Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 49(1)(2017) pp. 125–138

Multi-step frozen Jacobian iterative scheme for solving IVPs and BVPs based on higher order Fréchet derivatives

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Received: 16 July, 2016 / Accepted: 21 October, 2016 / Published online: 23 January, 2017

Abstract. A multi-step frozen Jacobian iterative scheme for solving system of nonlinear equations associated with IVPs (initial value problems) and BVPs (boundary value problems) is constructed. The multi-step iterative schemes consist of two parts, namely base method and a multi-step part. The proposed iterative scheme uses higher order Fréchet derivatives in the base method part and offers high convergence order (CO) $3s + 1$, here s is the number of steps. The increment in the CO per step is three, and we solve three upper and lower triangles systems per step in the multi-step part. A single inversion of the is not working in latexfrozen Jacobian is required and in fact, we avoid the direct inversion of the frozen Jacobian by computing the LU factors. The LU-factors are utilized in the multi-step part to solve upper and lower triangular systems repeatedly that makes the iterative scheme computationally efficient. We solve a set of IVPs and BVPs to show the validity, accuracy and efficiency of our proposed iterative scheme.

AMS (MOS) Subject Classification Codes: 34L30; 35G30; 35G31; 65H20
Key Words: Multi-step iterative schemes; Systems of nonlinear equations;Frozen Jacobian; Initial and boundary value problems; Higher order Fréchet derivatives.

1. Introduction

The closed form solution of a nonlinear problem is not always possible and then we seek a numerical solution. One of the possible ways to solve IVPs and BVPs is to solve associated discretized nonlinear problems. In most of the cases, we end with a system of nonlinear algebraic equations and it is an active area of research to develop a high CO iterative scheme for solving a system of nonlinear equations. In the majority of iterative schemes researchers employed the first order Fréchet derivative for the solution of the system of nonlinear equations because the computational cost of higher order Fréchet derivatives is high. But it is not the case always; we show that in many IVPs and BVPs, the computational cost of higher order Fréchet derivatives is almost equivalent to the single function evaluation and hence it is not a bad idea to propose higher order iterative schemes that consist of higher order Fréchet derivatives. In the case of scalar nonlinear equations, there is a conjecture (Kung-Traub conjecture) that states the relationship between the optimal CO and the number of function evaluations. If an iterative scheme without memory to find a simple root of a scalar nonlinear equation uses d function evaluations, then it can attain an optimal CO 2^{d-1} [10, 6]. But we do not have a conjecture about the optimal CO of an iterative method for solving a system of nonlinear equations. The direction of research in the development of iterative schemes for solving system of nonlinear equations is to construct iterative schemes with high CO but low computational cost. The frozen Jacobian multi-step iterative schemes offer efficient iterative schemes. The efficiency of frozen Jacobian iterative schemes is hidden in the single LU factorization of the frozen Jacobian and solution of upper and lower triangular systems. Actually, the solutions of upper and lower triangular systems are computationally efficient that makes the entire method computationally attractive. The majority of IVPs or BVPs after discretization can be written as

$$\mathbf{K}(\mathbf{q}) = \mathbf{A}\mathbf{q} + g(\mathbf{q}) + \mathbf{b} = \mathbf{0}\,, \tag{1. 1}$$

$\mathbf{A}$ is the differentiation approximation matrix, $g(\mathbf{q})$ is a nonlinear function, $\mathbf{b}$ is a constant vector and $\mathbf{0}$ is a zero vector. The higher order Fréchet derivatives of (1. 1) are listed as

$$\begin{aligned}\mathbf{K}'(\mathbf{q}) &= \mathbf{A} + \operatorname{diag}\left(g'(\mathbf{q})\right)\\ \mathbf{K}''(\mathbf{q}) &= \operatorname{diag}\left(g''(\mathbf{q})\right)\\ \mathbf{K}'''(\mathbf{q}) &= \operatorname{diag}\left(g'''(\mathbf{q})\right)\,,\end{aligned} \tag{1. 2}$$

where $\operatorname{diag}(\cdot)$ represents a diagonal matrix. The Fréchet derivative is the existence of the following limit

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{K}(\mathbf{q}+\mathbf{h}) - \mathbf{K}(\mathbf{q}) - \mathbf{B}\mathbf{h}||}{||\mathbf{h}||} = 0\,,$$

which ensures the Fréchet differentiability. The linear operator $\mathbf{B}$ is called the first order Fréchet derivative, and we denote it by $\mathbf{K}'(\mathbf{q})$. The higher order Fréchet derivatives can be computed recursively

$$\begin{aligned}&\mathbf{K}'(\mathbf{q}) = \text{Jacobian}\left(\mathbf{K}(\mathbf{q})\right),\\ &\mathbf{K}^{j}(\mathbf{q})\mathbf{u}^{j-1} = \text{Jacobian}\left(\mathbf{K}^{j-1}(\mathbf{q})\mathbf{v}^{j-1}\right), \quad j \geq 2\,,\end{aligned}$$

where $\mathbf{u}$ is a vector independent from $\mathbf{q}$. The classical Newton-Raphson iterative scheme for solving system of nonlinear equations can be expressed as

$$\text{NR} = \begin{cases} \mathbf{q}_0 = \text{initial guess} \\ \mathbf{q}_{n+1} = \mathbf{q}_n - \mathbf{K}'(\mathbf{q}_n)^{-1}\mathbf{K}(\mathbf{q}_n), \end{cases}$$

where $\det(\mathbf{K}'(\mathbf{q}_n)) \neq 0$. The CO of NR method is two. The multi-step frozen Jacobian Newton-Raphson iterative scheme can be described as

