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Approximate Nonlinear Self-Adjointness and Approximate Conservation Laws of the Gardner Equation

Narges Pourrostami
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran ,Iran.
Email: pourrostami@phd.pnu.ac.ir

Mehdi Nadjafikhah
School of Mathematics, Department of Pure Mathematics, Iran University of Science and Technology, Narmak-16, Tehran, Iran
Email: m_nadjafikhah@iust.ac.ir

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Abstract. In this paper, we prove that the Gardner equation with the small parameter is approximately nonlinear self-adjoint. This property is important for constructing approximate conservation laws associated with approximate symmetries. We utilize first-order approximate symmetries for constructing approximate conservation laws.

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1. Introduction

Canonical form of the Kortewege-de Vrise (KdV) equation is, $u_t - 6uu_x + u_{xxx} = 0$. This PDE is a mathematical model for describing weakly nonlinear long waves. Gardner et al.(1967-1974) published several papers about KdV equation. In (1968), "Miura transformation" was intruduced by Miura, in ([7],[8])

$$u = v^2 + v_x, \tag{1. 1}$$

to determine an infinite number of conservation law. If we put $v = 1/2\epsilon^{-1} + \epsilon w$ where ϵ is an arbitrary real parameter, then Miura transformation becomes:

$$u = 1/4\epsilon^{-2} + w + \epsilon w_x + \epsilon^2 w^2 \tag{1. 2}$$

However, since any arbitrary constant is a trivial solution of KdV equation, it may be removed by a Galilean transformation, so we just consider ?Gardner transfos-mation?, means,

$$u = w + \epsilon w_x + \epsilon^2 w^2, \tag{1. 3}$$

where ϵ is an arbitrary real parameter. Substituting the above transformation in KdV equation shows that w satisfies in ”Gardner equation”, $w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx} = 0$, for all ϵ. (see [2]):

$$\begin{aligned} 0 &= u_t - 6uu_x + u_{xxx} \qquad (1.\ 4)\\ &= w_t + \epsilon w_{xt} + 2\epsilon^2 ww_t - 6(w + \epsilon w_x + \epsilon^2 w^2)(w_x + \epsilon w_{xx} + 2\epsilon^2 ww_x) \\ &\quad + w_{xxx} + \epsilon w_{xxxx} + 2\epsilon^2 (ww_x)_{xx} \\ &= (1 + \epsilon \frac{\partial}{\partial x} + 2\epsilon^2 w)\{w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx}\}. \end{aligned}$$

In other words, if we consider

$$(1 + \epsilon \frac{\partial}{\partial x} + \epsilon^2 w^2) F(x, t, \epsilon, w) = 0, \tag{1. 5}$$

we have $F = h(t, \epsilon, w)e^{-2\epsilon x w - \frac{x}{\epsilon}}$, where h is an arbitrary function. As a special case, for $h \equiv 0$, we have a Gardner equation.

If we put $\epsilon = \epsilon^2$ for small real parameter ϵ, it becomes:

$$w_t - 6(w + \epsilon w^2)w_x + w_{xxx} = 0, \tag{1. 6}$$

for all ϵ. Approximate symmetries of Eq.(1. 6) are analysed with a method introduced by Baikov, Gazizov and Ibragimov, in [1].

The method of nonlinear self-adjointness and new conservation law theorem was introduced by Ibragimov in [3]. Consequently, conservation laws which cannot be obtained by Noether theorem, are constructed using this method. This method can be extendend to differential equation with small parameter.

In this paper, we calculate approximately adjoint equation to ”Gardner equation” and then we construct approximate conservation laws using approximate symmetries and carry out all computations to first order of approximation with respect to ϵ.

2. PRELIMINARIES

In this section, we recall the procedure in [3],[4],[5]. We consider a system of m (linear or nonlinear) differential equations,

$$F_\alpha(x, u, u_{(1)}, .., u_{(s)}) = 0 \qquad \alpha = 1, .., m, \tag{2. 7}$$

where $x = (x^1, .., x^n)$ and $u = (u^1, .., u^m)$ are independent and dependent variables, and $u_{(1)} = \partial u^\alpha / \partial x^i$, $u_{(2)} = \partial^2 u^\alpha / \partial x^i \partial x^j$. The equation adjoint to (2. 7) are written in the form:

$$F^*_\alpha(x, u, v, u_{(1)}, v_{(1)}, .., u_{(s)}, v_{(s)}) = \frac{\delta L}{\delta u^\alpha} = 0 \qquad \alpha = 1, .., m, \tag{2. 8}$$

