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Derivations of a Subclass of Filiform Leibniz Algebras

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Abstract. In this paper we study the derivations of a subclass of Leibniz algebras. We tabulate the basis derivations of this class in low-dimensional cases. Then we construct a basis of the derivation algebra for algebras from this class.

AMS (MOS) Subject Classification Codes: 15AXX; 16S50; 20GXX
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1. Introduction

During the last century, the theory of Lie algebras has become a center of interest for mathematicians as well as physicists. This theory extensively studied by many authors, like([7], [8]). In 1993 French mathematician J-Loday discovered a generalization of Lie algebra called a (left) Liebinz algebra, where every left multiplication operator is a derivation([5], [9]). There is the concept of right Leibniz algebra, where every right multiplication operator also is a derivation. The anti-symmetry condition of the Lie algebra ($[a, a] = 0$) not necessary to be hold in a Leibniz algebra, hence Leibniz algebra is a non commutative analogue of the Lie algebra. Derivations of low-dimensional Leibniz algebras were studied by Rakhimov and AL-Nashri up to dimension eight([1], [2], [12], [13]). In this work we describe the derivations of a subclass of filiform Leibniz algebras. The outline of this paper is as follows: In Section 2 we present some preliminary results on the Leibniz algebras, basic definitions and properties are given. In Section 3 we study the derivations of the subclass mentioned above in low dimensional cases. Finally, we present the program used in the course of our calculations.

2. Preliminary Results

In this section we provide some basic definitions and properties of Leibniz algebras, and focus on so-called filiform Leibniz algebras.

Definition 2.1. *(See*[10]*) An algebra L over a field K is called a Leibniz algebra if its bilinear operation $[.,.]$ satisfies the following Leibniz identity:*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],\ \forall\, x,y,z \in L.$$

From now onwards, all algebras are assumed to be over the fields of complex numbers $\mathbb{C}$. Throughout the paper, we denote by L the Leibniz algebra, and let us use the following sequence: $L^1 = L, L^{k+1} = [L^k, L], k \geqslant 1$.

Definition 2.2. *(See*[3]*) A Leibniz algebra L is nilpotent if there exists $s \in \mathbb{N}$ such that $L^1 \supset L^2 \supset \ldots \supset L^s = 0$.*

Definition 2.3. *(See*[6]*) A Leibniz algebra L is filiform if $dim(L^i) = n-i$, where $n = dimL$ and $2 \leqslant i \leqslant n$.*

Definition 2.4. *(See*[4]*) A $\mathbb{C}$-linear transformation d of a Leibniz algebra L is called a derivation of L if*

$$d([x,y]) = [d(x),y] + [x,d(y)] \quad \forall\, x,y \in L.$$

The set of all derivations of an algebra L is denoted by $Der(L)$. We denote by $Leib_n$ the set of all (n+1)-dimensional filiform Leibniz algebras.

Theorem 2.1. *Any $(n+1)$-dimensional complex Filiform Leibniz algebra L admits a basis $\{e_0, e_1, \ldots, e_n\}$ called adapted, such that the table of multiplication of L has the following form, where non defined products are zero:*

$$FLeib_{n+1} = \begin{cases} [e_0,e_0] = e_2, \\ [e_i,e_0] = e_{i+1}, \quad i \in [\![1;n-1]\!], \\ [e_0,e_1] = \sum\limits_{k=3}^{n} \alpha_k e_k, \\ [e_j,e_1] = \sum\limits_{k=3}^{n-j+1} \alpha_k e_{j+k-1},\ \ j \in [\![1;n-2]\!], \end{cases}$$

for $\alpha_3, \alpha_4, \ldots, \alpha_n \in \mathbb{C}$.

Lemma 2.1. *(See*[11]*) Let $d \in Der(L_n)$. In this case $d = \sum\limits_{i=0}^{n-1} d_i$ where $d_k \in End(L_n)$ and $d_k(L_i) \subseteq L_{i+k}$ for $i \in [\![1;n]\!]$.*

3. SUBCLASS OF $Fleib_{n+1}$

We denote by $L_n(\mathbb{C})$ a subclass of $Fleib_{n+1}$ defined by

$$L_n(\mathbb{C}) = \begin{cases} [e_0,e_0] = e_2, \\ [e_i,e_0] = e_{i+1},\ \ i \in [\![1;n-1]\!], \\ [e_0,e_1] = e_{n-1}, \\ [e_1,e_1] = e_{n-1}, \\ [e_2,e_1] = e_n, \end{cases}$$

where $[\![n;m]\!]$ denotes all integers between n and m. Let us denote by dim(L) the dimension of L and dimDer(L) denote the dimension of Der(L), then we have the following tables.

Dim(L)	Basic derivations	dimDer(L)
6	$d_1(e_0) = e_0 + 2e_1, d_1(e_1) = 3e_1,$ $d_1(e_2) = 4e_2 + 2e_4, d_1(e_3) = 5e_3 + 4e_5,$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 5$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 4$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 3$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 2$ $d_5(e_0) = e_5, d_6(e_1) = e_5.$	6
7	$d_1(e_0) = e_0 + 3e_1, d_1(e_1) = 4e_1,$ $d_1(e_2) = 5e_2 + 3e_5, d_1(e_3) = 6e_3 + 5e_6,$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 6$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 5$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 4$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 3$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 2$ $d_6(e_0) = e_6, d_7(e_1) = e_6.$	7
8	$d_1(e_0) = e_0 + 4e_1, d_1(e_1) = 5e_1,$ $d_1(e_2) = 6e_2 + 4e_6, d_1(e_3) = 7e_3 + 6e_7,$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 7$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 6$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 5$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 4$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 3$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 2$ $d_7(e_0) = e_7, d_8(e_1) = e_7.$	8
9	$d_1(e_0) = e_0 + 5e_1, d_1(e_1) = 6e_1,$ $d_1(e_2) = 7e_2 + 5e_6, d_1(e_3) = 8e_3 + 7e_8,$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 8$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 7$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 6$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 5$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 4$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 3$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 2$ $d_8(e_0) = e_8, d_9(e_1) = e_8.$	9
10	$d_1(e_0) = e_0 + 6e_1, d_1(e_1) = 7e_1,$ $d_1(e_2) = 8e_2 + 6e_6, d_1(e_3) = 9e_3 + 8e_9,$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 9$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 8$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 7$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 6$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 5$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 4$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 3$ $d_8(e_0) = e_8, d_8(e_i) = e_{i+7}, \quad 1 \leqslant i \leqslant 2$ $d_9(e_0) = e_9, d_{10}(e_1) = e_9.$	10