$$\text{MNR} = \begin{cases} \text{NOS} & = s \\ \text{CO} & = s+1 \\ \text{FE} & = s \\ \text{JE} & = 1 \\ \text{LUF} & = 1 \\ \text{LUTS} & = s \end{cases} \begin{cases} \text{Base method} \rightarrow & \begin{cases} \mathbf{q}_0 = \text{initial guess} \\ \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_1 = \mathbf{K}(\mathbf{q}_0) \\ \mathbf{q}_1 = \mathbf{q}_0 - \boldsymbol{\phi}_1 \end{cases} \\ \text{Multi-step part} \rightarrow & \begin{cases} \text{for } i = 1, s-1 \\ \quad \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_{i+1} = \mathbf{K}(\mathbf{q}_i) \\ \quad \mathbf{q}_{i+1} = \mathbf{q}_i - \boldsymbol{\phi}_{i+1} \\ \text{end} \\ \mathbf{q}_0 = \mathbf{q}_s\,, \end{cases} \end{cases}$$

where NOS, FE, JE, LUF and LUTS are number of steps, function evaluations, Jacobian evaluations, number of LU-factorizations and number of solutions of upper and lower triangular systems respectively. The per step increment in the CO of MNR is one in the multi-step part. The computational efficiency of MNR method is better than that of NR because the computation of the Jacobian and its LU factorization for solving upper and lower triangular systems are expensive. We are interested in constructing an iterative scheme that solves the system of nonlinear equations associated with IVPs and BVPs and offers a good computational efficiency. The computational efficiency means we achieve high CO with low computational cost. Many researchers [11, 8, 1, 12, 2, 3, 13, 7, 9] have proposed the high CO iterative scheme for solving the system of nonlinear equations.

2. The proposed iterative scheme

We construct an iterative scheme IZFZA with CO $3s+1$. The per step increment in the CO is three in the multi-step part. If we assume that the computational cost of second and third order Fréchet derivatives is equivalent to that of $\mathbf{K}(\cdot)$ then the total number of function evaluations is $s+2$. The Iterative method IZFZA is an efficient iterative scheme because we solve three upper and lower triangular systems per multi-step.

$$
\text{IZFZA} = \begin{cases} \text{NOS} & = s \\ \text{CO} & = 3s+1 \\ \mathbf{K}(\cdot)\ \text{evaluations} & = s \\ \mathbf{K}'(\cdot)\ \text{evaluations} & = 2 \\ \mathbf{K}''(\cdot)\ \text{evaluations} & = 1 \\ \mathbf{K}'''(\cdot)\ \text{evaluations} & = 1 \\ \text{LUF} & = 1 \\ \text{M-V multipliations} & = 2(s-1) \\ \text{V-V multiplications} & = s+7 \\ \text{LUTS} & = 3s+1 \end{cases}
\left\{ \begin{array}{l}
\text{Base method} \longrightarrow \begin{cases} \mathbf{q}_0 = \text{initial guess} \\ \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_1 = \mathbf{K}(\mathbf{q}_0) \\ \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_2 = \mathbf{K}''(\mathbf{q}_0)\,\boldsymbol{\phi}_1^2 \\ \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_3 = \mathbf{K}''(\mathbf{q}_0)\,\boldsymbol{\phi}_1\boldsymbol{\phi}_2 \\ \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_4 = \mathbf{K}'''(\mathbf{q}_0)\,\boldsymbol{\phi}_1^3 \\ \mathbf{q}_1 = \mathbf{q}_0 - \boldsymbol{\phi}_1 - \frac{\boldsymbol{\phi}_2+\boldsymbol{\phi}_3}{2} + \frac{\boldsymbol{\phi}_4}{6} \end{cases} \\
\text{Multi-step part} \rightarrow \begin{cases} \text{for } i = 2, s \\ \quad \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_5 = \mathbf{K}(\mathbf{q}_{i-1}) \\ \quad \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_6 = \mathbf{K}'(\mathbf{q}_1)\,\boldsymbol{\phi}_5 \\ \quad \mathbf{K}'(\mathbf{q}_0)\,\boldsymbol{\phi}_7 = \mathbf{K}'(\mathbf{q}_1)\,\boldsymbol{\phi}_6 \\ \quad \mathbf{q}_i = \mathbf{q}_{i-1} - 3(\boldsymbol{\phi}_5 - \boldsymbol{\phi}_6) - \boldsymbol{\phi}_7 \\ \text{end} \\ \mathbf{q}_0 = \mathbf{q}_s\,, \end{cases}
\end{array}\right.
$$

where M-V and V-V stand from matrix-vector and vector-vector respectively. The iterative scheme IZFZA for solving (1. 1) can be written as

$$
\text{IZFZA} = \begin{cases} \text{NOS} & = s \\ \text{CO} & = 3s+1 \\ \mathbf{K}(\cdot)\ \text{evaluations} & = s+2 \\ \mathbf{K}'(\cdot)\ \text{evaluations} & = 2 \\ \text{LUF} & = 1 \\ \text{M-V multiplications} & = 2(s-1) \\ \text{V-V multiplications} & = s+7 \\ \text{LUTS} & = 3s+1 \end{cases}
\left\{ \begin{array}{l}
\text{Base method} \longrightarrow \begin{cases} \mathbf{q}_0 = \text{initial guess} \\ \mathbf{B} = \mathbf{A} + \text{diag}\left(g'(\mathbf{q}_0)\right) \\ \mathbf{B}\,\boldsymbol{\phi}_1 = \mathbf{A}\,\mathbf{q}_0 + g(\mathbf{q}_0) + \mathbf{b} \\ \mathbf{B}\,\boldsymbol{\phi}_2 = g''(\mathbf{q}_0) \odot \boldsymbol{\phi}_1 \odot \boldsymbol{\phi}_1 \\ \mathbf{B}\,\boldsymbol{\phi}_3 = g''(\mathbf{q}_0) \odot \boldsymbol{\phi}_1 \odot \boldsymbol{\phi}_2 \\ \mathbf{B}\,\boldsymbol{\phi}_4 = g'''(\mathbf{q}_0) \odot \boldsymbol{\phi}_1 \odot \boldsymbol{\phi}_1 \odot \boldsymbol{\phi}_1 \\ \mathbf{q}_1 = \mathbf{q}_0 - \boldsymbol{\phi}_1 - \frac{\boldsymbol{\phi}_2+\boldsymbol{\phi}_3}{2} + \frac{\boldsymbol{\phi}_4}{6} \end{cases} \\
\text{Multi-step part} \rightarrow \begin{cases} \text{for } i = 2, s \\ \quad \mathbf{B}\,\boldsymbol{\phi}_5 = \mathbf{A}\,\mathbf{q}_{i-1} + g(\mathbf{q}_{i-1}) + \mathbf{b} \\ \quad \mathbf{B}\,\boldsymbol{\phi}_6 = \mathbf{A}\,\boldsymbol{\phi}_5 + g'(\mathbf{q}_1) \odot \boldsymbol{\phi}_5 \\ \quad \mathbf{B}\,\boldsymbol{\phi}_7 = \mathbf{A}\,\boldsymbol{\phi}_6 + g'(\mathbf{q}_1) \odot \boldsymbol{\phi}_6 \\ \quad \mathbf{q}_i = \mathbf{q}_{i-1} - 3(\boldsymbol{\phi}_5 - \boldsymbol{\phi}_6) - \boldsymbol{\phi}_7 \\ \text{end} \\ \mathbf{q}_0 = \mathbf{q}_s\,, \end{cases}
\end{array}\right.
$$

