Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 49(3)(2017) pp. 1-13

Generalization, Refinement and Reverses of the Right Fejér Inequality for Convex Functions

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Received: 18 April, 2017 / Accepted: 11 July, 2017 / Published online: 28 July, 2017

Abstract. In this paper we establish a generalization of the right Fejér inequality for general Lebesgue integral on measurable spaces as well as a positive *lower bound* and some *upper bounds* for the difference

$$\frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx,$$

where $h:[a,b]\to\mathbb{R}$ is a convex function and $g:[a,b]\to[0,\infty)$ is an integrable weight. Applications for discrete means are also provided.

AMS (MOS) Subject Classification Codes: Primary 26D15; Secondary 26D20.
Key Words: Convex functions, Integral inequalities, Jensen's type inequalities, Fejér type inequalities, Lebesgue integral, Hermite-Hadamard type inequalities, Special means.

1. Introduction

The *Hermite-Hadamard* integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for Special Means, in Information Theory and in Probability Theory and Statistics.

For related results, see for instance the research papers [1]-[6], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], the monograph online [7] and the references therein.

In 1906, Fejér [8], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1.1 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,g(x)\,dx$, where h is a convex function in the interval (a,b) and g is a positive function in the same interval such that*

$$g(a+t) = g(b-t)\,, \quad 0 \le t \le \frac{1}{2}(b-a)\,,$$

i.e., $y = g(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b h(x)\,g(x)\,dx \le \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx. \qquad (1.\ 1)$$

If h is concave on (a,b), then the inequalities reverse in (1. 1).

Clearly, for $g(x) \equiv 1$ on $[a,b]$ we get (HH).

Motivated by the above result, we establish in this paper a generalization of the right inequality (1. 1) for general Lebesgue integral on measurable spaces as well as a positive *lower bound* and some *upper bounds* for the difference

$$\frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx,$$

where g and h are as above.

Applications for discrete means are also provided.

2. General Results

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \le \Phi'_+(x) \le \Phi'_-(y) \le \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \ge \Phi(a) + (x-a)\,\varphi(a) \ \text{ for any } x, a \in I. \qquad (2.\ 2)$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \le \varphi(x) \le \Phi'_+(x) \ \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ – algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e.(almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\, w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$.

The following result holds:

Theorem 2.1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$ and $f : \Omega \to \mathbb{R}$ satisfying the condition*

$$-\infty < m \leq f \leq M < \infty \tag{2. 3}$$

μ-a.e. on Ω and so that $\Phi \circ f,\ f \in L_w(\Omega,\mu)$. Then we have

$$\begin{aligned} 0 &\leq \frac{\Phi(m)+\Phi(M)}{2} + \frac{\Phi(M)-\Phi(m)}{M-m}\int_\Omega \left(f - \frac{m+M}{2}\right) w d\mu \\ &\quad - \int_\Omega (\Phi\circ f)\, w d\mu \\ &\leq \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\int_\Omega (M-f)(f-m)\, w d\mu \\ &\leq \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\left(M - \int_\Omega f w d\mu\right)\left(\int_\Omega f w d\mu - m\right) \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]. \end{aligned} \tag{2. 4}$$

Proof. By the convexity of Φ we have

$$\begin{aligned} \Phi(t) &= \Phi\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) \\ &\leq \frac{M-t}{M-m}\Phi(m) + \frac{t-m}{M-m}\Phi(M) \\ &= \frac{\Phi(m)+\Phi(M)}{2} + \left(\frac{M-t}{M-m} - \frac{1}{2}\right)\Phi(m) + \left(\frac{t-m}{M-m} - \frac{1}{2}\right)\Phi(M) \\ &= \frac{\Phi(m)+\Phi(M)}{2} - \Phi(m)\left(\frac{t - \frac{m+M}{2}}{M-m}\right) + \Phi(M)\left(\frac{t - \frac{m+M}{2}}{M-m}\right) \\ &= \frac{\Phi(m)+\Phi(M)}{2} + \frac{\Phi(M)-\Phi(m)}{M-m}\left(t - \frac{m+M}{2}\right) \end{aligned}$$

for any $t \in [m, M]$.