TABLE 1. Derivation of $L \in L_n(\mathbb{C})$ for $n \in [\![6;10]\!]$.

Dim(L)	Basic derivations	dimDer(L)
11	$d_1(e_0) = e_0 + 7e_1, d_1(e_1) = 8e_1,$ $d_1(e_2) = 9e_2 + 7e_6, d_1(e_3) = 10e_3 + 9e_{10},$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 10$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 9$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 8$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 7$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 6$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 5$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 4$ $d_8(e_0) = e_8, d_8(e_i) = e_{i+7}, \quad 1 \leqslant i \leqslant 3$ $d_9(e_0) = e_9, d_9(e_i) = e_{i+8}, \quad 1 \leqslant i \leqslant 2$ $d_{10}(e_0) = e_{10}, d_{11}(e_1) = e_{10}.$	11
12	$d_1(e_0) = e_0 + 8e_1, d_1(e_1) = 9e_1,$ $d_1(e_2) = 10e_2 + 8e_6, d_1(e_3) = 11e_3 + 10e_{11},$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 11$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 10$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 9$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 8$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 7$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 6$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 5$ $d_8(e_0) = e_8, d_8(e_i) = e_{i+7}, \quad 1 \leqslant i \leqslant 4$ $d_9(e_0) = e_9, d_9(e_i) = e_{i+8}, \quad 1 \leqslant i \leqslant 3$ $d_{10}(e_0) = e_{10}, d_{10}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 2$ $d_{11}(e_0) = e_{11}, d_{12}(e_1) = e_{11}.$	12
13	$d_1(e_0) = e_0 + 9e_1, d_1(e_1) = 10e_1,$ $d_1(e_2) = 11e_2 + 8e_6, d_1(e_3) = 12e_3 + 11e_{12},$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 12$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 11$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 10$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 9$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 8$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 7$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 6$ $d_8(e_0) = e_8, d_8(e_i) = e_{i+7}, \quad 1 \leqslant i \leqslant 5$ $d_9(e_0) = e_9, d_9(e_i) = e_{i+8}, \quad 1 \leqslant i \leqslant 4$ $d_{10}(e_0) = e_{10}, d_{10}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 3$ $d_{11}(e_0) = e_{11}, d_{11}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 2$ $d_{12}(e_0) = e_{12}, d_{13}(e_1) = e_{12}.$	13

TABLE 2. Derivations of $L \in L_n(\mathbb{C}), n \in [\![11; 13]\!]$.

Dim(L)	Basic derivations	dimDer(L)
14	$d_1(e_0) = e_0 + 10e_1, d_1(e_1) = 11e_1,$ $d_1(e_2) = 12e_2 + 9e_6, d_1(e_3) = 13e_3 + 12e_{13},$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 13$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 12$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 11$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 10$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 9$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 8$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 7$ $d_8(e_0) = e_8, d_8(e_i) = e_{i+7}, \quad 1 \leqslant i \leqslant 6$ $d_9(e_0) = e_9, d_9(e_i) = e_{i+8}, \quad 1 \leqslant i \leqslant 5$ $d_{10}(e_0) = e_{10}, d_{10}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 4$ $d_{11}(e_0) = e_{11}, d_{11}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 3$ $d_{12}(e_0) = e_{12}, d_{12}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 2$ $d_{13}(e_0) = e_{13}, d_{14}(e_1) = e_{13}.$	14
15	$d_1(e_0) = e_0 + 11e_1, d_1(e_1) = 12e_1,$ $d_1(e_2) = 13e_2 + 10e_6, d_1(e_3) = 14e_3 + 13e_{14},$ $d_1(e_i) = (n + i - 3)e_i, \quad 4 \leqslant i \leqslant 14$ $d_2(e_0) = e_2, d_2(e_i) = e_{i+1}, \quad 1 \leqslant i \leqslant 13$ $d_3(e_0) = e_3, d_3(e_i) = e_{i+2}, \quad 1 \leqslant i \leqslant 12$ $d_4(e_0) = e_4, d_4(e_i) = e_{i+3}, \quad 1 \leqslant i \leqslant 11$ $d_5(e_0) = e_5, d_5(e_i) = e_{i+4}, \quad 1 \leqslant i \leqslant 10$ $d_6(e_0) = e_6, d_6(e_i) = e_{i+5}, \quad 1 \leqslant i \leqslant 9$ $d_7(e_0) = e_7, d_7(e_i) = e_{i+6}, \quad 1 \leqslant i \leqslant 8$ $d_8(e_0) = e_8, d_8(e_i) = e_{i+7}, \quad 1 \leqslant i \leqslant 7$ $d_9(e_0) = e_9, d_9(e_i) = e_{i+8}, \quad 1 \leqslant i \leqslant 6$ $d_{10}(e_0) = e_{10}, d_{10}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 5$ $d_{11}(e_0) = e_{11}, d_{11}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 4$ $d_{12}(e_0) = e_{12}, d_{12}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 3$ $d_{13}(e_0) = e_{13}, d_{13}(e_i) = e_{i+9}, \quad 1 \leqslant i \leqslant 2$ $d_{14}(e_0) = e_{14}, d_{15}(e_1) = e_{14}.$	15

TABLE 3. Derivations of $L \in L_n(\mathbb{C}), n \in [\![14; 15]\!]$.