where $\odot$ is the component-wise multiplication of two vectors of same length.

3. Convergence analysis

Theorem 3.1. *Let* $\mathbf{K} : \Gamma \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ *be a Fréchet differentiable function (with required order of differentiability) on an open convex neighborhood* Γ *of* $\mathbf{q}^* \in \mathbb{R}^n$ *with* $\mathbf{K}(\mathbf{q}^*) = \mathbf{0}$ *and* $\det(\mathbf{K}'(\mathbf{q}^*)) \neq 0$, *where* $\mathbf{K}'(\mathbf{q})$ *denotes the Fréchet derivative of* $\mathbf{K}(\mathbf{q})$. *Let* $\mathbf{A}_1 = \mathbf{K}'(\mathbf{q}^*)$ *and* $\mathbf{A}_j = \frac{1}{j!}\,\mathbf{K}'(\mathbf{q}^*)^{-1}\,\mathbf{K}^{(j)}(\mathbf{q}^*)$, *for* $j \geq 2$, *where* $\mathbf{K}^{(j)}(\mathbf{q})$ *denotes the* j*-th order Fréchet derivative of* $\mathbf{K}(\mathbf{q})$. *Then, with an initial* *guess in the neighborhood of* $\mathbf{q}^*$*, the sequence* $\{\mathbf{q}_m\}$ *generated by IZFZA converges to* $\mathbf{q}^*$ *with local order of convergence at least eight and error*

$$\mathbf{e}_4 = \mathbf{L}{\mathbf{e}_0}^7 + O\left({\mathbf{e}_0}^8\right) ,$$

where $\mathbf{e}_0 = \mathbf{q}_0 - \mathbf{q}^*$*,* ${\mathbf{e}_0}^p = \overbrace{(\mathbf{e}_0, \mathbf{e}_0, \ldots, \mathbf{e}_0)}^{p\ times}$*, and* $\mathbf{L} = -24\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 - 16\mathbf{A}_2^4\mathbf{A}_3 + 8\mathbf{A}_2^3\mathbf{A}_4 + 40\mathbf{A}_2^6$ *is a 7-linear function, i.e.* $\mathbf{L} \in \mathbb{L}\,\overbrace{(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^n, \cdots, \mathbb{R}^n)}^{7\ times}$ *with* $\mathbf{L}{\mathbf{e}_0}^7 \in \mathbb{R}^n$*.*

Proof. We define the error at the n-th step $\mathbf{e}_n = \mathbf{q}_n - \mathbf{q}^*$. To complete the convergence proof, we performed the detailed computations by using Maple and details are provided below in sequence.

$$\begin{aligned}\mathbf{K}'(\mathbf{q}_0)^{-1} = \Big(&\mathbf{I} - 2\mathbf{A}_2\mathbf{e}_0 + \Big(-3\mathbf{A}_3 + 4\mathbf{A}_2^2\Big)\mathbf{e}_0^2 + \Big(-4\mathbf{A}_4 + 6\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_2\mathbf{A}_3\\
&- 8\mathbf{A}_2^3\Big)\mathbf{e}_0^3 + \Big(-5\mathbf{A}_5 - 12\mathbf{A}_2^2\mathbf{A}_3 - 12\mathbf{A}_3\mathbf{A}_2^2 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 8\mathbf{A}_4\mathbf{A}_2\\
&+ 9\mathbf{A}_3^2 + 8\mathbf{A}_2\mathbf{A}_4 + 16\mathbf{A}_2^4\Big)\mathbf{e}_0^4 + \Big(-6\mathbf{A}_6 - 16\mathbf{A}_2^2\mathbf{A}_4 - 32\mathbf{A}_2^5\\
&- 16\mathbf{A}_4\mathbf{A}_2^2 - 18\mathbf{A}_3^2\mathbf{A}_2 - 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 18\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 18\mathbf{A}_2\mathbf{A}_3^2\\
&+ 24\mathbf{A}_3\mathbf{A}_2^3 + 24\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 24\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 10\mathbf{A}_5\mathbf{A}_2 + 12\mathbf{A}_4\mathbf{A}_3\\
&+ 12\mathbf{A}_3\mathbf{A}_4 + 10\mathbf{A}_2\mathbf{A}_5 + 24\mathbf{A}_2^3\mathbf{A}_3\Big)\mathbf{e}_0^5 + \cdots + O\left(\mathbf{e}_0^9\right)\Big)\mathbf{A}_1^{-1}\end{aligned}$$

$$\mathbf{K}(\mathbf{q}_0) = \mathbf{A}_1\Big(\mathbf{e}_0 + \mathbf{A}_2\mathbf{e}_0^2 + \mathbf{A}_3\mathbf{e}_0^3 + \mathbf{A}_4\mathbf{e}_0^4 + \mathbf{A}_5\mathbf{e}_0^5 + \mathbf{A}_6\mathbf{e}_0^6 + \mathbf{A}_7\mathbf{e}_0^7 + \mathbf{A}_8\mathbf{e}_0^8 + O\left(\mathbf{e}_0^9\right)\Big)$$