This inequality implies that

$$\Phi(f(x)) \leq \frac{\Phi(m)+\Phi(M)}{2} + \frac{\Phi(M)-\Phi(m)}{M-m}\left(f(x) - \frac{m+M}{2}\right) \tag{2. 5}$$

for any $x \in \Omega$.

If we multiply (2. 5) by $w \geq 0$ μ-a.e and integrate on Ω, we get the first inequality in (2. 4).

We will prove now that

$$\frac{M-t}{M-m}\Phi(m)+\frac{t-m}{M-m}\Phi(M)-\Phi(t) \tag{2. 6}$$
$$\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}(M-t)(t-m)$$
$$\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right],$$

for any $t\in[m,M]$, and if $\Phi'_{-}(M)$ and $\Phi'_{+}(m)$ are finite, the inequalities in (2. 6) are sharp.

By the convexity of Φ we have $\Phi(t)-\Phi(M)\geq\Phi'_{-}(M)(t-M)$ for any $t\in(m,M)$. If we multiply this inequality with $t-m\geq 0$, we deduce

$$(t-m)\Phi(t)-(t-m)\Phi(M)\geq\Phi'_{-}(M)(t-M)(t-m),\quad t\in(m,M). \tag{2. 7}$$

Similarly, we get

$$(M-t)\Phi(t)-(M-t)\Phi(m)\geq\Phi'_{+}(m)(t-m)(M-t),\quad t\in(m,M). \tag{2. 8}$$

Adding (2. 7) to (2. 8) and dividing by $M-m$, we deduce

$$\Phi(t)-\frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m}\geq\frac{(M-t)(t-m)}{M-m}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right],$$

for any $t\in(m,M)$, which proves the first inequality in (2. 6) for $t\in(m,M)$.

The second inequality in (2. 6) is obvious by the elementary fact that

$$\alpha\beta\leq\left(\frac{\alpha+\beta}{2}\right)^2 \text{ for real } \alpha,\beta. \tag{2. 9}$$

If $t=m$ or $t=M$, the inequality also holds.

Now, assume that (2. 6) holds with D and E greater than zero, i.e.,

$$\begin{aligned}\Phi_{\Phi}(t) &\leq D\cdot\frac{(M-t)(t-m)}{M-m}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]\\ &\leq E(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]\end{aligned}$$

for any $t\in[m,M]$. If we choose $t=\frac{m+M}{2}$, then we get

$$\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\leq\frac{1}{4}D(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right] \tag{2. 10}$$
$$\leq E(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right].$$

Consider $\Phi:[m,M]\to\mathbb{R}$, $\Phi(t)=\left|t-\frac{m+M}{2}\right|$. Then Φ is convex, $\Phi(m)=\Phi(M)=\frac{M-m}{2}$, $\Phi\left(\frac{m+M}{2}\right)=0$, $\Phi'_{-}(M)=1$, $\Phi'_{+}(m)=-1$ and by (2. 10) we deduce

$$\frac{M-m}{2}\leq\frac{1}{2}D(M-m)\leq 2E(M-m),$$

which implies that $D\geq 1$ and $E\geq\frac{1}{4}$.

From (2. 6) we have

$$\begin{aligned}&\frac{M-f(x)}{M-m}\Phi(m)+\frac{f(x)-m}{M-m}\Phi(M)-\Phi(f(x)) \\ &\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}(M-f(x))(f(x)-m)\end{aligned} \tag{2. 11}$$

for μ -a.e. $x \in \Omega$.

If we multiply (2. 11) by $w \geq 0$ μ-a.e and integrate on Ω, we get the second inequality in (2. 4).

Observe that the function $g : [m, M] \to \mathbb{R}$, $g(t) = (M-t)(t-m)$ is a concave function on $[m, M]$. Then by Jensen's inequality for concave functions we have

$$\int_{\Omega}(M-f)(f-m)\,wd\mu \leq \left(M-\int_{\Omega} fwd\mu\right)\left(\int_{\Omega} fwd\mu - m\right),$$

which proves the third inequality in (2. 4).

