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A Nonstandard Finite Difference Scheme for a SEI Epidemic Model

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Abstract. In this paper we propose a nonstandard finite difference scheme for an epidemic model which considers the effect of media coverage on the spread of some infectious diseases. We show that this scheme preserves equilibrium points of the corresponding continuous system. Furthere more we study the qualitative properties of the system, such as, positivity, stability of the equilibria and Neimark-Sacker bifurcation. The results demonstrates that the discretized epidemic model is dynamically consistent with the continuous system.

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1. INTRODUCTION

Mathematical models play a key role in analysing the spread and control of infectious disease, taking into account various factors of the disease. When an infectious disease spreads in a population, peoples choose various precautionary actions such as wearing masks, avoiding public places, avoiding travel with sickness, frequent hand washing,... Media coverage and education help the promotions of this precautions and reduce the contact rate of individuals, hence affect the spreading and control of the disease. The impact of media awareness have been studied by many researchers using mathematical modeling,

see [4,10,14,18,23]. Cui et. al. in [4] study the effect of media coverages on the spread of some infectious diseases. They consider the following categories in the population: $S(t)$, the number of susceptibles, $E(t)$, the number of exposed but not infectious individuals, and $I(t)$, the number of infected individuals who are infectious.
By assuming that the total population obey logistic growth, their model takes the following form:

$$\begin{aligned} \frac{dS}{dt} &= bS\left(1-\frac{S}{K}\right) - \mu e^{-mI}SI, \\ \frac{dE}{dt} &= \mu e^{-mI}SI - (c+d)E, \\ \frac{dI}{dt} &= cE - \gamma I, \end{aligned} \tag{1. 1}$$

in which all parameters are positive, and $\beta(I) = \mu e^{-mI}$ is the contact transmission rate, which is not only related to the spreading ability of the disease, but also related to the alertness of each susceptible individual of the population. The parameter m reflects the impact of media coverage to the contact transmission, see [4] for more details.
In this paper we apply nonstandard disceretization procedure to system (1.1). First in section 2, we construct a nonstandard finite difference scheme for (1.1) and study its properties such as positivity, steady states and boundedness of total population. In section 3 we study stability and bifurcations of the disceretized model in some cases. Our results demonstrate that the discretized epidemic model is dynamically consistent with the continuous model, hence it can be applied in numerical studies of the continuous model.

2. The NSFD scheme

A mathematical model for infectious disease such as (1.1) can be denoted by a first order system of ordinary differential equations:

$$\begin{aligned} \frac{dX}{dt} &= f(X,k), \\ X(t_0) &= X_0 \end{aligned} \tag{2. 2}$$

where $f:\mathbb{R}^m \to \mathbb{R}^m$, $X = X(t) : [0,+\infty) \to \mathbb{R}^m$ with initial condition $X_0 \in \mathbb{R}^m$ and $k = (k_1, k_2, ...)$ represents the system parameters.
To transform a continuous system into a discrete system, the continuous variable $t \in [0,+\infty)$ must be replaced by the discrete variable $n \in \mathbb{N}$, and the variable $X(t)$ must be replaced by the discrete values X_n. This transformation yields a difference equation of the form $G(X_{n+1}, X_n) = 0$, where $G : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. In some cases, X_{n+1} is given explicitly in terms of X_n:

$$X_{n+1} = F(X_n) \tag{2. 3}$$

where $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$.
There are several methods for discretization of differential equations such as forward Euler and Runge Kutta methods. The forward Euler method is one of the oldest methods. In this method one consider the following substitions

$$X \to X_n$$

,

$$\frac{dX}{dt} \to \frac{X_{n+1} - X_n}{\Delta t}$$

where Δt is the step size and $X_{n+1} \approx X(t + n\Delta t)$. Mickens, [12], showed that the forward Euler transformation and Runge Kutta methods, leads to numerical instabilities that do not appear in the original differential equation. To avoid these instabilities, Mickens suggest what is known as the nonstandard finite difference method, based on the concept of dynamic consistency, [13], as indicated in [11].
The first order ODE, $\frac{dX}{dt} = f(X, k)$ and the difference equation, $X_{n+1} = F(X_n, k, \Delta t)$, are dynamically consistent, when the difference equation posses the same steady states, stability, bifurcation and chaos of the original differential equation.
A finite difference method is called an NSFD schemes if it has the conditions described in [11], in which one of the main conditions is that nonlinear terms must be replaced by non local terms. Mickens nonstandard discretization method have been used to various kinds of problems, see [1-3,5-6,8,11-13,16-17,19-21]. We apply this method to epidemic model (1.1).
Let $h > 0$ denote the time step size. For the construction of NSFD scheme, we use the following substitions to model (1.1): in the first equation, $S \to S_n$, $S^2 \to S_{n+1}S_n$, $I \to I_n$ and $SI \to S_{n+1}I_n$. In the second equation, $E \to E_{n+1}$ and $SI \to S_{n+1}I_n$. Finally in the third equation, $E \to E_{n+1}$ and $I \to I_{n+1}$. Using this substitions we obtain the following system of difference equations,

$$\begin{aligned} \frac{S_{n+1} - S_n}{h} &= bS_n - \frac{b}{K} S_{n+1} S_n - \mu e^{-mI_n} S_{n+1} I_n, \\ \frac{E_{n+1} - E_n}{h} &= \mu e^{-mI_n} S_{n+1} I_n - (c + d) E_{n+1}, \\ \frac{I_{n+1} - I_n}{h} &= cE_{n+1} - \gamma I_{n+1}. \end{aligned}$$

