written in more simplified form as

$$(\eta - F_1)(\eta^3 + A_1\eta^2 + A_2\eta + A_3) = 0$$

Where

$$A_1 = -(F_{11} + F_5 - F_9)$$
$$A_2 = ((F_9F_{11} + F_5(F_{11} + F_9) - (F_8F_6))$$
$$A_3 = -((F_5F_9F_{11}) + (F_8F_6F_{11}) + (F_8F_7F_{10}))$$

Clearly first Eigen value $\eta = F_1 = 0.9996 < 1$ for all combinations of controls and for other Eigen values we consider the polynomial $\eta^3 + A_1\eta^2 + A_2\eta + A_3 = 0$. By substituting all parametric values from Table.1 with combination of $u_1 = 1, u_2 = 1$ and $h = 0.05$ we get

1. p(1)=1+$A_1 + A_2 + A_3 = 0.0542 > 0$

2.$(-1)^3p(-1) = 1 - A_1 + A_2 - A_3 = 2.4161 > 0$

3.$1 - (A_3)^2 > |A_2 - A_1A_3|$ because $1 - (A_3)^2 = 0.9703$ and $|A_2 - A_1A_3| = 0.0727$

Similarly for all combinations of u_1 and u_2 and for all step sizes h above three conditions are satisfied. Thus by Schur-Cohn Stability Criteria all Eigen values lies in the unit circle and this theorem confirms the local asymptotic stability of our developed NSFD scheme. Secondly for stability analysis at Endemic Equilibria we find the jacobian of system of equations (4.19-4.22) at endemic equilibria points. So firstly we find EE of system (2.6-2.9). For endemic equilibria we put $\dot{S} = \dot{E} = \dot{I} = \dot{C} = 0$ in system (2.6-2.9) and replacing (S, E, I, C) by (S^*, E^*, I^*, C^*) then the system of differential equations yields the following endemic equilibria points.

Case-1 $u_1 = 0, u_2 = 0$ (No Vaccination, No Control)

$$S_1^* = \frac{(\nu + \lambda_1)(\nu + \lambda_2)(\nu + \lambda_3 - \nu p_1)}{\rho\lambda_1(\nu + \lambda_3 + \theta p_3\lambda_2 - \nu p_1)}$$
$$E_1^* = \frac{\rho\theta(\nu + \lambda_2)C_1^*S_1^*}{(\nu + \lambda_1)(\nu + \lambda_2) - \rho\lambda_1 S_2^*}$$
$$I_1^* = \frac{\rho\theta\lambda_1 C_1^*S_1^*}{(\nu + \lambda_1)(\nu + \lambda_2) - \rho\lambda_1 S_2^*}$$
$$C_1^* = \frac{p_3\lambda_1\lambda_2(\nu + \lambda_4 - \nu p_2)S_4^*(R_0 - 1)}{(\nu + \lambda_3 - \nu p_1)W}$$

Where

$$W = ((\nu + \lambda_4 - \nu p_2)(\nu + \lambda_2 + \lambda_1) + \lambda_1\lambda_2) + p_3\lambda_1\lambda_2(\nu p_1 + \lambda_4 - \nu p_2)$$

Case-2 $u_1 = 0, u_2 \neq 0$ (No Vaccination, Control)

$$S_2^* = \frac{(\nu + \lambda_1)(\nu + \lambda_2)(\nu + \lambda_3 - \nu p_1 + u_2)}{\rho\lambda_1(\nu + \lambda_3 + \theta p_3\lambda_2 - \nu p_1) + u_2}$$
$$E_2^* = \frac{\rho\theta(\nu + \lambda_2)C_2^*S_2^*}{(\nu + \lambda_1)(\nu + \lambda_2) - \rho\lambda_1 S_2^*}$$
$$I_2^* = \frac{\rho\theta\lambda_1 C_2^*S_2^*}{(\nu + \lambda_1)(\nu + \lambda_2) - \rho\lambda_1 S_2^*}$$

$$C_2^* = \frac{p_3\lambda_1\lambda_2(\nu+\lambda_4-\nu p_2)S_4^*(R_0-1)}{(\nu+\lambda_3-\nu p_1+u_2)W}$$

Case-3 $u_1 \neq 0, u_2 = 0$ (Vaccination, No Control)

$$S_3^* = \frac{(\nu+\lambda_1)(\nu+\lambda_2)(\nu+\lambda_3-\nu p_1)}{\rho\lambda_1(\nu+\lambda_3+\theta p_3\lambda_2-\nu p_1)}$$

$$E_3^* = \frac{\rho\theta(\nu+\lambda_2)C_3^*S_3^*}{(\nu+\lambda_1)(\nu+\lambda_2)-\rho\lambda_1 S_3^*}$$

$$I_3^* = \frac{\rho\theta\lambda_1 C_3^*S_3^*}{(\nu+\lambda_1)(\nu+\lambda_2)-\rho\lambda_1 S_3^*}$$

$$C_3^* = \frac{p_3\lambda_1\lambda_2(\nu+u_1+\lambda_4-\nu p_2)S_4^*(R_0-1)}{(\nu+\lambda_3-\nu p_1)W}$$