where, $v = (v^1, .., v^m)$ are new dependent variables. Here L is called formal Lagrangian for equation (2. 7), and given by $L = \sum_{B=1}^{m} v^\beta F_\beta(x, u, u_{(1)}, .., u_{(s)})$, and $\delta/\delta u^\alpha = \partial/\partial u^\alpha + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \partial/\partial u^\alpha_{i_1..i_s}$, is the variational derivative where D_i is the operator of total differentiation. The system (2. 7) is said to be nonlinearly self-adjoint if the adjoint system (2. 8) is satisfied for all the solutions of (2. 7) after a substitution,

$$v^\alpha = \phi^\alpha(x, u) \qquad \alpha = 1, .., n, \tag{2. 9}$$

under the condition that not all ϕ^α vanish identically. This definition is equvalent to the condition,

$$F^*_\alpha(x, u, \phi(x, u), .., u_{(s)}, \phi_{(s)}) = \lambda^\beta_\alpha F_\beta(x, u, .., u_{(s)}) \quad \alpha = 1, .., m, \tag{2. 10}$$

where λ^β_α are indeterminate variable coefficients (λ^β_α don‘t become infinite on solutions of the equation (2. 7)). When our system is perturbed system (system with small parameter) and if we use,

$$v^\alpha = \phi^\alpha(x, u) + \epsilon\psi^\alpha(x, u) \qquad \alpha = 1, .., n, \tag{2. 11}$$

such that not all ϕ^α and ψ^α are identically equal to zero instead of condition (2. 9), the perturbed system is called approximate nonlinear self-adjointness, and we can find approximate conservation laws associated with approximate symmetry with the following theorem. We have a main theorem [5]:

Theorem 2.1. *Any infinitesimal symmetry*

$$X = \xi^?(x, u, u_{(1)}, \cdots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \cdots)\frac{\partial}{\partial u^\alpha}, \tag{2. 12}$$

of a nonlinearly self-adjoint system leads to a conservation law $D_i(C^i) = 0$ *constructed by the formula,*

$$\begin{aligned} C^i &= \xi^i L + W^\alpha\left[\frac{\partial L}{\partial u^\alpha_i} - D_j\Big(\frac{\partial L}{\partial u^\alpha_{ij}}\Big) + D_j D_k\Big(\frac{\partial L}{\partial u^\alpha_{ijk}}\Big) - \cdots\right] \\ &\quad + D_j(W^\alpha)\left[\frac{\partial L}{\partial u^\alpha_{ij}} - D_k\Big(\frac{\partial L}{\partial u^\alpha_{ijk}}\Big) + \cdots\right] + D_j D_k(W^\alpha)\left[\frac{\partial L}{\partial u^\alpha_{ijk}} - \cdots\right], \end{aligned} \tag{2. 13}$$

where $W^\alpha = \phi^\alpha - \xi^j u^\alpha_j$*, and* L *is the formal Lagrangian.*

3. Approximate self-adjointness

We write Eq.(2. 7) in the form:

$$F \equiv u_t - 6(u + \epsilon u^2)u_x + u_{xxx} = 0. \tag{3. 14}$$

Formal Lagrangian for Eq.(3. 14) is:

$$L \equiv v(u_t - 6(u + \epsilon u^2)u_x + u_{xxx}). \tag{3. 15}$$

Then the following equation,

$$F^* \equiv v_t - 6(u + \epsilon u^2)v_x + v_{xxx} = 0, \tag{3. 16}$$

is approximately adjoint to Eq.(3. 14) . We look for the substitution

$$v(t,x,u,\epsilon)\simeq\phi(t,x,u)+\epsilon\psi(t,x,u), \tag{3. 17}$$

that is satisfying in nonlinear self-adjointness condition

$$F^*|_{v(t,x,u,\epsilon)\simeq\phi(t,x,u)+\epsilon\psi(t,x,u)}\simeq\lambda F(t,x,u,\epsilon). \tag{3. 18}$$

After substitutiong (3. 14) and (3. 16) into (3. 18), we conclude that:

$$\lambda=\phi_u, \tag{3. 19}$$

and we have,

$$\begin{aligned}&3\phi_{xxu}u_x+3\phi_{xuu}u_x^2+3\phi_{xu}u_{xx}+\phi_{xxx}\\&+3\phi_{uu}u_xu_{xx}+\phi_{uuu}u_x^3+\phi_t+\epsilon\psi_t+\epsilon\psi_{xxx}\\&-6(u+\epsilon u^2)\phi_x-6\epsilon(u+\epsilon u^2)\psi_x+\epsilon\psi_{uuu}u_x^3\\&+3\epsilon\psi_{uxx}u_x+\epsilon\psi_{xuu}u_x^2+3\epsilon\psi_{xu}u_{xx}+3\epsilon\psi_{uu}u_xu_{xx}=0.\end{aligned} \tag{3. 20}$$

For calculating ϕ, we consider the non contain ϵ terms in (3. 20) and for calculating ψ, we consider in (3. 20), only the linear terms in ϵ. Then $\phi_{uu}=0$, $\phi_{ux}=0$ and $6u\phi_x-\phi_t-\phi_{xuu}=0$ lead to:

$$\phi=A_1(6tu+x)+A_2u+A_3. \tag{3. 21}$$

Accordingly, $\psi_{uu}=0$, $\psi_{ux}=0$ and $6\psi_xu-\psi_t-\psi_{xxx}+6A_1u^2=0$ lead to

$$\phi=(H_1t++H_2)u+\frac{1}{6}H_1x+H_3, \tag{3. 22}$$

and $A_1=0$.