The above difference system is implicit, but it can be rearranged to its explicit form:

$$\begin{aligned} S_{n+1} &= \frac{(1 + bh)S_n}{1 + \frac{b}{K} h S_n + \mu h I_n e^{-mI_n}}, \\ E_{n+1} &= \frac{E_n + \mu h S_{n+1} I_n e^{-mI_n}}{1 + (c + d)h}, \\ I_{n+1} &= \frac{I_n + chE_{n+1}}{1 + \gamma h}, \end{aligned} \tag{2. 4}$$

An important characteristic of dynamical systems, espesially in those from biology, is that solutions must remain non negative in order to have well posedness.
From the positivity of parameters of (2.1), it is clear that this method preserves positivity of solutions, in fact if $S_0 \geq 0, E_0 \geq 0$ and $I_0 \geq 0$ then $S_n \geq 0, E_n \geq 0$ and $I_n \geq 0$ for all $n \geq 1$.

For a discrete dynamical system defined by (2.2) a steady state $\bar{X} \in \mathbb{R}^m$ respects to the condition $F(\bar{X}) = \bar{X}$, therefore in (2.3) we must solve the following system:

$$\begin{aligned} S &= \frac{(1+bh)S}{1+\frac{b}{K}hS+\mu hIe^{-mI}}, \\ E &= \frac{E+\mu hSIe^{-mI}}{1+(c+d)h}, \\ I &= \frac{I+chE}{1+\gamma h}. \end{aligned} \tag{2. 5}$$

Clearly $(0,0,0)$ and $(K,0,0)$ are equilibrium points of (2.3). Furthere more (2.3) has endemic equilibrium points.
At first as in the continuous case [4], we define the basic reproduction number by

$$R_0 = \frac{\mu c K}{\gamma(c+d)}.$$

Now if $m = 0$, when $R_0 > 1$, (2.3) has the endemic steady state, (S_0^*, E_0^*, I_0^*) as follows:

$$\begin{aligned} S_0^* &= \frac{\gamma(c+d)}{\mu c} = \frac{K}{R_0}, \\ E_0^* &= \frac{b\gamma^2(c+d)}{\mu^2 c^2 K}(R_0 - 1), \\ I_0^* &= \frac{b\gamma(c+d)}{\mu^2 c K}(R_0 - 1). \end{aligned}$$

If $m > 0$, we have:

$$\begin{aligned} S^* &= K(1 - \frac{\mu}{b} I^* e^{-mI^*}) := g(I^*), \\ E^* &= \frac{\gamma}{c} I^*, \\ S^* &= \frac{\gamma(c+d)}{c\mu} e^{mI^*} := h(I^*). \end{aligned}$$

Consider

$$\delta := \frac{\mu}{b}, \quad m_0 := \frac{8\mu}{bR_0} = \frac{8\delta}{R_0}.$$

$$I^* = \frac{mR_0 + 4\delta \pm \sqrt{mR_0(mR_0 - 8\delta)}}{8m\delta}. \tag{2. 6}$$

We have the following result which is the same as Proposition (4.1) proved in [4], about the endemic equilibrium points of (1.1). This result shows that differential system (1.1) and its discretization (2.3) have the same number of equilibrium points.

Proposition 2.1. *Let m_0 be defined as above. When $R_0 > 1$, the system has at least one and at most three positive steady states. Furthermore,*

- *if $0 < m < m_0$, (2.3) has a unique steady state;*
- *if $m > m_0$, the model has three endemic steady state;*
- *if $m = m_0$, (2.3) has one endemic steady state of multiplicity at least 2.*

In this model total population is not constant, we prove its boundedness.

Lemma 2.2. *For any solution* (S_n, E_n, I_n) *of the system (2.3), the total population,* $N_n = S_n + E_n + I_n$*, satisfies*

$$\limsup_{n\to\infty} N_n \leq \frac{bK}{l},$$

where $l = \min\{b, d, \gamma\}$.