Case-4 Generalised Case $u_1 \neq 0, u_2 \neq 0$ (No Vaccination, No Control)

$$S_4^* = \frac{(\nu+\lambda_1)(\nu+\lambda_2)(\nu+\lambda_3-\nu p_1+u_2)}{\rho\lambda_1(\nu+\lambda_3+\theta p_3\lambda_2-\nu p_1+u_2)}$$

$$E_4^* = \frac{\rho\theta(\nu+\lambda_2)C_3^*S_3^*}{(\nu+\lambda_1)(\nu+\lambda_2)-\rho\lambda_1 S_3^*}$$

$$I_4^* = \frac{\rho\theta\lambda_1 C_3^*S_3^*}{(\nu+\lambda_1)(\nu+\lambda_2)-\rho\lambda_1 S_3^*}$$

$$C_4^* = \frac{p_3\lambda_1\lambda_2(\nu+u_1+\lambda_4-\nu p_2)S_4^*(R_0-1)}{(\nu+\lambda_3-\nu p_1+u_2)W}$$

Where

$$R_0 = \rho(FV^{-1}) = \frac{\lambda_1\rho S_{dfe}(\nu+\lambda_3+u_2+p_3\lambda_2\theta-\nu p_1)}{(\nu+\lambda_1)(\nu+\lambda_2)(\nu+\lambda_3+u_2-\nu p_1)}$$

With

$$S_{dfe} = \frac{\nu+(\lambda_4-\nu p_2)}{\nu+u_1+\lambda_4-\nu p_2}$$

Is known as reproduction number which can also be calculated by using recipe of Ven den Driessche and Watmough [35]. We can reduce the value of R_0 by applying both vaccination and treatment. When we insert the parametric values from Table.1 in above endemic equilibria points, these implies endemic point values demonstrated in Table.3 below

Table 3: Endemic Equilibria Points of HBV Model

Controls	R_0	S	C	E	I
$u_1 = 0, u_2 = 0$	2.5288 >1	0.3954	0.00027	0.0552	0.0825
$u_1 = 0, u_2 = 0$	2.5229 >1	0.3964	0.0024	0.0551	0.0824
$u_1 = 1, u_2 = 0$	1.0508 >1	0.3954	0.0002193	0.0044	0.0066
$u_1 = 1, u_2 = 0$	1.0483 >1	0.3964	0.0001794	0.0042	0.0063

And Jacobian of equation (4.19-4.22) at endemic equilibria points are given by

$$J_{NSFD}at(S_1, E_1, I_1, C_1) = (0.3954, 0.0552, 0.0825, 0.0027)$$

$$J_{NSFD}at(S_2, E_2, I_2, C_2) = (0.3964, 0.0551, 0.0824, 0.0024)$$

$$J_{NSFD}at(S_3, E_3, I_3, C_3) = (0.3954, 0.0044, 0.0066, 0.0002193)$$

$$J_{NSFD}at(S_4, E_4, I_4, C_4) = (0.3964, 0.0042, 0.0063, 0.0001794)$$

At all equilibria points we find the absolute maximum Eigen value of the Jacobian matrix which should be less than unity and meet the linearized stability theorem. We have find these values numerically and plotted the spectral radius versus step size for four cases($u_1 = 0, u_2 = 0; u_1 = 0, u_2 = 1; u_1 = 1, u_2 = 0; u_1 = 1, u_2 = 1$) and are shown in Figure.9. Value of spectral radius in all cases remain less than unity which ensure that all Eigen values of the jacobian matrix remain less than unity and ultimately confirms the local stability of NSFD scheme.

5. Results and Discussion

The traditional numerical schemes like Euler method and RK-4 Method generally produce negative, oscillatory, unstable and diverging results when we simulate biological models at large step sizes. Therefore we want a numerical schemes which produces positive, stable and converging results at all step sizes. In this research work we proposed an unconditional nonstandard finite difference scheme for HBV mathematical model which carry these qualities. It produces positive and converging results for this model at large step sizes where Euler and RK-4 methods produce diverging results. We have simulated the HBV model using three numerical schemes by varying step sizes as shown in Figures(2-8). Along vertical axis we have taken the fractions of compartmental populations because we have taken total populations equals to unity and along horizontal axis we have taken time in years. We are interested to study the behavior of these compartmental populations at large step sizes. Figure.2 to Figure.8 demonstrate when we increase the step sizes, Euler method diverges at step size nearly 0.4 and RK-4 method diverges at step size 0.5 but NSFD Scheme remain converging for all large step sizes. The detailed variation of outcomes of three numerical techniques with varying step sizes have been demonstrated in Table-2.Moreover we have proved the stability of our proposed NSFD scheme by using Linearized Stability Theorem and Schur-Cohn Stability Criteria. We have stated the basic theorems and applied them on our proposed NSFD scheme (2.15-2.18). We have found the Eigen values of the characteristics equation for jacobian of discretized model. For DFE all Eigen values are less than unity which confirms that all Eigen values lie in unit circle. This confirms the local asymptotic stability of our proposed NSFD Scheme. For EE we have used numerical codes to find the spectral radius versus step size. The simulationsin Figure.9 show that spectral radius remain less than unity for all four cases. As spectral radius is maximum value of Eigen values thus all Eigen values for endemic case are also less than unity. Both theorems confirmed the local asymptotic stability for our discretized NSFD scheme (2.15-2.18) at DFE and EE.Thus NSFD scheme is also computationally efficient than other two traditional numerical schemes.

6. CONCLUSION

We have proposed a Nonstandard Finite Difference Scheme which carries stable, positive and unconditional solutions, for a model of HBV transmission dynamics. We have numerically solved this model using Euler method, RK-4 method and NSFD scheme. Simulations of the model show that NSFD is better than other two numerical techniques because NSFD always produces positive, stable and converging results at large step sizes where as other two techniques diverges at large step sizes. Finally Schur-Cohn stability criteria and Linearized stability theorem both confirmthe local asymptotic stability of our proposed NSFD scheme.

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8. CONFLICT OF INTEREST

All authors have no conflict of interest to disclose.

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