$$\begin{aligned}\boldsymbol{\phi}_1 = \mathbf{e}_0 &- \mathbf{A}_2\mathbf{e}_0^2 + \Big(-2\mathbf{A}_3 + 2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(-3\mathbf{A}_4 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2 - 4\mathbf{A}_2^3\Big)\mathbf{e}_0^4\\
&+ \Big(-4\mathbf{A}_5 + 6\mathbf{A}_2\mathbf{A}_4 + 4\mathbf{A}_4\mathbf{A}_2 - 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_3^2 + 8\mathbf{A}_2^4 - 6\mathbf{A}_3\mathbf{A}_2^2\\
&- 8\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-5\mathbf{A}_6 - 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4\\
&+ 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 + 12\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 12\mathbf{A}_3\mathbf{A}_2^3 - 9\mathbf{A}_3^2\mathbf{A}_2\\
&- 12\mathbf{A}_2\mathbf{A}_3^2 - 16\mathbf{A}_2^5 - 12\mathbf{A}_2^2\mathbf{A}_4 - 8\mathbf{A}_4\mathbf{A}_2^2 + 16\mathbf{A}_2^3\mathbf{A}_3\Big)\mathbf{e}_0^6 + O\left(\mathbf{e}_0^7\right)\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\phi}_2 =\ & 2\mathbf{A}_2\mathbf{e}_0^2 + \Big(-8\mathbf{A}_2^2 + 6\mathbf{A}_3\Big)\mathbf{e}_0^3 + \Big(12\mathbf{A}_4 - 20\mathbf{A}_2\mathbf{A}_3 + 26\mathbf{A}_2^3 - 18\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 \\
& + \Big(60\mathbf{A}_2^2\mathbf{A}_3 - 76\mathbf{A}_2^4 - 36\mathbf{A}_2\mathbf{A}_4 + 52\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 32\mathbf{A}_4\mathbf{A}_2 + 54\mathbf{A}_3\mathbf{A}_2^2 \\
& + 20\mathbf{A}_5 - 42\mathbf{A}_3^2\Big)\mathbf{e}_0^5 + \Big(-142\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 146\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 150\mathbf{A}_3\mathbf{A}_2^3 \\
& + 86\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 102\mathbf{A}_3^2\mathbf{A}_2 + 92\mathbf{A}_4\mathbf{A}_2^2 + 116\mathbf{A}_2\mathbf{A}_3^2 + 120\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 168\mathbf{A}_2^3\mathbf{A}_3 \\
& + 102\mathbf{A}_2^2\mathbf{A}_4 + 208\mathbf{A}_2^5 - 56\mathbf{A}_2\mathbf{A}_5 - 72\mathbf{A}_3\mathbf{A}_4 - 72\mathbf{A}_4\mathbf{A}_3 - 50\mathbf{A}_5\mathbf{A}_2 + 30\mathbf{A}_6\Big)\mathbf{e}_0^6 \\
& + \Big(372\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 380\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 388\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 396\mathbf{A}_3\mathbf{A}_2^4 + 128\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 \\
& + 162\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 140\mathbf{A}_5\mathbf{A}_2^2 + 168\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 308\mathbf{A}_2^2\mathbf{A}_3^2 - 316\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& - 324\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 222\mathbf{A}_3^3 + 188\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 200\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 192\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 \\
& + 198\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 448\mathbf{A}_2^4\mathbf{A}_3 - 276\mathbf{A}_2^3\mathbf{A}_4 - 224\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 544\mathbf{A}_2^6 - 264\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 \\
& - 236\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 - 270\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 276\mathbf{A}_3^2\mathbf{A}_2^2 - 248\mathbf{A}_4\mathbf{A}_2^3 + 152\mathbf{A}_2^2\mathbf{A}_5 \\
& - 80\mathbf{A}_2\mathbf{A}_6 - 108\mathbf{A}_3\mathbf{A}_5 - 120\mathbf{A}_4^2 - 110\mathbf{A}_5\mathbf{A}_3 - 72\mathbf{A}_6\mathbf{A}_2 + 42\mathbf{A}_7\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\phi}_3 =\ & 4\mathbf{A}_2^2\mathbf{e}_0^3 + \Big(-28\mathbf{A}_2^3 + 12\mathbf{A}_2\mathbf{A}_3 + 12\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(24\mathbf{A}_4\mathbf{A}_2 + 132\mathbf{A}_2^4 \\
& + 24\mathbf{A}_2\mathbf{A}_4 - 76\mathbf{A}_2^2\mathbf{A}_3 - 68\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 72\mathbf{A}_3\mathbf{A}_2^2 + 36\mathbf{A}_3^2\Big)\mathbf{e}_0^5 + \Big(292\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& + 296\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 312\mathbf{A}_3\mathbf{A}_2^3 - 124\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 168\mathbf{A}_3^2\mathbf{A}_2 - 136\mathbf{A}_4\mathbf{A}_2^2 \\
& - 180\mathbf{A}_2\mathbf{A}_3^2 - 192\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 336\mathbf{A}_2^3\mathbf{A}_3 - 144\mathbf{A}_2^2\mathbf{A}_4 - 516\mathbf{A}_2^5 \\
& + 40\mathbf{A}_2\mathbf{A}_5 + 72\mathbf{A}_3\mathbf{A}_4 + 72\mathbf{A}_4\mathbf{A}_3 + 40\mathbf{A}_5\mathbf{A}_2\Big)\mathbf{e}_0^6 + \Big(-1076\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 \\
& - 1080\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 - 1100\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 - 1152\mathbf{A}_3\mathbf{A}_2^4 - 196\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 \\
& -300\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 - 220\mathbf{A}_5\mathbf{A}_2^2 - 312\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 + 724\mathbf{A}_2^2\mathbf{A}_3^2 \\
& + 740\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 780\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 - 432\mathbf{A}_3^3 - 324\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 \\
& - 360\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 - 336\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 - 360\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 - 1256\mathbf{A}_2^4\mathbf{A}_3 \\
& + 612\mathbf{A}_2^3\mathbf{A}_4 + 508\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 + 1800\mathbf{A}_2^6 + 636\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 + 520\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 \\
& + 672\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 672\mathbf{A}_3^2\mathbf{A}_2^2 + 568\mathbf{A}_4\mathbf{A}_2^3 - 232\mathbf{A}_2^2\mathbf{A}_5 + 60\mathbf{A}_2\mathbf{A}_6 \\
& + 120\mathbf{A}_3\mathbf{A}_5 + 144\mathbf{A}_4^2 + 120\mathbf{A}_5\mathbf{A}_3 + 60\mathbf{A}_6\mathbf{A}_2\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\phi}_4 =\ & 6\mathbf{A}_2^2\mathbf{e}_0^3 + \Big(24\mathbf{A}_4 - 12\mathbf{A}_2\mathbf{A}_3 - 18\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(60\mathbf{A}_5 + 36\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& + 54\mathbf{A}_3\mathbf{A}_2^2 - 48\mathbf{A}_2\mathbf{A}_4 - 54\mathbf{A}_3^2 - 72\mathbf{A}_4\mathbf{A}_2 + 24\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(120\mathbf{A}_6 \\
& - 72\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 108\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 150\mathbf{A}_3\mathbf{A}_2^3 + 144\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 144\mathbf{A}_3^2\mathbf{A}_2 \\
& + 216\mathbf{A}_4\mathbf{A}_2^2 + 108\mathbf{A}_2\mathbf{A}_3^2 + 144\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 48\mathbf{A}_2^3\mathbf{A}_3 + 96\mathbf{A}_2^2\mathbf{A}_4 \\
& - 120\mathbf{A}_2\mathbf{A}_5 - 126\mathbf{A}_3\mathbf{A}_4 - 168\mathbf{A}_4\mathbf{A}_3 - 180\mathbf{A}_5\mathbf{A}_2\Big)\mathbf{e}_0^6 + \Big(144\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 \\
& + 216\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 300\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 396\mathbf{A}_3\mathbf{A}_2^4 + 360\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 342\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 \\
& + 540\mathbf{A}_5\mathbf{A}_2^2 + 432\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 216\mathbf{A}_2^2\mathbf{A}_3^2 - 288\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 372\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 \\
& + 342\mathbf{A}_3^3 + 336\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 480\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 252\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 306\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 \\