Proof. From (2.3) we have

$$S_{n+1} = \frac{(1+bh)S_n}{1 + \frac{b}{K}hS_n + \mu h I_n e^{-mI_n}} \leq \frac{(1+bh)S_n}{1+\frac{b}{K}hS_n}.$$

Let $S_n = \frac{1}{z_n}$, we obtain

$$z_{n+1} \geq \frac{z_n}{1+bh} + \frac{bh}{K(1+bh)}.$$

Hence,

$$z_n \geq \frac{1}{(1+bh)^n} z_0 + \left[1 - \frac{1}{(1+bh)^n}\right]\frac{1}{K}.$$

which yields

$$\limsup_{n\to\infty} S_n \leq K.$$

This relation and (2.3) implies:

$$\begin{aligned} \frac{N_{n+1}-N_n}{h} &= \left(b - \frac{b}{K}S_{n+1}\right)S_n - dE_{n+1} - \gamma I_{n+1} \\ &\leq \left(b - \frac{b}{K}S_{n+1}\right)K - dE_{n+1} - \gamma I_{n+1}, \end{aligned}$$

for large n.
Now we consider $l = \min\{b, d, \gamma\}$, therefore,

$$\frac{N_{n+1}-N_n}{h} \leq bK - lN_{n+1},$$

hence,

$$N_{n+1} \leq \frac{1}{1+lh}N_n + \frac{bhK}{1+lh}.$$

This relation implies

$$\limsup_{n\to\infty} N_n \leq \frac{bK}{l}.$$

□

From this lemma we see that discrete model (2.3) has a compact positively invariant set

$$D = \left\{(S,E,I) \in R^3 \middle| \quad 0 \leq S+E+I \leq \frac{bK}{l}, \quad l = \min\{b,d,\gamma\}\right\},$$

In other words, the above NSFD scheme defines a discrete dynamical system on the region D.

3. Stability and bifurcation

In this section we investigate stability of equilibrium points and existence of Neimark-Sacker bifurcation. To check the locall stability of steady states of equation (2.3), we use the linearization theorem, see [9], which ensures that a steady state $\bar{X}$ is:
(a): Locally asymptotically stable if and only if $\mid \lambda \mid < 1$ for all $\lambda \in \sigma(J)$.
(b): Unstable if and only if $\mid \lambda \mid > 1$ for some $\lambda \in \sigma(J)$.
In which $J = DF(\bar{X})$ is the Jacobian matrix of the system at the equilibrium point $\bar{X}$.
For the simplicity of calculation, we denote the functions of (2.3) as follows:

$$F_1(S,E,I) = \frac{(1+bh)S}{1+\frac{b}{K}hS+\mu hIe^{-mI}}$$

$$F_2(S,E,I) = \frac{E+\mu hF_1(S,E,I)Ie^{-mI}}{1+(c+d)h}$$

$$F_3(S,E,I) = \frac{I+chF_2(S,E,I)}{1+\gamma h}$$

The Jacobian matrix of (2.3) is given by:

$$J(S,E,I) = \begin{pmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I} \\ \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial E} & \frac{\partial F_3}{\partial I} \end{pmatrix},$$

Lemma 3.1. *The equilibrium point* $(0,0,0)$ *is unstable.*

Proof. We have:

$$J(0,0,0) = \begin{pmatrix} 1+bh & 0 & 0 \\ 0 & \frac{1}{1+(c+d)h} & 0 \\ 0 & \frac{ch}{1+\gamma h} & \frac{1}{1+\gamma h} \end{pmatrix},$$

and the eigenvalues of $J(0,0,0)$ are $1+bh$, $\frac{1}{1+(c+d)h}$ and $\frac{1}{1+\gamma h}$. Since $1+bh > 1$, $(0,0,0)$ is unstable. □

Theorem 3.2. *The disease free equilibrium* $(K,0,0)$ *is asymptotically stable if* $R_0 < 1$, *and unstable if* $R_0 > 1$.

Proof. We have the following Jacobian matrix:

$$J(K,0,0) = \begin{pmatrix} 1-\frac{bh}{1+bh} & 0 & -\frac{\mu Kh}{1+bh} \\ 0 & \frac{1}{1+(c+d)h} & \frac{\mu Kh}{1+(c+d)h} \\ 0 & \frac{ch}{1+\gamma h} & \frac{1}{1+\gamma h} \end{pmatrix}.$$

It is clear that $1-\frac{bh}{1+bh} < 1$. Consider the following submatrix

$$B = \begin{pmatrix} \frac{1}{1+(c+d)h} & \frac{\mu Kh}{1+(c+d)h} \\ \frac{ch}{1+\gamma h} & \frac{1}{1+\gamma h} \end{pmatrix}.$$

The so-called Jury condition, [7], states that for a 2×2 matrix B a necessary and sufficient condition in order to have, $\mid \lambda \mid < 1$ for all $\lambda \in \sigma(B)$, is $\mid trB \mid < 1 + \det B < 2$. An easy computation shows $\det B < 1$. Furthere more $trB < 1 + \det B$ is the same as the relation

$$\begin{aligned} & [1 + (c+d)h + 1 + \gamma h]\,(1 + \gamma h)\,[1 + (c+d)h] \\ < \quad & [1 + (c+d)h]^2 (1 + \gamma h)^2 + [1 + (c+d)h](1 + \gamma h)(1 - \mu c K h^2) \end{aligned}$$

which is equivalent to

$$2 + (c+d)h + \gamma h < [1 + (c+d)h](1 + \gamma h) + (1 - \mu c K h^2)$$

this relation is true if and only if

$$\gamma(c+d)h^2 - \mu c K h^2 > 0$$

which is the same as $R_0 < 1$. □

Now we prove global stability of $(K, 0, 0)$ when $m = 0$.