In above tables, the first column contains algebras from $L_n(\mathbb{C})$ with $n \in [\![6; 15]\!]$. The second column gives the derivations of algebras $L \in L_n(\mathbb{C})$. The third column indicates the dimension of the derivation algebras $Der(L)$ for $L \in L_n(\mathbb{C})$ where $n \in [\![6; 15]\!]$.

Remarks:

In the next lines, we add some points to give more information about our study.

- A basis of $Der(L)$, where $L \in L_n(\mathbb{C})$, can be found and this will be the main purpose of the next pages.
- Since $\dim L_n(\mathbb{C}) = n + 1$, for $n \geqslant 5$, the dimension of $Der(L)$ can be computed.

Next, we will present some statements that are the main task of our paper.

Definition 3.1. *Let* $n \geqslant 4$ *and* $x = \sum_{i=0}^{n} \lambda_i e_i$*. We denote by* $t_1(x) = \sum_{i=0}^{n} \lambda_i t_1(e_i)$*, where*

$$\begin{cases} t_1(e_0) = e_0 + (n-3)e_1, \\ t_1(e_1) = (n-2)e_1, \\ t_1(e_2) = (n-1)e_2 + (n-3)e_{n-1}, \\ t_1(e_3) = (n)e_3 + (n-1)e_n, \\ t_1(e_i) = (n+i-3)e_i \ \ , i \in [\![4; n]\!], \end{cases}$$

and we define d_0 *such that*

$$d_0(e_i) = \begin{cases} \alpha_0\, e_0 + \alpha_0(n-3)e_1, & i = 0, \\ \alpha_1(n-2)\, e_1, & i = 1, \\ \alpha_2(n-1)\, e_2 + \alpha_2(n-3)\, e_{n-1}, & i = 2, \\ \alpha_3 n\, e_3 + (n-1)\alpha_3\, e_n & i = 3, \\ \gamma_i(n-3+i)\, e_i, & i \in [\![4; n]\!]. \end{cases} \quad (3.\ 1)$$

Lemma 3.1. *We have* $d_0 = \alpha_0 t_1$*.*

Proof. Consider $L \in L_n(\mathbb{C})$ and $d_0 \in Der(L)$ given by (3. 1), where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and γ_i $(4 \leqslant i \leqslant n)$ are parameters. Consider the derivation conditions

$$d_0([e_i, e_j]) = [d_0(e_i), e_j] + [e_i, d_0(e_j)], \quad i, j = 0, 1, 2, \ldots, n.$$

Using the definition of $L_n(\mathbb{C})$ and d_0, we have the following cases:

- First Case: Let $(i, j) = (0, 0)$.

Then

$$d_0\big([e_0, e_0]\big) = \big[d_0(e_0), e_0\big] + \big[e_0, d_0(e_0)\big].$$

Using Theorem 2.1

$$d_0(e_2) = \big[\alpha_0\, e_0 + \alpha_0(n-3)e_1, e_0\big] + \big[e_0, \alpha_0\, e_0 + \alpha_0(n-3)e_1\big].$$

Therefore

$$\alpha_2(n-1)\, e_2 + \alpha_2(n-3)\, e_{n-1} = \alpha_0(n-1)\, e_2 + \alpha_0(n-3)\, e_{n-1}$$

and then

$$\alpha_2(n-3) = \alpha_0(n-3).$$

Therefore

$$\alpha_0 = \alpha_2. \quad (3.\ 2)$$

- Second Case: Let $(i, j) = (1, 0)$.

Then

$$d_0([e_1, e_0]) = [d_0(e_1), e_0] + [e_1, d_0(e_0)],$$

and so

$$d_0(e_2) = [\alpha_1(n-2)\, e_1, e_0] + [e_1, \alpha_0\, e_0 + \alpha_0(n-3)e_1]$$

and

$$\alpha_2(n-1)\, e_2 + \alpha_2(n-3)\, e_{n-1} = \alpha_1(n-2)\, e_2 + \alpha_0\, e_2 + \alpha_0(n-3)e_{n-1}$$

$$\alpha_2(n-1) = \alpha_1(n-2) + \alpha_0$$

we get

$$\alpha_2 = \alpha_1.$$

- Third Case: Let $(i, j) = (2, 0)$.

Then

$$d_0([e_2, e_0]) = [d_0(e_2), e_0] + [e_2, d_0(e_0)].$$

Using Theorem 2.1, we get

$$d_0(e_3) = [\alpha_2(n-1)\, e_2 + \alpha_2(n-3)\, e_{n-1}, e_0] + [e_2, \alpha_0\, e_0 + \alpha_0(n-3)e_1]$$

this leads to

$$\alpha_3 n\, e_3 + (n-1)\alpha_3\, e_n = \alpha_2(n-1)\, e_3 + \alpha_2(n-3)\, e_n + \alpha_0\, e_3 + \alpha_0(n-3)e_n.$$

And that

$$\alpha_3 n\, e_3 + (n-1)\alpha_3\, e_n = (n-1+1)\alpha_1\, e_3 + (n-1)\alpha_2\, e_n$$

$$\alpha_3 = \alpha_1 \;\; and \;\; \alpha_3 = \alpha_2.$$

Thus

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3. \tag{3. 3}$$

- Fourth Case: Let $(i, j) = (3, 0)$.