& + 96\mathbf{A}_2^4\mathbf{A}_3 - 192\mathbf{A}_2^3\mathbf{A}_4 - 288\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 288\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 432\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 \\
& - 354\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 408\mathbf{A}_3^2\mathbf{A}_2^2 - 600\mathbf{A}_4\mathbf{A}_2^3 + 240\mathbf{A}_2^2\mathbf{A}_5 - 240\mathbf{A}_2\mathbf{A}_6 \\
& - 252\mathbf{A}_3\mathbf{A}_5 - 312\mathbf{A}_4^2 - 390\mathbf{A}_5\mathbf{A}_3 - 360\mathbf{A}_6\mathbf{A}_2 + 210\mathbf{A}_7\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{e}_1 = &\Big(\mathbf{A}_4 + 5\mathbf{A}_2^3 - 2\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(20\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 24\mathbf{A}_3\mathbf{A}_2^2 - 36\mathbf{A}_2^4 \\
&+ 4\mathbf{A}_5 - 8\mathbf{A}_2\mathbf{A}_4 - 12\mathbf{A}_3^2 - 12\mathbf{A}_4\mathbf{A}_2 + 20\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-99\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
&- 105\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 118\mathbf{A}_3\mathbf{A}_2^3 + 51\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 66\mathbf{A}_3^2\mathbf{A}_2 + 66\mathbf{A}_4\mathbf{A}_2^2 + 62\mathbf{A}_2\mathbf{A}_3^2 \\
&+ 72\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 108\mathbf{A}_2^3\mathbf{A}_3 + 49\mathbf{A}_2^2\mathbf{A}_4 + 170\mathbf{A}_2^5 - 20\mathbf{A}_2\mathbf{A}_5 - 30\mathbf{A}_3\mathbf{A}_4 \\
&- 36\mathbf{A}_4\mathbf{A}_3 - 30\mathbf{A}_5\mathbf{A}_2 + 10\mathbf{A}_6\Big)\mathbf{e}_0^6 + \Big(400\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 410\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 \\
&+ 430\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 468\mathbf{A}_3\mathbf{A}_2^4 + 104\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 138\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 140\mathbf{A}_5\mathbf{A}_2^2 \\
&+ 156\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 268\mathbf{A}_2^2\mathbf{A}_3^2 - 284\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 314\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 180\mathbf{A}_3^3 \\
&+ 140\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 176\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 132\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 150\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 452\mathbf{A}_2^4\mathbf{A}_3 \\
&- 224\mathbf{A}_2^3\mathbf{A}_4 - 206\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 660\mathbf{A}_2^6 - 252\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 230\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 \\
&- 278\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 284\mathbf{A}_3^2\mathbf{A}_2^2 - 276\mathbf{A}_4\mathbf{A}_2^3 + 96\mathbf{A}_2^2\mathbf{A}_5 - 40\mathbf{A}_2\mathbf{A}_6 \\
&- 60\mathbf{A}_3\mathbf{A}_5 - 76\mathbf{A}_4^2 - 80\mathbf{A}_5\mathbf{A}_3 - 60\mathbf{A}_6\mathbf{A}_2 + 20\mathbf{A}_7\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\phi}_5 = &\Big(\mathbf{A}_4 + 5\mathbf{A}_2^3 - 2\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(4\mathbf{A}_5 + 26\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 24\mathbf{A}_3\mathbf{A}_2^2 - 46\mathbf{A}_2^4 \\
&- 10\mathbf{A}_2\mathbf{A}_4 - 12\mathbf{A}_3^2 - 12\mathbf{A}_4\mathbf{A}_2 + 24\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(10\mathbf{A}_6 - 151\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
&- 153\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 133\mathbf{A}_3\mathbf{A}_2^3 + 75\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 75\mathbf{A}_3^2\mathbf{A}_2 + 66\mathbf{A}_4\mathbf{A}_2^2 + 86\mathbf{A}_2\mathbf{A}_3^2 \\
&+ 78\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 156\mathbf{A}_2^3\mathbf{A}_3 + 69\mathbf{A}_2^2\mathbf{A}_4 + 262\mathbf{A}_2^5 - 28\mathbf{A}_2\mathbf{A}_5 - 33\mathbf{A}_3\mathbf{A}_4 - 36\mathbf{A}_4\mathbf{A}_3 \\
&- 30\mathbf{A}_5\mathbf{A}_2\Big)\mathbf{e}_0^6 + \Big(702\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 716\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 696\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 606\mathbf{A}_3\mathbf{A}_2^4 \\
&+ 164\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 174\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 140\mathbf{A}_5\mathbf{A}_2^2 + 168\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 440\mathbf{A}_2^2\mathbf{A}_3^2 \\
&- 440\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 386\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 216\mathbf{A}_3^3 + 212\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 184\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 \\
&+ 198\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 180\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 764\mathbf{A}_2^4\mathbf{A}_3 - 362\mathbf{A}_2^3\mathbf{A}_4 - 356\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 \\
&-1184\mathbf{A}_2^6 - 402\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 362\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 - 356\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
&- 356\mathbf{A}_3^2\mathbf{A}_2^2 - 296\mathbf{A}_4\mathbf{A}_2^3 + 152\mathbf{A}_2^2\mathbf{A}_5 - 60\mathbf{A}_2\mathbf{A}_6 - 72\mathbf{A}_3\mathbf{A}_5 \\
&- 80\mathbf{A}_4^2 - 80\mathbf{A}_5\mathbf{A}_3 - 60\mathbf{A}_6\mathbf{A}_2 + 20\mathbf{A}_7\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\phi}_6 = &\Big(\mathbf{A}_4 + 5\mathbf{A}_2^3 - 2\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(32\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 24\mathbf{A}_3\mathbf{A}_2^2 - 56\mathbf{A}_2^4 \\
&+ 4\mathbf{A}_5 - 12\mathbf{A}_2\mathbf{A}_4 - 12\mathbf{A}_3^2 - 12\mathbf{A}_4\mathbf{A}_2 + 28\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(-215\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
&- 201\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 148\mathbf{A}_3\mathbf{A}_2^3 + 99\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 84\mathbf{A}_3^2\mathbf{A}_2 + 66\mathbf{A}_4\mathbf{A}_2^2 + 110\mathbf{A}_2\mathbf{A}_3^2 \\
&+ 84\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 212\mathbf{A}_2^3\mathbf{A}_3 + 93\mathbf{A}_2^2\mathbf{A}_4 + 374\mathbf{A}_2^5 - 36\mathbf{A}_2\mathbf{A}_5 - 36\mathbf{A}_3\mathbf{A}_4 \\
&- 36\mathbf{A}_4\mathbf{A}_3 - 30\mathbf{A}_5\mathbf{A}_2 + 10\mathbf{A}_6\Big)\mathbf{e}_0^6 + \Big(1132\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 1118\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 \\
&+ 992\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 774\mathbf{A}_3\mathbf{A}_2^4 + 224\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 210\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 140\mathbf{A}_5\mathbf{A}_2^2 \\
&+ 180\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 660\mathbf{A}_2^2\mathbf{A}_3^2 - 608\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 470\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 252\mathbf{A}_3^3 \\
&+ 284\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 192\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 270\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 216\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 1188\mathbf{A}_2^4\mathbf{A}_3 \\
&- 548\mathbf{A}_2^3\mathbf{A}_4 - 554\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 1932\mathbf{A}_2^6 - 570\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 494\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 \\
&- 452\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 428\mathbf{A}_3^2\mathbf{A}_2^2 - 316\mathbf{A}_4\mathbf{A}_2^3 + 224\mathbf{A}_2^2\mathbf{A}_5 - 80\mathbf{A}_2\mathbf{A}_6 \\
&- 84\mathbf{A}_3\mathbf{A}_5 - 84\mathbf{A}_4^2 - 80\mathbf{A}_5\mathbf{A}_3 - 60\mathbf{A}_6\mathbf{A}_2 + 20\mathbf{A}_7\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)
\end{aligned}
$$