The last part follows by (2. 9). □

Corollary 2.2. *With the assumptions of Theorem 2.1 and if*

$$\int_{\Omega}\left(f-\frac{m+M}{2}\right)wd\mu = 0, \tag{2. 12}$$

then we have

$$\begin{aligned}0 &\leq \frac{\Phi(m)+\Phi(M)}{2}-\int_{\Omega}(\Phi\circ f)\,wd\mu \\ &\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\int_{\Omega}(M-f)(f-m)\,wd\mu \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right].\end{aligned} \tag{2. 13}$$

Remark 2.3. *Let $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $g : [a,b] \to [0,\infty)$ a Lebesgue integrable function on $[a,b]$. Then from Theorem 2.1 we have*

$$\begin{aligned}0 &\leq \frac{h(a)+h(b)}{2}+\frac{h(b)-h(a)}{b-a}\cdot\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx \\ &-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx \\ &\leq \frac{h'_{-}(b)-h'_{+}(a)}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b(b-x)(x-a)\,g(x)\,dx \\ &\leq \frac{h'_{-}(b)-h'_{+}(a)}{b-a} \\ &\times\left(b-\frac{1}{\int_a^b g(x)\,dx}\int_a^b xg(x)\,dx\right)\left(\frac{1}{\int_a^b g(x)\,dx}\int_a^b xg(x)\,dx-a\right) \\ &\leq \frac{1}{4}(b-a)\left[h'_{-}(b)-h'_{+}(a)\right].\end{aligned} \tag{2. 14}$$

If we assume that $g:[a,b]\to\mathbb{R}$ *is integrable and symmetric on the interval* $[a,b]$ *, then*

$$\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx=0. \tag{2. 15}$$

The converse is obviously not true.

Now, if $g:[a,b]\to[0,\infty)$ *is integrable and satisfies the condition (2. 15), then from (2. 14) we get*

$$\begin{aligned}0&\le\frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx \\ &\le\frac{h'_-(b)-h'_+(a)}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b(b-x)(x-a)\,g(x)\,dx \\ &\le\frac{1}{4}(b-a)\left[h'_-(b)-h'_+(a)\right].\end{aligned} \tag{2. 16}$$

The above inequality (2. 16) provides both a generalization and a reverse for the right Fejér inequality (1. 1) as announced in the introduction.

Example 2.4. *The first two inequalities in (2. 16) can be written as*

$$\begin{aligned}0&\le\frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx-\int_a^b h(x)\,g(x)\,dx \\ &\le\frac{h'_-(b)-h'_+(a)}{b-a}\int_a^b(b-x)(x-a)\,g(x)\,dx.\end{aligned} \tag{2. 17}$$

If in this inequality we make $g:[a,b]\to[0,\infty)$, $g(x)=(b-x)(x-a)$ *and since*

$$\int_a^b(b-x)(x-a)\,dx=\frac{1}{6}(b-a)^3,\quad\int_a^b(b-x)^2(x-a)^2\,dx=\frac{1}{30}(b-a)^5,$$

hence

$$\begin{aligned}0&\le\frac{h(a)+h(b)}{12}(b-a)^3-\int_a^b(b-x)(x-a)\,h(x)\,dx \\ &\le\frac{1}{30}(b-a)^4\left[h'_-(b)-h'_+(a)\right],\end{aligned} \tag{2. 18}$$

for any convex function $h:[a,b]\to\mathbb{R}$.