Then

$$d_0([e_3, e_0]) = [d_0(e_3), e_0] + [e_3, d_0(e_0)]$$

and so

$$\begin{aligned} d_0(e_4) &= [\alpha_3 n\, e_3 + (n-1)\alpha_3\, e_n, e_0] + [e_3, \alpha_0\, e_0 + \alpha_0(n-3)e_1] \\ (n+1)\gamma_4\, e_4 &= \alpha_3 n\, e_4 + \alpha_0\, e_4 \\ &= (n+1)\alpha_0 e_4. \end{aligned}$$

Thus $\gamma_4 = \alpha_0.$

- Fifth Case: Let $(i, j) \in ([\![4; n]\!], 0)$,

then

$$d_0\Big(\Big[\sum_{i=4}^{n} e_i, e_0\Big]\Big) = \Big[d_0(\sum_{i=4}^{n} e_i), e_0\Big] + \Big[\sum_{i=4}^{n} e_i, d_0(e_0)\Big].$$

Using Theorem 2.1, we have

$$\begin{aligned} d_0\Big(\sum_{i=4}^{n} e_{i+1}\Big) &= \Big[\sum_{i=4}^{n} \gamma_i(n-3+i)\, e_i, e_0\Big] \\ &+ \Big[\sum_{i=4}^{n} e_i, \alpha_0\, e_0 + \alpha_0(n-3)e_1\Big] \\ d_0\Big(\sum_{i=4}^{n} e_{i+1}\Big) &= \sum_{i=4}^{n} \gamma_i(n-3+i)\, e_{i+1} + \sum_{i=4}^{n} \alpha_0\, e_{i+1} \\ \sum_{i=4}^{n}(n-3+i+1)\gamma_{i+1}\, e_{i+1} &= \sum_{i=4}^{n} \gamma_i(n-3+i)\, e_{i+1} + \sum_{i=4}^{n} \alpha_0\, e_{i+1}. \end{aligned}$$

If $i = 4$, we obtain

$$(n+2)\gamma_5\, e_5 = (n+1)\gamma_4\, e_5 + \alpha_0\, e_5 \Rightarrow \gamma_5 = \gamma_4.$$

$$\gamma_5 \quad = \quad \gamma_4. \tag{3. 4}$$

The same procedure gives

$$\gamma_{i+1} \quad = \quad \gamma_i, \;\; i \in [\![4; n]\!].$$

Finally, we have

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \gamma_i, \;\; i \in [\![4; n]\!]. \tag{3. 5}$$

Now let us compute $d_0(x)$

$$\begin{aligned}
d_0\Big(\sum_{i=0}^{n} \lambda_i\, e_i\Big) &= \lambda_0\, d_0(e_0) + \lambda_1\, d_0(e_1) + \lambda_2\, d_0(e_2) + \lambda_3\, d_0(e_3) + \sum_{i=4}^{n} \lambda_i\, d_0(e_i) \\
&= \lambda_0(\alpha_0\, e_0 + \alpha_0(n-3)e_1) + \lambda_1(\alpha_1(n-2)\, e_1) \\
&+ \lambda_2(\alpha_2(n-1)\, e_2 + \alpha_2(n-3)\, e_{n-1}) \\
&+ \lambda_3(\alpha_3 n\, e_3 + (n-1)\alpha_3\, e_n) \\
&+ \sum_{i=4}^{n} \lambda_i(n-3+i)\gamma_i\, e_i \\
&= \alpha_0[\lambda_0(e_0 + (n-3)e_1) + \lambda_1(n-2)e_1 \\
&+ \lambda_2((n-1)e_2 + (n-3)e_{n-1}) \\
&+ \lambda_3(ne_3 + (n-1)e_n) + \sum_{i+4}^{n} \lambda_i(n-3+i)e_i] \\
&= \alpha_0\, t_1(x). \qquad (3.\ 6)
\end{aligned}$$

Therefore

$$d_0 = \alpha_0 t_1.$$

□

Lemma 3.2. *Consider* $d_k \in Der(L), \;\; L \in L_n(\mathbb{C})$, *for* $k \in [\![1; n-2]\!]$, *and* $n \geqslant 3$, *such that* d_k *is defined by*

$$d_k(e_i) = \begin{cases} \tau_0\, e_{k+1}, \;\; i = 0, \\ \tau_i\, e_{k+i}, i \in [\![1; n-k]\!]. \end{cases} \tag{3. 7}$$

where $\tau_0, \tau_i \in \mathbb{R}$ *and* $i \in [\![1; n-k]\!]$. *Then*

$$d_k(e_0) = e_{k+1}$$

and

$$d_k(e_i) = e_{k+i}, \;\; i \in [\![1; n-k]\!].$$

Proof. Consider the family of derivations

$$d_k([e_i, e_j]) = [d_k(e_i), e_j] + [e_i, d_k(e_j)], \ \ i, j = 0, 1, 2, \ldots, n.$$

Consider the following case:

- First Case: Let $(i, j) = (0, 0)$.

Then

$$d_k([e_0, e_0]) = [d_k(e_0), e_0] + [e_0, d_k(e_0)].$$

Using Theorem 2.1, we get

$$d_k(e_2) = [\tau_0\, e_{k+1}, e_0] + [e_0, \tau_0\, e_{k+1}].$$

This implies

$$\tau_2\, e_{k+2} = \tau_0\, e_{k+2} \Rightarrow \tau_2 = \tau_0. \tag{3. 8}$$

- Second Case: Let $(i, j) = (0, 1)$.

By the definition:

$$d_k([e_0, e_1]) \ \ = \ \ [d_k(e_0), e_1] + [e_0, d_k(e_1)].$$

Then

$$d_k(e_n) \ \ = \ \ [\tau_0\, e_{k+1}, e_1] + [e_0, \tau_1\, e_{k+1}].$$

This gives

$$\tau_0 \neq 0 \ \ and \ \ \tau_1 \neq 0.$$

- Third Case: Let $(i, j) \in ([\![1; n-k]\!], 0)$.