Theorem 3.3. *Let $m = 0$, if $R_0 < 1$, then the disease free equilibrium $(K, 0, 0)$ is globally asymptotically stable.*

Proof. Without loss of generality, we set $h = 1$. Consider the map $F : R_+^2 \to R_+^2$ with

$$F(E, I) = (f_1, f_2)(E, I) = \left(\frac{E + \mu K I}{1 + c + d}, \frac{I + cE}{1 + \gamma} \right)$$

Define

$$V(E, I) = c(1 + c + d)E + (1 + \gamma)(c + d)I.$$

Now V is positive definite at $(0, 0)$, and

$$\begin{aligned} \Delta V(E, I) &= V(F(E, I)) - V(E, I) \\ &= c(1 + c + d)(f_1(E, I) - E) + (1 + \gamma)(c + d)(f_2(E, I) - I) \\ &= c(1 + c + d)\left(\frac{E + \mu K I}{1 + c + d} - E \right) + (1 + \gamma)(c + d)\left(\frac{I + cE}{1 + \gamma} - I \right) \\ &= c(1 + c + d)\left(\frac{\mu K I - (c + d)E}{1 + c + d} \right) + (1 + \gamma)(c + d)\left(\frac{cE - \gamma I}{1 + \gamma} \right) \\ &= [\mu c K - \gamma(c + d)]I = \gamma(c + d)(R_0 - 1)I. \end{aligned}$$

Obviously when $R_0 < 1$, $\Delta V(E, I) \leq 0$. Furthere more, $\Delta V(E, I) = 0$ if and only if $I = 0$. Let

$$G = \left\{ (E, I) \in R^2 \middle| \quad 0 \leq E + I \leq \frac{bK}{l}, \quad l = \min\{b, d, \gamma\} \right\}.$$

The maximal invariant subset of

$$\left\{ (E, I) \in G \middle| \, \Delta V(E, I) = 0 \right\}$$

is the singleton $\{(0, 0)\}$ and LaSalle's invariance principle implies global asymptotic stability of equilibrium point $(0, 0)$. Hence we have

$$\lim_{n \to \infty} E_n = 0, \quad \lim_{n \to \infty} I_n = 0,$$

whenever $R_0 < 1$. Now since $\limsup_{n\to\infty} S_n \leq K$, it is sufficient to prove

$$\liminf_{n\to\infty} S_n \geq K.$$

Since $\lim_{n\to\infty} I_n = 0$, for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$,

$$I_n < \varepsilon.$$

The first equation of (2.3), implies

$$S_{n+1} = \frac{(1+b)S_n}{1 + \frac{b}{K}S_n + \mu I_n} > \frac{(1+b)S_n}{1 + \frac{b}{K}S_n + \mu\varepsilon},$$

for all $n \geq n_0$.
Let $S_n = \frac{1}{\omega_n}$, then

$$\omega_{n+1} < \frac{1+\mu\varepsilon}{1+b}\omega_n + \frac{b}{(1+b)K},$$

for $n \geq n_0$. Therefore

$$\omega_n < \left(\frac{1+\mu\varepsilon}{1+b}\right)^n \omega_0 + \left[1 - \left(\frac{1+\mu\varepsilon}{1+b}\right)^n\right] \frac{b}{K(b-\mu\varepsilon)},$$

for all $n \geq n_0$, which implies

$$\limsup_{n\to\infty} \omega_n \leq \frac{1}{K}.$$

Hence

$$\liminf_{n\to\infty} S_n \geq K$$

therefore $\lim_{n\to\infty} S_n = K$. This completes the proof of the theorem. □

Now we investigate asymptotic stability of endemic equilibrium (S_0^*, E_0^*, I_0^*) when $R_0 > 1$.

Theorem 3.4. *Let $m = 0$ and $h^2 < \frac{1}{2\gamma\alpha}$, then there exists $R_{H_0} > 1$ such that when $1 < R_0 < R_{H_0}$, the endemic equilibrium (S_0^*, E_0^*, I_0^*) is asymptotically stable.*

Proof. The Jacobian matrix at (S_0^*, E_0^*, I_0^*) has the following form,

$$J(S_0^*, E_0^*, I_0^*) = \begin{pmatrix} 1 - \frac{bh}{(1+bh)R_0} & 0 & -\frac{\mu K h}{(1+bh)R_0} \\ \frac{bh(R_0-1)}{(1+(c+d)h)R_0} & \frac{1}{1+(c+d)h} & \frac{\mu K h}{(1+(c+d)h)R_0} \\ 0 & \frac{ch}{1+\gamma h} & \frac{1}{1+\gamma h} \end{pmatrix}.$$