We have the following result as well:

Theorem 2.5. *Let* $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ *be a convex function on* $[m,M]$ *and* $f:\Omega\to\mathbb{R}$ *satisfying the condition (2. 3)* μ*-a.e. on* Ω *and so that* $\Phi\circ f,\ f\in L_w(\Omega,\mu)$. *Then we* *have*

$$
\begin{aligned}
0 &\leq \left(1-\frac{2}{M-m}\int_{\Omega}\left|t-\frac{M+m}{2}\right|wd\mu\right) \qquad (2.\ 19)\\
&\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\leq \frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\int_{\Omega}\left(f-\frac{m+M}{2}\right)wd\mu\\
&-\int_{\Omega}(\Phi\circ f)\,wd\mu\\
&\leq \left(1+\frac{2}{M-m}\int_{\Omega}\left|t-\frac{M+m}{2}\right|wd\mu\right)\\
&\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right].
\end{aligned}
$$

Proof. We recall the following result obtained by the author in [5] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$
\begin{aligned}
0 &\leq n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right] \qquad (2.\ 20)\\
&\leq \frac{1}{P_n}\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\\
&\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right],
\end{aligned}
$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2. 20) that

$$
\begin{aligned}
0 &\leq 2\min\{\lambda,1-\lambda\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \qquad (2.\ 21)\\
&\leq \lambda\Phi(x)+(1-\lambda)\Phi(y)-\Phi(\lambda x+(1-\lambda)y)\\
&\leq 2\max\{\lambda,1-\lambda\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

If we replace in (2. 21) $C=[x,y]=[m,M]$ and $\lambda=\frac{M-t}{M-m}$, then we get:

$$0\leq 2\min\left\{\frac{M-t}{M-m},\frac{t-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \tag{2. 22}$$
$$\leq\frac{M-t}{M-m}\Phi(m)+\frac{t-m}{M-m}\Phi(M)-\Phi(t)$$
$$\leq 2\max\left\{\frac{M-t}{M-m},\frac{t-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right].$$

The inequality (2. 22) implies that

$$0\leq 2\min\left\{\frac{M-f(x)}{M-m},\frac{f(x)-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \tag{2. 23}$$
$$\leq\frac{M-f(x)}{M-m}\Phi(m)+\frac{f(x)-m}{M-m}\Phi(M)-\Phi(f(x))$$
$$\leq 2\max\left\{\frac{M-f(x)}{M-m},\frac{f(x)-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$

for any $x\in\Omega$.

If we multiply (2. 23) by $w\geq 0$ μ-a.e and integrate on Ω, we get

$$0\leq 2\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \tag{2. 24}$$
$$\times\int_\Omega\min\left\{\frac{M-f}{M-m},\frac{f-m}{M-m}\right\}wd\mu$$
$$\leq\frac{M-\int_\Omega fwd\mu}{M-m}\Phi(m)+\frac{\int_\Omega fwd\mu-m}{M-m}\Phi(M)-\int_\Omega(\Phi\circ f)\,wd\mu$$
$$\leq 2\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\times\int_\Omega\max\left\{\frac{M-f}{M-m},\frac{f-m}{M-m}\right\}wd\mu.$$

Using the elementary facts

$$\min\{\alpha,\beta\}=\frac{1}{2}(\alpha+\beta-|\alpha-\beta|),\ \ \max\{\alpha,\beta\}=\frac{1}{2}(\alpha+\beta+|\alpha-\beta|),$$

where $\alpha,\beta\in\mathbb{R}$, then we have

$$\int_\Omega\min\left\{\frac{M-f}{M-m},\frac{f-m}{M-m}\right\}wd\mu=\frac{1}{2}-\frac{1}{M-m}\int_\Omega\left|f-\frac{M+m}{2}\right|wd\mu \tag{2. 25}$$

and

$$\int_\Omega\max\left\{\frac{M-f}{M-m},\frac{f-m}{M-m}\right\}wd\mu=\frac{1}{2}-\frac{1}{M-m}\int_\Omega\left|f-\frac{M+m}{2}\right|wd\mu. \tag{2. 26}$$