$$\begin{aligned}\boldsymbol{\phi}_7 =& \Big(\mathbf{A}_4 + 5\mathbf{A}_2^3 - 2\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(38\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 24\mathbf{A}_3\mathbf{A}_2^2 - 66\mathbf{A}_2^4 \\ &+ 4\mathbf{A}_5 - 14\mathbf{A}_2\mathbf{A}_4 - 12\mathbf{A}_3^2 - 12\mathbf{A}_4\mathbf{A}_2 + 32\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^5 + \Big(- 291\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\ &- 249\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 163\mathbf{A}_3\mathbf{A}_2^3 + 123\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 93\mathbf{A}_3^2\mathbf{A}_2 + 66\mathbf{A}_4\mathbf{A}_2^2 + 134\mathbf{A}_2\mathbf{A}_3^2 \\ &+ 90\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 276\mathbf{A}_2^3\mathbf{A}_3 + 121\mathbf{A}_2^2\mathbf{A}_4 + 506\mathbf{A}_2^5 - 44\mathbf{A}_2\mathbf{A}_5 - 39\mathbf{A}_3\mathbf{A}_4 \\ &- 36\mathbf{A}_4\mathbf{A}_3 - 30\mathbf{A}_5\mathbf{A}_2 + 10\mathbf{A}_6\Big)\mathbf{e}_0^6 + \Big(1714\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 1616\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 \\ &+ 1318\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 + 972\mathbf{A}_3\mathbf{A}_2^4 + 284\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 246\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 140\mathbf{A}_5\mathbf{A}_2^2 \\ &+ 192\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 928\mathbf{A}_2^2\mathbf{A}_3^2 - 788\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 566\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 288\mathbf{A}_3^3 \\ &+ 356\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 200\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 348\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 + 258\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 1740\mathbf{A}_2^4\mathbf{A}_3 \\ &- 790\mathbf{A}_2^3\mathbf{A}_4 - 800\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 2944\mathbf{A}_2^6 - 756\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 626\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 \\ &- 566\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 500\mathbf{A}_3^2\mathbf{A}_2^2 - 336\mathbf{A}_4\mathbf{A}_2^3 + 312\mathbf{A}_2^2\mathbf{A}_5 - 100\mathbf{A}_2\mathbf{A}_6 \\ &- 96\mathbf{A}_3\mathbf{A}_5 - 88\mathbf{A}_4^2 - 80\mathbf{A}_5\mathbf{A}_3 - 60\mathbf{A}_6\mathbf{A}_2 + 20\mathbf{A}_7\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)\end{aligned}$$