Let $\alpha = c + d$,

$$\begin{aligned} H_1 &= 1 + bh, \\ H_2 &= 1 + \alpha h, \\ H_3 &= 1 + \gamma h, \\ H_4 &= b\gamma\alpha h^3, \\ H_5 &= 1 - \gamma\alpha h^2. \end{aligned}$$

Jacobian matrix has the characteristic equation $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$, with the following coefficients:

$$\begin{aligned} a_0 &= -\det J = -\frac{R_0H_1H_5 - bhH_5 + H_4 - R_0H_4}{R_0H_1H_2H_3}, \\ a_1 &= (\mathrm{tr}J)^2 - \mathrm{tr}J^2 = \frac{R_0H_1H_3 - bhH_3 + R_0H_1H_2 - bhH_2 + R_0H_1H_5}{R_0H_1H_2H_3}, \\ a_2 &= -\mathrm{tr}J = -\frac{R_0H_1H_2H_3 - bhH_2H_3 + R_0H_1H_3 + R_0H_1H_2}{R_0H_1H_2H_3}. \end{aligned}$$

Now Jury condition states that for a 3×3 matrix J, the necessary and sufficient condition in order to have, $\mid \lambda \mid< 1$ for all $\lambda \in \sigma(J)$, is $\mid a_0 \mid< 1$, $\mid a_0 + a_2 \mid< 1 + a_1$ and $a_1 - a_0a_2 < 1 - a_0^2$.

We prove this relations. First we show $|a_0| = |\det J| < 1$. The inequality $\det J < 1$ is equivalent to

$$\frac{R_0H_1H_5 - bhH_5 + H_4 - R_0H_4}{R_0H_1H_2H_3} < 1$$

which is true if and only if

$$R_0H_1H_5 + H_4 < R_0H_1H_2H_3 + bhH_5 + R_0H_4.$$

Now since $R_0 > 1$ and $H_5 < H_2H_3$, this relation is true, therefore $\det J < 1$. The relation $\det J > -1$ is equivalent to

$$R_0H_1H_2H_3 + R_0H_1H_5 + H_4 > bhH_5 + R_0H_4$$

which is true, since $H_4 < H_1H_2H_3$.
Now we show $|a_0 + a_2| < 1 + a_1$. we have

$$1 + a_0 + a_1 + a_2 = \frac{bhH_5 + (R_0 - 1)H_4 - bhH_3 - bhH_2 + bhH_2H_3}{R_0H_1H_2H_3}.$$

Since $R_0 > 1$ and $H_5 - H_3 - H_2 + H_2H_3 > 0$, therefore $1 + a_0 + a_1 + a_2 > 0$.
Moreover,

$$\begin{aligned} 1 - a_0 + a_1 - a_2 \quad = \quad & \frac{2R_0H_1H_2H_3 + 2R_0H_1H_5 - bhH_5 + H_4 - R_0H_4 + 2R_0H_1H_3}{R_0H_1H_2H_3} \\ & + \frac{-bhH_3 + 2R_0H_1H_2 - bhH_2 - bhH_2H_3}{R_0H_1H_2H_3}. \end{aligned}$$

Therefore $1 - a_0 + a_1 - a_2 > 0$ is equivalent to

$$R_0 > \frac{bh(H_5 + H_3 + H_2 + H_2H_3) - H_4}{2H_1(H_5 + H_3 + H_2 + H_2H_3) - H_4} := Q.$$

Since $Q < 1 < R_0$, the above inequality is true. Hence $1 - a_0 + a_1 - a_2 > 0$, therefore $|a_0 + a_2| < 1 + a_1$.

Finally, we investigate the relation $a_1 - a_0a_2 < 1 - a_0^2$. This relation is equivalent to the inequality

$$AR_0^2 + BR_0 + C < 0,$$

with the following coefficients,

$$\begin{aligned} A &= H_1^2H_2H_3^2 + H_1^2H_2^2H_3 + H_1H_2H_3H_4 - H_1^2H_3H_5 + H_1H_3H_4 \\ &\quad -H_1^2H_2H_5 + H_1H_2H_4 - H_1^2H_2^2H_3^2 + H_1^2H_5^2 + H_4^2 - 2H_1H_4H_5, \\ B &= -bhH_1H_2H_3^2 - bhH_1H_2^2H_3 - 3H_1H_2H_3H_4 + 2bhH_1H_2H_3 \\ &\quad -bhH_2H_3H_4 - 2H_1H_3H_4 + bhH_1H_3 - 2H_1H_2H_4 + bhH_1H_2 \\ &\quad +4H_1H_4H_5 - 4H_4^2 + 2bhH_4 - 2bhH_1H_5, \\ C &= 2bhH_2H_3H_4 - (bh)^2H_2H_3 + 4H_4^2 - 4bhH_4 + (bh)^2. \end{aligned}$$