Making use of (2. 24)-(2. 26) we deduce the desired result (2. 19). □

Corollary 2.6. *With the assumptions of Theorem 2.5 and if the condition (2. 12) is satisfied, then*

$$0 \leq \left(1-\frac{2}{M-m}\int_{\Omega}\left|t-\frac{M+m}{2}\right|wd\mu\right) \tag{2. 27}$$
$$\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\leq \frac{\Phi(m)+\Phi(M)}{2}-\int_{\Omega}(\Phi\circ f)\,wd\mu$$
$$\leq \left(1+\frac{2}{M-m}\int_{\Omega}\left|t-\frac{M+m}{2}\right|wd\mu\right)$$
$$\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right].$$

Remark 2.7. *Let $h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function and $g:[a,b]\to[0,\infty)$ a Lebesgue integrable function on $[a,b]$. Then from Theorem 2.5 we have*

$$0 \leq \left(1-\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right) \tag{2. 28}$$
$$\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right]$$

$$\leq \frac{h(a)+h(b)}{2}+\frac{h(b)-h(a)}{b-a}\cdot\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx$$
$$-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx$$

$$\leq \left(1+\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right)$$
$$\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right].$$

Now, if $g:[a,b]\to[0,\infty)$ *is integrable and satisfies the condition (2. 15), then from (2. 28) we get*

$$
\begin{aligned}
0 &\leq \left(1-\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right) \\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right] \\
&\leq \frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx \\
&\leq \left(1+\frac{2}{b-a}\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right) \\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right].
\end{aligned}
\tag{2. 29}
$$

The above inequality (2. 29) provides a generalization, a refinement and a new reverse for the right Fejér inequality (1. 1) as claimed in the introduction.

Example 2.8. *The inequality (2. 29) may be written as*

$$
\begin{aligned}
0 &\leq \left(\int_a^b g(x)\,dx-\frac{2}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right) \\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right] \\
&\leq \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx-\int_a^b h(x)\,g(x)\,dx \\
&\leq \left(\int_a^b g(x)\,dx+\frac{2}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx\right) \\
&\quad\times\left[\frac{h(a)+h(b)}{2}-h\left(\frac{a+b}{2}\right)\right].
\end{aligned}
\tag{2. 30}
$$

If in (2. 30) we take $g(x)=\left|x-\frac{a+b}{2}\right|,\ x\in[a,b]$, *and since*

$$
\int_a^b\left|x-\frac{a+b}{2}\right|dx=\frac{1}{4}(b-a)^2,\quad \int_a^b\left(x-\frac{a+b}{2}\right)^2dx=\frac{1}{12}(b-a)^2,
$$

hence

$$0 \leq \frac{1}{12}(b-a)^2\left[\frac{h(a)+h(b)}{2} - h\left(\frac{a+b}{2}\right)\right] \tag{2. 31}$$

$$\leq \frac{h(a)+h(b)}{8}(b-a)^2 - \int_a^b \left|x - \frac{a+b}{2}\right| h(x)\,dx$$

$$\leq \frac{5}{12}(b-a)^2\left[\frac{h(a)+h(b)}{2} - h\left(\frac{a+b}{2}\right)\right],$$

for any convex function $h : [a,b] \to \mathbb{R}$.

3. Applications for Discrete Inequalities

Let $\mathbf{x} = (x_1, ..., x_n)$ be an n-tuple with $x_i \in \mathbb{R}$, $i \in \{1, ..., n\}$ and $\mathbf{p} = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

If $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a convex function on $[m, M]$, then for any $\mathbf{x} = (x_1, ..., x_n)$ with $x_i \in (m, M) \subset \mathbb{R}$, $i \in \{1, ..., n\}$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n)$ we have from (2. 4) and (2. 19) for the discrete measure that

$$0 \leq \frac{\Phi(m)+\Phi(M)}{2} + \frac{\Phi(M)-\Phi(m)}{M-m}\sum_{i=1}^n p_i\left(x_i - \frac{m+M}{2}\right) - \sum_{i=1}^n p_i \Phi(x_i) \tag{3. 32}$$

$$\leq \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\sum_{i=1}^n p_i (M-x_i)(x_i-m)$$

$$\leq \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\left(M - \sum_{i=1}^n p_i x_i\right)\left(\sum_{i=1}^n p_i x_i - m\right)$$