Proposition 3.1. *The approximate substitution is,*

$$v=A_2u+A_3+\epsilon[(A_4t++A_5)u+\frac{1}{6}A_4x+A_6], \tag{3. 23}$$

where A_i, $i=2,..,6$ are arbitrary constant. That makes the Gardner equation (3. 14) approximately self-adjoint.

4. Approximate conservation laws

Approximate symmetries of (3. 14) in [9] are:

$$\begin{aligned}&v_1=\partial_x,\qquad v_2=\partial_t,\qquad v_3=6t\partial_x+(2\epsilon u-1)\partial_u,\qquad v_4=\epsilon v_1,\\&v_5=\epsilon v_2,\qquad v_6=\epsilon\Big(6t\frac{\partial}{\partial x}-\partial_u\Big),\qquad v_7=\epsilon\Big(x\partial_x+3t\frac{\partial}{\partial t}-2u\partial_u\Big).\end{aligned} \tag{4. 24}$$

We can now construct approximate conservation laws

$$[D_t(C^1)+D_x(C^2)]\;|_{Eq.(3.14)}\approx 0, \tag{4. 25}$$

By applying the formula (2. 13). We perform all computations to first order of approximation with respect to ϵ. The conserved vector for (3. 14) is:

$$\begin{aligned}C^1&=Wv,\\C^2&=W(-6(u+\epsilon u^2)v+v_{xx})-v_xD_x(W)+vD_x^2(W).\end{aligned} \tag{4. 26}$$

We obtain $W_i = W$ for corresponding v_i, $i = 1..7$ as showen in Table 1. We can calculate the conserved vector C^1 and C^2 (4. 26) for the approximate symmetries $v_i, i = 1..7$ in (4. 24) in Table 2. We eliminate u_t with the help of (3. 14). We can consider the special cases for calculating approximate conservation laws by substituting variety constants instead of A_i. For instance, by considering $A_2 = 1$ and $A_3 = A_4 = A_5 = A_6 = 0$, we have one approximate conserved vector.

Table 1

approximate symmetry	correponding W_i
$v_1 = \partial_x$	$W_1 = -u_x$
$v_2 = \partial_t$	$W_2 = -u_t$
$v_3 = 6t\partial_x + (2\epsilon u - 1)\partial_u$	$W_3 = (2u\epsilon - 1) - 6tu_x$
$v_4 = \epsilon v_1$	$W_4 = -\epsilon u_x$
$v_5 = \epsilon v_2$	$W_5 = -\epsilon u_t$
$v_6 = \epsilon(6t\partial_x - \partial_u)$	$W_6 = -\epsilon - 6t\epsilon u_x$
$v_7 = \epsilon(x\partial_x + 3t\partial_t - 2u\partial_u)$	$W_7 = -2\epsilon u - 3t\epsilon u_t - \epsilon x u_x$