Using MATLAB, we have:

$$\begin{aligned} A &= bh^3(\gamma\alpha + 4b\gamma^2\alpha^2h^3 + 2\gamma^2\alpha h + b\gamma\alpha h + 2b\gamma^2\alpha h^2 + 2\gamma\alpha^2 h + 2b\gamma\alpha^2h^2 \\ &\quad +3\gamma^2\alpha^2h^2) > 0, \\ B &= -bh^3(b\gamma^2h + \gamma\alpha^2h + 3\gamma\alpha + 7\gamma^2\alpha^2h^2 + b\alpha^2h + 6\gamma^2\alpha h + 2b\gamma\alpha h + \gamma^2 + \alpha^2 \\ &\quad +7b\gamma^2\alpha h^2 + 7b\gamma\alpha^2h^2 + 12b\gamma^2\alpha^2h^3) < 0, \\ C &= b^2h^3(3\gamma\alpha h + \alpha + \gamma)(2\alpha\gamma h^2 - 1) < 0. \end{aligned}$$

Consider

$$R_{H_0} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{3. 7}$$

We have $A + B + C < 0$ and $B^2 - 4AC > 0$, hence $R_{H_0} > 1$ and the other root is less than 1. Therefore when $1 < R_0 < R_{H_0}$, the inequality $a_1 - a_0a_2 < 1 - a_0^2$ holds, and the equilibrium point is asymptotic stable. □

By the solution of $R_0 = R_{H_0}$, we have a threshold value $\mu_{H_0} = \frac{\gamma(c+d)}{cK}R_{H_0}$ on the parameter μ for the asymptotic stability of endemic equilibrium.
Now we prove the occurence of Neimark-Sacker bifurcation when $R_0 = R_{H_0}$. We use the following lemma from [22].

Lemma 3.5. *The equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ where $a_i \in R$, $i = 1, 2, 3$, has a pair of complex conjugate eigenvalues lying on the unit circle and the third eigenvalue lies inside the unit circle if and only if the following conditions hold,*

- $|a_0| < 1$,
- $|a_0 + a_2| < 1 + a_1$, *and*
- $a_1 - a_0a_2 = 1 - a_0^2$.

Theorem 3.6. *Let $m = 0$ and $h^2 < \frac{1}{2\gamma\alpha}$, if $R_0 = R_{H_0}$ (i.e. $\mu = \mu_{H_0}$), the endemic equilibrium (S_0^*, E_0^*, I_0^*) is unstable and (2.3) undergoes a Neimark-Sacker Bifurcation.*

Proof: If $R_0 = R_{H_0}$, we have

$$a_1 - a_0a_2 = 1 - a_0^2.$$

Hence conditions of Lemma 4.3 holds. Since $a_1 - a_0a_2 = 1 - a_0^2$, the equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ has the root $\lambda_1 = -a_0$ and this equation has the following form:

$$(\lambda + a_0)\left[\lambda^2 + (a_2 - a_0)\lambda + 1\right] = 0. \tag{3. 8}$$

Since $(a_2-a_0)^2-4<0$, there exists complex conjugate roots $\lambda_{2,3}=e^{\pm i\varphi}$ with $\varphi\in(0,\pi)$ and $\cos\varphi=\frac{1}{2}(a_0-a_2)$. In fact if $e^{i\varphi}$ is a root of $\lambda^2+(a_2-a_0)\lambda+1=0$, by separating the real and imaginary parts we have:

$$\begin{cases} \cos 2\varphi+(a_2-a_0)\cos\varphi &= -1, \\ \sin 2\varphi+(a_2-a_0)\sin\varphi &= 0. \end{cases}$$

Squaring and adding both sides of the above relations yields

$$(a_2-a_0)^2+2(a_2-a_0)\cos\varphi=0,$$

or

$$\cos\varphi=\frac{a_0-a_2}{2}.$$

Now we investigate the conditions of the generic Neimark-Sacker bifurcation theorem [8]. We can see that $a_0-a_2>0$, hence, $\cos\varphi=\frac{a_0-a_2}{2}>0$. Therefore, $\varphi\notin\left\{0,\pm\frac{\pi}{2},\pm\frac{2\pi}{3},\pi\right\}$. As a consequence $e^{ik\varphi}\neq 1$ for $k=1,2,3,4$.

Furthere more

$$\frac{d(\lambda^3+a_2\lambda^2+a_1\lambda)}{d\lambda}\frac{d\lambda}{d\mu}=\frac{d(-a_0)}{dR_0}\frac{dR_0}{d\mu}.$$

Therefore

$$\left[3\lambda^2+2a_2\lambda+a_1\right]\frac{d\lambda}{d\mu}=\left[\frac{bhH_5-H_4}{R_0^2H_1H_2H_3}\right]\frac{dR_0}{d\mu}.$$

Hence,

$$\frac{d\lambda}{d\mu}=\frac{cK}{\gamma(c+d)}\left[\frac{bhH_5-H_4}{R_0^2H_1H_2H_3}\right]\frac{1}{3\lambda^2+2a_2\lambda+a_1}.$$