Using Definition 2.4, we have

$$d_k\Big(\Big[\sum_{i=1}^{n-k} e_i, e_0\Big]\Big) \ \ = \ \ \Big[d_k\Big(\sum_{i=1}^{n-k} e_i\Big), e_0\Big] + \Big[\sum_{i=1}^{n-k} e_i, d_k(e_0)\Big].$$

Therefore

$$d_k\Big(\sum_{i=1}^{n-k} e_{i+1}\Big) \ \ = \ \ \Big[\sum_{i=1}^{n-k} \tau_i\, e_{k+i}, e_0\Big] + \Big[\sum_{i=1}^{n-k} e_i, \tau_0\, e_{k+i}\Big].$$

Then

$$\sum_{i=1}^{n-k} \tau_{i+1}\, e_{k+i+1} \ \ = \ \ \sum_{i=1}^{n-k} \tau_i\, e_{k+i+1}.$$

We obtain

$$\tau_{i+1} \ \ = \ \ \tau_i, \ \ i \in [\![1; n-k]\!]. \tag{3. 9}$$

From equations (8) and (9), we get

$$\tau_0 \ \ = \ \ \tau_i, \ \ i = 1, 2, 3, \ldots, n-k. \tag{3. 10}$$

Thus

$$\begin{aligned} d_k\Big(\sum_{i=0}^{n}\lambda_i\, e_i\Big) &= d_0(\lambda_0\, e_0) + \sum_{i=1}^{n} d_0(\lambda_i\, e_i) \\ &= \lambda_0(\tau_0\, e_{k+1}) + \sum_{i=1}^{n}\lambda_i(\tau_i\, e_{k+i}). \end{aligned}$$

Using equation (10), we get

$$d_k\Big(\sum_{i=0}^{n}\lambda_i\, e_i\Big) = \tau_0\Big(\lambda_0\, e_{k+1} + \sum_{i=1}^{n-k}\lambda_i\, e_{k+i}\Big).$$

Thus

$$d_k(e_0) = e_{k+1} \text{ and } d_k(e_i) = e_{k+i}, \ \ i \in [\![1, n-k]\!].$$

Hence, we complete the proof. □

Lemma 3.3. *Let x given by $x = \sum\limits_{i=0}^{n}\lambda_i e_i$. Then*

$$t_2(x) = \sum_{i=0}^{n}\lambda_i t_2(e_i).$$

Proof. Consider $d_{n-1} \in Der(L)$, where d_{n-1} is defined by

$$d_{n-1}(e_i) = \begin{cases} \pi_0\, e_n, & i = 0, \\ 0, & i \neq 0, \end{cases} \tag{3. 11}$$

for scalar π_0. Consider the family of derivations

$$d_{n-1}([e_i, e_j]) = [d_{n-1}(e_i), e_j] + [e_i, d_{n-1}(e_j)].$$

- First Case: Let $(i, j) = (0, 1)$.

Then

$$d_{n-1}([e_0, e_0]) = [d_{n-1}(e_0), e_0] + [e_0, d_{n-1}(e_0)].$$

Using the equation (11) and the definition of $L_n(\mathbb{C})$, we get

$$d_{n-1}(e_2) = [\pi_0\, e_n, e_0] + [e_0, \pi_0\, e_n].$$

This implies $\pi_0 \neq 0$.

- Second Case: Let $(i, j) = (1, 0)$.

Then

$$d_{n-1}([e_1, e_0]) = [d_{n-1}(e_1), e_0] + [e_1, d_{n-1}(e_0)].$$

Using the equation (11) and the definition of $L_n(\mathbb{C})$, we obtain

$$d_{n-1}(e_2) = [0, e_0] + [e_0, \pi_0\, e_n] = 0.$$

Thus

$$\begin{aligned} d_{n-1}\Big(\sum_{i=0}^{n}\lambda_i\, e_i\Big) &= d_{n-1}(\lambda_0\, e_0) + d_{n-1}(\lambda_1\, e_1) + \sum_{i=2}^{n} d_{n-1}(\lambda_i\, e_i) \\ &= \lambda_0(\pi_0\, e_n) \\ &= \pi_0(\lambda_0\, e_n) \\ &= \pi_0\, t_2. \end{aligned}$$

Then, we obtain

$$t_2(e_0) = e_n. \quad (3.\ 12)$$

This completes the proof. □

Lemma 3.4. *The mappings t_1, t_2 and d_k for $k \in [\![1, n-2]\!]$ are Linearly independent.*

Proof. Remember that, α_1, α_2 and β_k where $k \in [\![1, n-2]\!]$ are real numbers such that

$$\alpha_1 t_1 + \alpha_2 t_2 + \sum_{k=1}^{n-2} \beta_k d_k = 0.$$

Let $x \in Span\{e_0, e_1, e_2, e_3, \ldots, e_{n-1}\}$. Then

$$\alpha_1 t_1(x) + \alpha_2 t_2(x) + \sum_{k=1}^{n-2} \beta_k d_k(x) = 0.$$

Using the linearly independence of functions, it is sufficient to work with

$$x = e_i, \;\; i = 0, 1, \ldots, n-1.$$

So, we have the following linear system for $i = 0, 1, 2, 3, \ldots, n-1$

$$\alpha_1 t_1(e_i) + \alpha_2 t_2(e_i) + \sum_{k=1}^{n-2} \beta_k d_k(e_i) = 0.$$

We have a system of n equations, which we solve it by two cases:

- First Case: let $i = 0, 1, 2, 3, \ldots, n-1$, then

$$\sum_{i=0}^{n-1}\Big[\alpha_1 t_1(e_i)+\alpha_2 t_2(e_i)+\sum_{k=1}^{n-2}\beta_k d_k(e_i)\Big]$$
$$=\sum_{i=0}^{n-1}\Big[\alpha_1 t_1(e_i)+\alpha_2 t_2(e_i)+\beta_1 d_1(e_i)+\beta_2 d_2(e_i)+\beta_{n-2}d_{n-2}(e_i)\Big]$$
$$=\alpha_1 t_1(e_0)+\alpha_2 t_2(e_0)+\beta_1 d_1(e_0)+\beta_2 d_2(e_0)+\ldots+\beta_{n-2}d_{n-2}(e_0)$$
$$+[\alpha_1 t_1(e_1)+\alpha_2 t_2(e_1)+\alpha_3 t_3(e_1)+\beta_1 d_1(e_1)+\beta_2 d_2(e_1)+\ldots+\beta_{n-2}d_{n-2}(e_1)]$$
$$+[\alpha_1 t_1(e_2)+\alpha_2 t_2(e_2)+\alpha_3 t_3(e_2)+\beta_1 d_1(e_2)+\beta_2 d_2(e_2)+\ldots+\beta_{n-2}d_{n-2}(e_2)]$$
$$+\ldots$$
$$+\ldots$$
$$+[\alpha_1 t_1(e_{n-1})+\alpha_2 t_2(e_{n-1})+\alpha_3 t_3(e_{n-1})+\beta_1 d_1(e_{n-1})$$
$$+\beta_2 d_2(e_{n-1})+\ldots+\beta_{n-2}d_{n-2}(e_{n-1})]=0.$$

This implies

$$\begin{aligned}
&\alpha_1(e_0+(n-3)e_1)+\beta_1 e_2+\beta_2 e_3+\beta_3 e_4+\beta_4 e_5\\
&+\ldots+\beta_{n-3}e_{n-2}+\beta_{n-2}e_{n-1}\\
&+\alpha_2 e_n)+(\alpha_1(n-2)e_2+\beta_1 e_2+\beta_2 e_3+\beta_3 e_4+\beta_4 e_5+\ldots\\
&+\beta_{n-3}e_{n-2}+\beta_{n-2}e_{n-1})+(\alpha_1(n-1)e_2+\beta_1 e_3+\beta_2 e_4+\beta_3 e_5\\
&+\beta_4 e_6+\ldots+(\alpha_1(n-3)+\beta_{n-3})e_{n-1}+\beta_{n-2}e_n)\\
&+(\alpha_1(n)e_3+\beta_1 e_4+\beta_2 e_5+\beta_3 e_6+\beta_4 e_7+\ldots\\
&+(\alpha_1(n-3)+\beta_{n-4})e_{n-1}+(\beta_{n-3}+2\alpha_1(n-3))e_n)\\
&+\ldots\\
&+\ldots\\
&+(\alpha_1(2n-5)e_{n-2}+\beta_1 e_{n-1}+\beta_2 e_n)\\
&+(\alpha_1(2n-4)e_{n-1}+\beta_1 e_n)\\
&+(\alpha_1(2n-3)e_n)=0.
\end{aligned}$$

We obtain

$$
\begin{aligned}
&\alpha_1 e_0 \\
&+\alpha_1(n-3)e_1 \\
&+(\alpha_1(n-2)+2\beta_1+\alpha_1(n-1))e_2 \\
&+(\alpha_1(n)+\beta_1+2\beta_2)e_3 \\
&+(\alpha_1(n+1)+\beta_1+\beta_2+2\beta_3)e_4 \\
&+(\alpha_1(n+2)+\beta_1+\beta_2+\beta_3+2\beta_4)e_5 \\
&+(\alpha_1(n+3)+\beta_1+\beta_2+\beta_3+\beta_4+2\beta_5)e_6 \\
&+\dots \\
&+\dots \\
&+(\alpha_1(2n-4)+\beta_1+\beta_2+\beta_3+\ldots+\beta_{n-4}+\beta_{n-3}+2\alpha_1(n-3))e_{n-1} \\
&+(\alpha_1(2n-3)+\beta_1+\beta_2+\ldots+\beta_{n-4}+\beta_{n-3}+2\beta_{n-2}+\alpha_2+2\alpha_1(n-3))e_n \\
&=0.
\end{aligned}
$$

Here we have the following results:

1) $2\alpha_1 e_0 = 0$ which implies $\alpha_1 = 0$.

2) $(\alpha_1(n-2)+2\beta_1+\alpha_1(n-1))e_2 = 0,$ which implies $2\beta_1+\alpha_1 = 0.$

   Since $\alpha_1 = 0,$ then $\beta_1 = 0.$

3) $(2\beta_2+\beta_1+\alpha_1(n))e_3 = 0,$ which implies $2\beta_2+\beta_1+\alpha_1(n) = 0.$

   But $\alpha_1 = \beta_1 = 0,$ then $\beta_2 = 0.$

4) $(2\beta_3+\beta_2+\beta_1+\alpha_1(n+1))e_4 = 0,$ which implies

   $2\beta_3+\beta_2+\beta_1+\alpha_1(n+1) = 0.$ But $\alpha_1 = \beta_1 = \beta_2 = 0,$ then $\beta_3 = 0.$

5) $(2\beta_4 + \beta_3 + \beta_2 + \beta_1 + \alpha_1(n_2))e_5 = 0,$ which implies

$$2\beta_4 + \beta_3 + \beta_2 + \beta_1 + \alpha_1(n-2) = 0.$$

But $\alpha_1 = \beta_1 = \beta_2 = \beta_3 = 0,$ then $\beta_4 = 0.$

6) Similarly, $(\beta_{n-2} + \beta_{n-3} + \beta_{n-4} + \beta_{n-5} + \ldots + \beta_2 + \beta_1 + \alpha_1 + \alpha_2)e_n = 0,$

which implies $\beta_{n-2} + \beta_{n-3} + \ldots + \beta_2 + \beta_1 + \alpha_1 + \alpha_2 = 0.$

But $\alpha_1 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \ldots = \beta_{n-2} = 0$. Then $\alpha_2 = 0$.