$$\mathbf{e}_2 = \Big(- 24\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 - 16\mathbf{A}_2^4\mathbf{A}_3 + 8\mathbf{A}_2^3\mathbf{A}_4 + 40\mathbf{A}_2^6\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right) .$$

Which completes the proof. □

Now we present the proof of convergence of IZFZA via mathematical induction.

Theorem 3.2. *Prove that the CO of IZFZA method is* $3s + 1$ *for* $s \geq 2$.

Proof. All the computations are made under the assumption of Theorem 3.1. We know from Theorem 3.1 that the CO of IZFZA method is seven for $s = 2$ and the error equation is

$$\mathbf{e}_2 = \Big(- 24\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 - 16\mathbf{A}_2^4\mathbf{A}_3 + 8\mathbf{A}_2^3\mathbf{A}_4 + 40\mathbf{A}_2^6\Big)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right) . \quad (3.\ 3)$$

Now we assume that the CO of IZFZA is $3s + 1$ for $s \geq 2$, and we will prove that the CO of IZFZA is $3s + 4$ for $(s + 1)$th step. If the CO of IZFZA is $3s + 1$ then

$$\mathbf{e}_s = \mathbf{q}_s - \mathbf{q}^* \sim d_1\mathbf{e}_0^{3s+1} , \quad (3.\ 4)$$

where d_1 is the asymptotic constant and the symbol $\sim$ means the approximation. By using (3. 4), we perform the following steps to complete the proof.

$$\begin{aligned}\mathbf{K}(\mathbf{q}_0)^{-1} &\sim (\mathbf{I} - 2\mathbf{A}_2\mathbf{e}_0)\,\mathbf{A}_1^{-1} \\ \mathbf{K}(\mathbf{q}_s) &\sim \mathbf{A}_1 d_1\mathbf{e}_0^{3s+1} \\ \boldsymbol{\phi}_5 &\sim - 2\mathbf{A}_2 d_1\mathbf{e}_0^{3s+2} + d_1\mathbf{e}_0^{3s+1} \\ \mathbf{K}'(\mathbf{q}_1) &\sim \mathbf{A}_1 + \Big(- 6\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 10\mathbf{A}_1\mathbf{A}_2^4 + 2\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4 - 4\mathbf{A}_1\mathbf{A}_2^2\mathbf{A}_3\Big)\mathbf{e}_0^4 \\ \boldsymbol{\phi}_6 &\sim d_1\mathbf{e}_0^{3s+1} - 4\mathbf{A}_2 d_1\mathbf{e}_0^{3s+2} + 4\mathbf{A}_2^2 d_1\mathbf{e}_0^{3s+3} \\ \boldsymbol{\phi}_7 &\sim d_1\mathbf{e}_0^{3s+1} - 6\mathbf{A}_2 d_1\mathbf{e}_0^{3s+2} + 12\mathbf{A}_2^2 d_1\mathbf{e}_0^{3s+3} - 8\mathbf{A}_2^3 d_1\mathbf{e}_0^{3s+4} \\ \mathbf{e}_{s+1} &\sim 8\mathbf{A}_2^3 d1\mathbf{e}_0^{3s+4} .\end{aligned}$$

Which completes the proof. □

4. Numerical Testing

The verification of CO is important and we adopt the following definition of computational CO (CCO)

$$\text{CCO} = \frac{\log(||\mathbf{K}(\mathbf{q}_{k+1})||_\infty / ||\mathbf{K}(\mathbf{q}_k)||_\infty)}{\log(||\mathbf{K}(\mathbf{q}_k)||_\infty / ||\mathbf{K}(\mathbf{q}_{k-1})||_\infty)} . \tag{4. 5}$$

4.1. **Verification of computational CO.** Consider the following system of non-linear equations $\mathbf{K}(\mathbf{q}) = [K_1(\mathbf{q}), K_2(\mathbf{q}), K_3(\mathbf{q}), K_4(\mathbf{q})]^T = \mathbf{0}$,

$$\begin{aligned} K_1(\mathbf{q}) &= q_3\, q_2 + (q_2 + q_3)\, q_4 = 0 \\ K_2(\mathbf{q}) &= q_3\, q_1 + (q_1 + q_3)\, q_4 = 0 \\ K_3(\mathbf{q}) &= q_2\, q_1 + (q_1 + q_2)\, q_4 = 0 \\ K_4(\mathbf{q}) &= q_2\, q_1 + (q_1 + q_2)\, q_3 = 1\,. \end{aligned} \tag{4. 6}$$

Let $\mathbf{d} = [d_1, d_2, d_3, d_4]^T$ be a constant vector, and $\mathbf{K}'(\mathbf{q})$ and $\mathbf{K}''(\mathbf{q})\mathbf{d} = (\mathbf{K}'(\mathbf{q})\mathbf{d})'$ can be written as

$$\begin{aligned} \mathbf{K}'(\mathbf{q}) &= \begin{bmatrix} 0 & q_3+q_4 & q_2+q_4 & q_2+q_3 \\ q_3+q_4 & 0 & q_1+q_4 & q_1+q_3 \\ q_2+q_4 & q_1+q_4 & 0 & q_1+q_2 \\ q_2+q_3 & q_1+q_3 & q_1+q_2 & 0 \end{bmatrix} \\ \mathbf{K}''(\mathbf{q})\mathbf{d} &= \begin{bmatrix} 0 & d_3+d_4 & d_2+d_4 & d_2+d_3 \\ d_3+d_4 & 0 & d_1+d_4 & d_1+d_3 \\ d_2+d_4 & d_1+d_4 & 0 & d_1+d_2 \\ d_2+d_3 & d_1+d_3 & d_1+d_2 & 0 \end{bmatrix}, \end{aligned} \tag{4. 7}$$

and $\mathbf{K}'''(\mathbf{q})\mathbf{d}^3 = \mathbf{0}$. Table 1 shows that the computational COs are in agreement with the theoretical CO of the iterative scheme IZFZA.