We must show

$$\frac{d|\lambda(\mu)|^2}{d\mu}\Big|_{\mu=\mu_{H_0}}\neq 0.$$

In fact,

$$\begin{aligned}\frac{d|\lambda|^2}{d\mu}\Big|_{\mu=\mu_{H_0}} &= \left[\bar{\lambda}\frac{d\lambda}{d\mu}+\lambda\frac{d\bar{\lambda}}{d\mu}\right]\Big|_{\lambda=e^{i\varphi}} \\ &= 2Re\left[\bar{\lambda}\frac{d\lambda}{d\mu}\right]\Big|_{\lambda=e^{i\varphi}} \\ &= \frac{2cK}{\gamma(c+d)}\left[\frac{bhH_5-H_4}{R_0^2H_1H_2H_3}\right]Re\left[\frac{\bar{\lambda}}{3\lambda^2+2a_2\lambda+a_1}\right]\Big|_{\lambda=e^{i\varphi}} \\ &= \frac{2cK}{\gamma(c+d)}\left[\frac{bhH_5-H_4}{R_0^2H_1H_2H_3}\right]Re\left[\frac{e^{-i\varphi}}{3e^{2i\varphi}+2a_2e^{i\varphi}+a_1}\right] \\ &= \frac{2cK}{\gamma(c+d)}\left[\frac{bhH_5-H_4}{R_0^2H_1H_2H_3}\right]Re\left[\frac{1}{3e^{3i\varphi}+2a_2e^{2i\varphi}+a_1e^{i\varphi}}\right]. \quad (3.\ 9)\end{aligned}$$

and

$$Re\left[\frac{1}{3e^{3i\varphi}+2a_2e^{2i\varphi}+a_1e^{i\varphi}}\right] = \frac{3\cos 3\varphi+2a_2\cos 2\varphi+a_1\cos\varphi}{|3e^{3i\varphi}+2a_2e^{2i\varphi}+a_1e^{i\varphi}|^2}.$$

Furthere more,

$$3\cos 3\varphi+2a_2\cos 2\varphi+a_1\cos\varphi=12(\cos\varphi)^3-9\cos\varphi+4a_2(\cos\varphi)^2-2a_2+a_1\cos\varphi.$$

Now $\cos\varphi = \frac{a_0 - a_2}{2}$ and $a_1 = 1 - a_0^2 + a_0 a_2$ implies,

$$3\cos 3\varphi + 2a_2\cos 2\varphi + a_1\cos\varphi = 2a_0^3 - 8a_0 - 5a_0^2 a_2 + 4a_0 a_2^2 + 4a_2 - a_2^3.$$

Using MATLAB yields

$$\begin{aligned}
& 3\cos 3\varphi + 2a_2\cos 2\varphi + a_1\cos\varphi \\
= \; & bh^2(-3\gamma\alpha h + R_0\gamma\alpha h - \alpha - \gamma) \\
& \times(5R_0 b\gamma\alpha h^3 + 3R_0\gamma\alpha h^2 - 5b\gamma\alpha h^3 + 2R_0\alpha h - bh^2\alpha + 2R_0 bh^2\alpha \\
& \quad + R_0 + bh + R_0 bh + 2R_0\gamma h - bh^2\gamma + 2R_0 bh^2\gamma) \\
& \times(5R_0 b\gamma\alpha h^3 + 4R_0\gamma\alpha h^2 - 3b\gamma\alpha h^3 + 4R_0\alpha h - bh^2\alpha + 4R_0 bh^2\alpha \\
& \quad + 4R_0 + bh + 4R_0 bh + 4R_0\gamma h - bh^2\gamma + 4R_0 bh^2\gamma).
\end{aligned}$$

Clearly $R_0 > 1$ implies,

$$\begin{aligned}
&(5R_0 b\gamma\alpha h^3 + 3R_0\gamma\alpha h^2 - 5b\gamma\alpha h^3 + 2R_0\alpha h - bh^2\alpha + 2R_0 bh^2\alpha + R_0 + bh + R_0 bh \\
&\quad + 2R_0\gamma h - bh^2\gamma + 2R_0 bh^2\gamma) > 0,
\end{aligned}$$

and

$$\begin{aligned}
&(5R_0 b\gamma\alpha h^3 + 4R_0\gamma\alpha h^2 - 3b\gamma\alpha h^3 + 4R_0\alpha h - bh^2\alpha + 4R_0 bh^2\alpha + 4R_0 + bh + 4R_0 bh \\
&\quad + 4R_0\gamma h - bh^2\gamma + 4R_0 bh^2\gamma) > 0.
\end{aligned}$$

Moreover,

$$(-3\gamma\alpha h + R_0\gamma\alpha h - \alpha - \gamma) \neq 0.$$

In fact $(-3\gamma\alpha h + R_0\gamma\alpha h - \alpha - \gamma) = 0$, implies $R_0 = \frac{3\gamma\alpha h + \alpha + \gamma}{\gamma\alpha h}$ which contradicts the relation $R_0 = R_{H_0}$. Therefore,

$$3\cos 3\varphi + 2a_2\cos 2\varphi + a_1\cos\varphi \neq 0.$$

Furthere more $h^2 < \frac{1}{2\gamma\alpha}$ implies $bhH_5 - H_4 > 0$, hence

$$\frac{d|\lambda(\mu)|^2}{d\mu}\Big|_{\mu=\mu_{H_0}} \neq 0.$$

Now theorem 4.5 in [9] implies the occurence of Niemark-Sacker bifurcation when $R_0 = R_{H_0}$. □

Problem. In theorems 3.3 and 3.4 we study stability and Neimark-Sacker bifurcation of the endemic equilibriums of system with $m = 0$, when $m > 0$, the analysis of endemic equilibriums is steel open and needs furthere work.