- Second Case: let $i = n$, then

$$\alpha_1 t_1(e_n) + \alpha_2 t_2(e_n) + \sum_{k=1}^{n-2} \beta_k d_k(e_n) = 0$$

by using Lemmas 3.1-3.3. Here, from Case 1 and Case 2 we obtain

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \ldots = \beta_{n-2} = 0. \tag{3. 13}$$

These prove the mappings are linearly independent. □

Lemma 3.5. *The Linear mappings* $d_0, d_k, d_{n-1} \in Der(L_n(\mathbb{C})),\ 1 \leqslant k \leqslant n-2,$ *span* $Der(L_n(\mathbb{C}))$.

Proof. First we observe that by (1), (7) and (11),

$$d_k(x) = \begin{cases} \lambda_0(\alpha_0)\, e_0 + \lambda_1(\alpha_0(n-3) + \beta_1(n-2))\, e_1, \\ +\lambda_2(\alpha_1(n-1)\, e_2 + \alpha_2(n-3)\, e_{n-1}) \\ +\lambda_3(\eta_0 n\, e_3 + 2(n-3)\eta_1)\, e_n \\ + \sum\limits_{i=4}^{n} \lambda_i(\gamma_i(n-3+i))\, e_i, \text{ for } k = 0, \\ +\lambda_0\tau_0\, e_{k+1} + \sum\limits_{i=1}^{n-k} \lambda_k\tau_i\, e_{k+i}, \text{ for } 1 \leqslant k \leqslant n-2, \\ +\lambda_0\pi_0\, e_n, \text{ for } k = n-1. \end{cases} \tag{3. 14}$$

Hence, by using equations (10) and (14)

$$d_k(x) = \alpha_0[\lambda_0(e_0 + (n-3)e_1) + \lambda_1(n-2)e_1 + \lambda_2((n-2)e_2 + (n-3)e_{n-1})$$

$$+ \lambda_3(ne_3 + 2(n-3)e_n) + \sum_{i+4}^{n} \lambda_i(n-3+i)e_i] + \tau_0(\lambda_0\, e_{k+1} + \sum_{i=1}^{n-k} \lambda_i\, e_{k+i})$$

$$+ \pi_0(\lambda_0\, e_n).$$

Therefore

$$d = \alpha_0 t_1 + \pi_0 t_2 + \tau_0 d_k \text{ for } 1 \leqslant k \leqslant n-2.$$

Hence, we conclude that t_1, t_2 and d_k for $1 \leqslant k \leqslant n-2$, span $Der(L_n(\mathbb{C}))$. □

We note that $t_1,\ t_2$ and d_k for $1 \leqslant k \leqslant n-2$ are defined in Lemma 3.1 to Lemma 3.3.

Proposition 3.1. *Let* $Ln(\mathbb{C})$ *be* $[e_0, e_0] = e_2, [e_i, e_0] = e_{i+1},\ \ 1 \leqslant i \leqslant n-1$ *and* $[e_0, e_1] = e_{n-1}, [e_1, e_1] = e_{n-1}, [e_2, e_1] = e_n$. *Then* t_1, t_2 *and* d_k *for* $1 \leqslant k \leqslant n-2$ *are a basis of the space Der(*$L_n(\mathbb{C})$*) .*

Proof. : The proof follows from Lemma 3.4, and Lemma 3.5. □

Proposition 3.2. *The mapping* t_1, t_2 *and* d_k *for* $1 \leqslant k \leqslant n-2$ *are derivations .*

Proof. : Consider

$$x = \sum_{k=0}^{n} \alpha_k e_k, \quad y = \sum_{k=0}^{n} \beta_k e_k.$$

Then, by using the definition of the algebra $L_n(\mathbb{C})$, we get

$$\begin{aligned} x.y &= \beta_0(\alpha_0 + \alpha_1)e_2 + \alpha_2\beta_0 e_3 + \sum_{i=3}^{n-3} \beta_0 \alpha_i e_{i+1} \\ &+ (\beta_1(\alpha_0 + \alpha_1) + \beta_0 \alpha_{n-2})e_{n-1} + (\alpha_2 \beta_1 + \beta_0 \alpha_{n-1})e_n. \end{aligned} \tag{3. 15}$$

Then, using (3. 15) and Definition 3.1, we get

$$\begin{aligned} t_1(x.y) = (n-1)e_2 + ne_3 + \sum_{i=4}^{n-3}(n+i-3)e_i + (2n-5)e_{n-2} \\ +(3n-7)e_{n-1} + (4n-9)e_n. \end{aligned} \tag{3. 16}$$

Thus

$$\begin{aligned} t_1(x) &= \alpha_0 e_0 + (\alpha_0(n-3) + \alpha_1(n-2))e_1 + \alpha_2(n-1)e_2 + \alpha_3 n e_3 \\ &+ \sum_{i=4}^{n-2} \alpha_i(n+i-3)e_i + (\alpha_2 + \alpha_{n-1})(3n-7)e_{n-1} \\ &+ (\alpha_3 + \alpha_n)(4n+9)e_n. \end{aligned}$$

Hence

$$\begin{aligned} t_1(x).y &= [\beta_0\, \alpha_0 + \beta_0(\alpha_0(n-3) + \alpha_1(n-2))]e_2 + \beta_0\, \alpha_2(n-1)e_3 \\ &+ \sum_{i=2}^{n-2} \beta_0 \alpha_i (n+i-3)e_{i+1} + \beta_1(\alpha_0 + \alpha_0(n-3) + \alpha_1(n-2))e_{n-1} \\ &+ [\beta\alpha_2(n-1) + \beta_0(\alpha_2 + \alpha_{n-1})(3n-7)]e_n, \end{aligned} \tag{3. 17}$$

and thus

$$\begin{aligned} t_1(y) &= \beta_0\, e_0 + [\beta_0(n-3) + \beta_1(n-2)]\, e_1 + \beta_2(n-1)\, e_2 + \beta_3 n\, e_3 \\ &+ \sum_{i=4}^{n-2} \beta_i(n+i-3)e_i + [\beta_2(n-3) + \beta_{n-1}(2n-4)]e_{n-1} \\ &+ [2\beta_3(n-3) + \beta_n(2n-3)]e_n. \end{aligned}$$