$$\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right],$$

$$0 \leq \left(1 - \frac{2}{M-m}\sum_{i=1}^n p_i \left|x_i - \frac{m+M}{2}\right|\right)\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \tag{3. 33}$$

$$\leq \frac{\Phi(m)+\Phi(M)}{2} + \frac{\Phi(M)-\Phi(m)}{M-m}\sum_{i=1}^n p_i\left(x_i - \frac{m+M}{2}\right) - \sum_{i=1}^n p_i \Phi(x_i)$$

$$\leq \left(1 + \frac{2}{M-m}\sum_{i=1}^n p_i \left|x_i - \frac{m+M}{2}\right|\right)\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]$$

and the corresponding inequalities if $\sum_{i=1}^n p_i\left(x_i - \frac{m+M}{2}\right) = 0$.

If we write the inequalities (3. 32) and (3. 33) for the convex power function $\Phi(t) = t^p, p \in (-\infty, 0) \cup (1, \infty)$, we have

$$0 \leq \frac{m^p + M^p}{2} + \frac{M^p - m^p}{M - m} \sum_{i=1}^{n} p_i \left(x_i - \frac{m+M}{2} \right) - \sum_{i=1}^{n} p_i x_i^p \tag{3. 34}$$
$$\leq p \frac{M^{p-1} - m^{p-1}}{M - m} \sum_{i=1}^{n} p_i (M - x_i)(x_i - m)$$
$$\leq p \frac{M^{p-1} - m^{p-1}}{M - m} \left(M - \sum_{i=1}^{n} p_i x_i \right) \left(\sum_{i=1}^{n} p_i x_i - m \right)$$
$$\leq \frac{1}{4} p (M - m) \left(M^{p-1} - m^{p-1} \right)$$

and

$$0 \leq \left(1 - \frac{2}{M-m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m+M}{2} \right| \right) \left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2} \right)^p \right] \tag{3. 35}$$
$$\leq \frac{m^p + M^p}{2} + \frac{M^p - m^p}{M - m} \sum_{i=1}^{n} p_i \left(x_i - \frac{m+M}{2} \right) - \sum_{i=1}^{n} p_i x_i^p$$
$$\leq \left(1 + \frac{2}{M-m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m+M}{2} \right| \right) \left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2} \right)^p \right].$$

If we write the inequalities (3. 32) and (3. 33) for the convex function $\Phi(t) = -\ln t$, then we get

$$0 \leq \sum_{i=1}^{n} p_i \ln x_i - \ln G(m, M) - \frac{1}{L(m, M)} \sum_{i=1}^{n} p_i \left(x_i - \frac{m+M}{2} \right) \tag{3. 36}$$
$$\leq \frac{1}{G^2(m, M)} \sum_{i=1}^{n} p_i (M - x_i)(x_i - m)$$
$$\leq \frac{1}{G^2(m, M)} \left(M - \sum_{i=1}^{n} p_i x_i \right) \left(\sum_{i=1}^{n} p_i x_i - m \right) \leq \frac{1}{4} \frac{(M-m)^2}{mM},$$

and

$$0 \leq \left(1 - \frac{2}{M-m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m+M}{2} \right| \right) \ln \left(\frac{A(m, M)}{G(m, M)} \right) \tag{3. 37}$$
$$\leq \sum_{i=1}^{n} p_i \ln x_i - \ln G(m, M) - \frac{1}{L(m, M)} \sum_{i=1}^{n} p_i \left(x_i - \frac{m+M}{2} \right)$$
$$\leq \left(1 + \frac{2}{M-m} \sum_{i=1}^{n} p_i \left| x_i - \frac{m+M}{2} \right| \right) \ln \left(\frac{A(m, M)}{G(m, M)} \right),$$

where $A(m,M) := \frac{m+M}{2}$ is *the arithmetic mean,* $G(m,M) := \sqrt{mM}$ is the *geometric mean* and

$$L(m,M) := \frac{M-m}{\ln M - \ln m}$$

is the *logarithmic mean*.

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