4. Numerical simulation

In this section, we use the constructed nonstandard scheme to simulate solutions of (1.1) numerically. The parameter values are the same as in [4]: $K = 5000000$, $b = 0.001$, $\mu = 1.2 \times 10^{-8}$, $c = 0.1$, $d = 0.001$, and $\gamma = 0.05$. In addition, we choose $(S_0, E_0, I_0) =$

$(4999700, 200, 100)$, $h = 0.5$ and the following cases, $m = 0$, $m = 1 \times 10^{-6}$ and $m = 6 \times 10^{-6}$. In this case

$$R_0 = \frac{\mu c K}{\gamma(c+d)} = 1.188.$$

Hence, if $m = 0$, then we have the unique positive equilibrium (S_0^*, E_0^*, I_0^*) where

$$\begin{aligned} S_0^* &= \frac{\gamma(c+d)}{\mu c} = \frac{K}{R_0} = 4208333, \\ E_0^* &= \frac{b\gamma^2(c+d)}{\mu^2 c^2 K}(R_0 - 1) = 6584, \\ I_0^* &= \frac{b\gamma(c+d)}{\mu^2 c K}(R_0 - 1) = 13194. \end{aligned}$$

Moreover, if $m = 1 \times 10^{-6}$ and $m = 6 \times 10^{-6}$, then we have unique endemic equilibriums $(4261135, 6235, 12468)$ and $(4457313, 4790, 9580)$ respectively. Figures (1), (2) and (3) shows that the population approaches the equilibrium value.

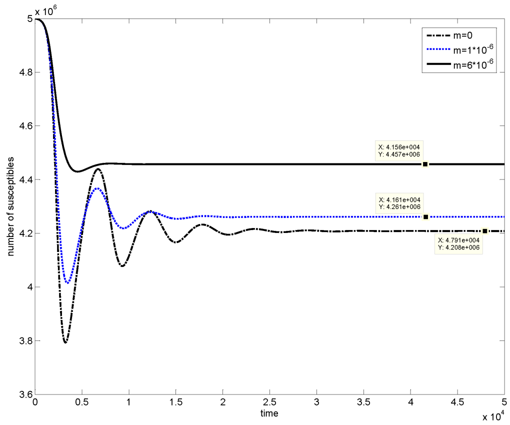

FIGURE 1. The component S of the system which converges to its equilibrium value.

5. CONCLUSION

In this paper we have constructed an NSFD scheme using Mickens discretization method. This scheme were developed in order to use in numerical solution of an SEI model. This SEI model has been used to study the effect of media on the spread of infectious disease. We proved that this scheme preserves positivity, fixed points, their stability nature and bifurcations. In other word the NSFD scheme developed here has dynamic consistency with its continuous counterpart.

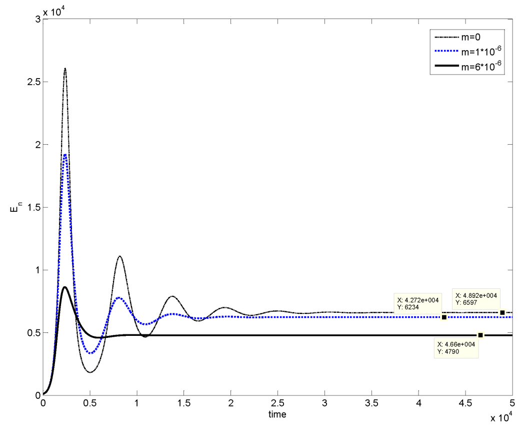

FIGURE 2. The component E of the system which converges to its equilibrium value.

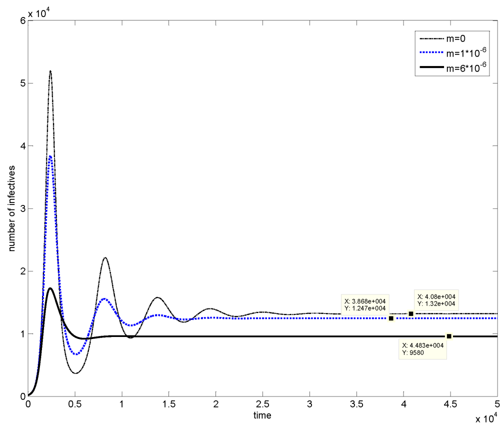

FIGURE 3. The component I of the system which converges to its equilibrium value.

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