Therefore

$$x.t_1(y) = \beta_0\left(\alpha_0+\alpha_1\right)e_2 + \sum_{i=2}^{n-3}\beta_0\alpha_i e_{i+1} + (\beta_0\alpha_{n-1}$$

$$+ \quad (\alpha_0+\alpha_1)(\beta_0(n-3)+\beta_1(n-2))e_{n-1} + \alpha_0\alpha_{n-1}$$

$$+ \quad \alpha_2(\beta_0(n-3)+\beta_1(n-2))\,e_n. \tag{3. 18}$$

By adding (17) to (18), we obtain (16). This implies that t_1 is a derivation. Now, we should show that t_2 is also a derivation. $t_2(x)=\alpha_0 e_n$ and $t_2(y)=\beta_0 e_n$. Similarly, we have $t_2(x).y=0$, $x.t_2(y)=0$ and $t_2(x.y)=0$ and thus t_2 is a derivation. Now, since

$$\sum_{k=1}^{n-1} d_k(x) \quad = \quad \sum_{k=1}^{n-1}\sum_{i=1}^{n}\alpha_i d_k(e_i),$$

thus

$$\sum_{k=1}^{n-1} d_k(x) \quad = \quad (\alpha_0+\alpha_1)\sum_{k=1}^{n-1}e_{k+1} + \alpha_2\sum_{k=1}^{n-2}e_{k+2} + \alpha_3\sum_{k=1}^{n-3}e_{k+3}$$

$$+\ldots+\alpha_{n-2}\sum_{k=1}^{2}e_{k+n-2}+\alpha_{n-1}e_n.$$

Then

$$\Big[\sum_{k=1}^{n-1} d_k(x), y\Big] \quad = \quad \beta_0(\alpha_0+\alpha_1)\sum_{k=1}^{n-2}e_{k+2}$$

$$+ \quad \beta_0\sum_{i=2}^{n-3}\alpha_i\sum_{k=1}^{n-i-1}e_{k+n-2} + \big(\beta_0\alpha_{n-2}+\beta_1(\alpha_0+\alpha_1)\big)e_n. \tag{3. 19}$$

In addition

$$\sum_{k=1}^{n-1} d_k(y) \quad = \quad (\beta_0+\beta_1)\sum_{k=1}^{n-1}e_{k+1}$$

$$+ \quad \beta_2\sum_{k=1}^{n-2}e_{k+2}+\beta_3\sum_{k=1}^{n-3}e_{k+3}+\ldots+\beta_{n-2}\sum_{k=1}^{2}e_{k+n-2}+\beta_{n-1}e_n.$$

And we have

$$\Big[x, \sum_{k=1}^{n-1} d_k(y)\Big] = 0. \tag{3. 20}$$

Also

$$\begin{aligned}\sum_{k=1}^{n-1} d_k(x.y) &= \beta_0\,(\alpha_0+\alpha_1)\sum_{k=1}^{n-2} e_{k+2}\\ &+ \beta_0\sum_{i=2}^{n-3}\alpha_i\sum_{k=1}^{n-i-1} e_{k+i+1}\\ &+ \big(\beta_0\alpha_{n-2}+\beta_1(\alpha_0+\alpha_1)\big)e_n. \qquad (3.\ 21)\end{aligned}$$

By adding (19) to (20), we get (21). Thus d_k is a derivation. This completes the proof of the proposition. □

The following Maple code can be used to get the most formula proved in our paper.

```
restart:
n:=8:
eiej:=proc(ii::nonnegint, jj::nonnegint)
global n;
local i:=ii, j:=jj,
v:=Vector[row](n+1);
#`v[n+1] for e0 `
if i=n+1 then i:=0 fi;
if j=n+1 then j:=0 fi;
if  i=0 and j=1 then v[n-1]:=1
elif i=0 and j=0 then v[2]:=1
elif i=1 and j=1 then v[n-1]:=1
elif i=2 and j=1 then v[n]:=1
elif i<n and j=0 then v[i+1]:=1  fi;
eval(v) end:
eiej(2,0)=eiej(2,n+1), eiej(1,1), eiej(2,1);
`&x` := (v,w) -> add(add( v[i]*w[j]*eiej(i,j),
i=1..n+1),j=1..n+1);e[0]:=Vector[row](n+1,{n+1=1}):
 #I define the basis to make computation `
for i to n do   e[i]:=Vector[row](n+1,{i=1})
od:  v2e:=v -> v[n+1]*`%e`[0]+ add(v[i]*`%e`[i],i=1..n);
 #display using inert %e[i], optional `
t1:=v -> v[n+1]*(e[0]+(n-3)*e[1]) + v[1]*(n-2)*e[1]+
add(v[i]*((n+i-3)*e[i]+(i-1)*(n-3)*e[n-3+i]),i=2..3)+
add(v[i]*(n+i-3)*e[i],i=4..n);
x:=add(alpha[k]*e[k],k=0..n):
y:=add(beta[k]*e[k],k=0..n):
v2e(x);  value(`%`);
t1(x);#` Compute the value of t1 on x`
v2e(`%`);
t1(x) &x y;
```

```
v2e(‘%‘);
t1(y);
v2e(‘%‘);
x &x y;
v2e(‘%‘);
x &x t1(y);
v2e(‘%‘);
t1(x &x y)– t1(x)&x y–x &x t1(y);
simplify(v2e(‘%‘));
t1(x &x y);   v2e(‘%‘);
t1(x)&x y+x &x t1(y);
v2e(‘%‘);
```

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