Table 2

case	C^1	C^2
v_1	$-u_x \;\; A_2u + A_3 + \epsilon((A_4t + A_5)u$ $+1/6A_4x + A_6)$	$-u_x \;\; - 6A_2u - 6A_3 - 6\epsilon((A_4t + A_5)u$ $+1/6A_4x + A_6) + A_2u_{xx}$ $+\epsilon((A_4t + A_5)u_{xx}) \;\; + u_{xx} \;\; A_2u_x$ $+\epsilon((A_4t + A_5)u_x + 1/6A_4)$ $-u_{xxx} \;\; A_2u + A_3$ $+\epsilon((A_4t + A_5)u + 1/6A_4x + A_6)$
v_2	$- 6\epsilon u^2u_xu_{xxx}(A_2u + A_3)$ $+ \;\; - 6uu_xu_{xxx}(A_2u + A_3$ $+\epsilon((A_4 + A_5)u + 1/6A_4x + A_6)$	$\epsilon \;\; ((A_4t + A_5)u_x + 1/6A_4)$ $(36uu_x^3 + 36u^2u_xu_{xx} - 6uu_xu_{xxxx}$ $-6u_x^2u_{xxx} - 6uu_{xx}u_{xxx}$ $+u_{xxx}u_{xxxx}) \;\; + \;\; 36A_2u^2u_x^3$ $+36A_2u^3u_xu_{xx} - 6u^2u_xA_2u_{xxxx}$ $+72A_2u^2u_x^4 + 36A_2u^3u_x^2u_{xx}$ $-12A_2uu_x^3u_{xxx} - 6A_2u_xu^2u_{xx}u_{xxx}$ $- 6A_2u - 6A_3 + A_2u_{xx}$ $+ \;\; 36A_2uu_x^4 + 36A_2u^2u_x^2u_{xx}$ $-6A_2uu_x^2u_{xxxx} - 6A_2u_x^3u_{xxx}$ $-6A_2uu_xu_{xx}u_{xxx} + A_2u_xu_{xxx}u_{xxxx}$ $- 6A_2u - 6A_36\epsilon((A_4t + A_5)u$ $+1/6A_4x + A_6) + A_2u_{xx}$ $+\epsilon((A_4t + A_5)u_{xx})$ $+ \;\; \epsilon((A_4t + A_5)u + 1/6A_4x + A_6)$ $- 6u_{xx}u_x - 12u_xu_{xx}$ $-6uu_{xxx} + u_{xxxxx} \;\; + \;\; A_2u + A_3$ $(-12\epsilon u_x^2 + (-12\epsilon u - 6)u_{xx})u_x$ $+2(-12\epsilon u - 6)u_xu_{xx}$ $+(-6\epsilon u^2 - 6u)u_{xxx} + u_{xxxxx}$

v_3	$2\epsilon u(A_2u + A_3)\ +\ (-6tu_x - 1)$ $\times(A_2u + A_3 + \epsilon((A_4 + A_5)u$ $+1/6A_4x + A_6))$	$\epsilon\ \ - 2u_x - 12tuu_{xx} - 12tu_xu_2$ $- 6A_2u - 6A_3 + A_2u_{xx})$ $-(A_2u_x\ \ +\ \ - 6A_2u - 6A_3$ $-6\epsilon((A_4t + A_5)u + 1/6A_4x + A_6)$ $+A_2u_{xx} + \epsilon((A_4t + A_5)u_{xx}))$ $-(A_2u_x + \epsilon((A_4t + A_5)u_x + 1/6A_4)$ $36t^2u_2u_{xx} + 6tu_{xx}$ $+\ A_2u + A_3\ \ \ \epsilon u_{xx}$ $+\ A_2u + A_3 + \epsilon((A_4t + A_5)u$ $+1/6A_4x + A_6)\ \ \ - 6tu_{xxx}$
v_4	$-\epsilon u_x\ \ A_2u + A_3$	$-\epsilon u_x\ \ - 6A_2u$ $-6A_3 + A_2u_{xx}\ \ +\ \ A_2u_x\ \ \epsilon u_{xx}$ $-\ \ A_2u + A_3\ \ \epsilon u_{xxx}$
v_5	$-\epsilon\ \ 6uu_x - u_{xxx}\ \ \ \ A_2u + A_3$	$-\epsilon\ \ 6uu_x - u_{xxx}$ $- 6A_2u - 6A_3 + A_2u_{xx}\ \ -\ \ A_2u_x$ $- \epsilon 6u_x^2 - \epsilon 6uu_{xx} + \epsilon u_{xxxx}$ $+\epsilon\ \ A_2u + A_3$ $- 6u_{xx}u_x - 12u_xu_{xx}$ $-6uu_{xxx} + u_{xxxxx}$
v_6	$- 6t\epsilon u_x - \epsilon\ \ \ A_2u + A_3$	$- 6t\epsilon u_x - \epsilon\ \ \ - 6A_2u - 6A_3$ $+A_2u_{xx}\ \ + 6\epsilon tu_{xx}\ \ A_2u_x$ $-6\epsilon tu_{xxx}\ \ A_2u + A_3$
v_7	$A_2u + A_3\ \ \ \ - 2\epsilon u - 3t\epsilon$ $(6uu_x - u_{xxx}) - \epsilon xu_x$	$\epsilon\ \ - 2u - 3t(6uu_x - u_{xxx}) - xu_x$ $- 6A_2u - 6A_3 + A_2u_{xx}$ $-\epsilon\ \ A_2u_x\ \ \ - u_x$ $+(-2 - 3t6u_x)u_x + (-3t(6u) - x)u_{xx}$ $+3tu_{xxxx}\ \ + \epsilon\ \ A_2u + A_3$ $- u_{xx} - 18tu_{xx}u_x + (-3 - 36tu_x)u_{xx}$ $+(-18tu - x)u_{xxx} + 3tu_{xxxxx}$

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