Iter \ Steps		$s=2$	$s=3$	$s=4$	$s=5$
1	$\|\|\mathbf{K}(\mathbf{q}_k)\|\|_\infty$	2.42e-6	9.19e-9	2.78e-11	9.42e-14
2	-	4.71e-42	1.67e-84	8.94e-142	7.28e-214
3	-	5.05e-292	2.00e-841	3.55e-1838	1.19e-3415
CCO		7	10	13	16
Theoretical CO $(3s+1)$		7	10	13	16

Table 1. IZFZA : verification of CO for the problem (4. 6).

4.2. **Lane-Emden problem.** Next we consider the Lane-Emden boundary value problem

$$u''(x) + \frac{2}{x}\, u'(x) + u(x)^p = 0\,, \quad u'(0) = 0\,, \quad u(0) = 1\,. \tag{4. 8}$$

In Tables 2, 3, 4, 5, we computed the computational CO for $p \in \{2, 3, 4, 5\}$ and use different number of steps. By performing three iterations, we found that the CCO confirms the theoretical COs. In Figure 1, we plotted the numerical solution of Lane-Emden equation for different indices ranging from two to five.

Iter \ s		2	3	4	5
1	$\|\|\mathbf{K}(\mathbf{q}_k)\|\|_\infty$	4.12e-1	1.60e-2	2.70e-4	2.50e-6
2	-	9.58e-26	5.88e-61	2.37e-110	1.18e-172
3	-	7.40e-216	6.91e-664	1.14e-1498	5.97e-2853
CCO ($p = 2$)		7.71	10.32	13.09	16.11
Theoretical CO ($3s + 1$)		7	10	13	16

Table 2. IZFZA : verification of CO for the problem (4. 8) over the domain [0,3], number of grid points 50.

Iter \ s		2	3	4	5
1	$\|\|\mathbf{K}(\mathbf{q}_k)\|\|_\infty$	8.62e-1	4.70e-2	2.75e-3	8.53e-5
2	-	2.69e-15	2.09e-38	1.45e-74	3.69e-123
3	-	8.91e-133	3.23e-433	2.81e-1041	6.93e-2059
CCO ($p = 3$)		8.10	10.86	13.56	16.35
Theoretical CO ($3s + 1$)		7	10	13	16

Table 3. IZFZA : verification of CO for the problem (4. 8) over the domain [0,3], number of grid points 50.

Iter \ s		2	3	4	5
1	$\|\|\mathbf{K}(\mathbf{q}_k)\|\|_\infty$	2.50	3.91e-1	2.83e-2	1.12e-2
2	-	2.58e-14	1.94e-29	2.41e-51	3.41e-84
3	-	2.06e-116	3.41e-330	3.18e-731	3.51e-1435
CCO($p = 4$)		7.30	10.63	13.86	16.57
Theoretical CO ($3s + 1$)		7	10	13	16

Table 4. IZFZA : verification of CO for the problem (4. 8) over the domain [0,3], number of grid points 50.

Iter \ s		2	3	4	5
1	$\lvert\lvert\mathbf{K}(\mathbf{q}_k)\rvert\rvert_\infty$	2.25e+1	9.78	2.24	1.70e-1
2	-	2.42e-6	8.09e-19	4.92e-38	9.67e-59
3	-	2.74e-66	8.89e-240	5.68e-550	1.20e-1025
CCO ($p = 5$)		8.60	11.57	13.59	16.89
Theoretical CO ($3s + 1$)		7	10	13	16

TABLE 5. IZFZA : verification of CO for the problem (4. 8) over the domain [0,3], number of grid points 50.

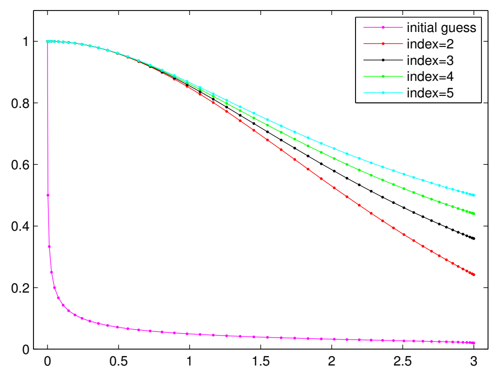

FIGURE 1. IZFZA: plot of Lane-Emden equation with different indices, number of gird points 50.

4.3. **3-D Poisson nonlinear problem.** The nonlinear Poisson-Dirichlet boundary value problem can be describe as

$$u_{xx} + u_{yy} + u_{zz} + g(u) = p(x, y, z), \qquad (x, y, z) \in (0, 1)^3 \tag{4. 9}$$

where $p(x, y, z)$ is the source term and $g(u) = u^s$ is a nonlinear function. We assume Dirichlet boundary conditions. Using Chebyshev pseudo-spectral collocation method [4, 5], we perform the following discretization of (4. 9):

$$\begin{aligned}
&\mathbf{K}(\mathbf{U}) = ((\mathbf{T}_{xx} \otimes \mathbf{I}_y \otimes \mathbf{I}_z) + (\mathbf{I}_x \otimes \mathbf{T}_{yy} \otimes \mathbf{I}_z) + (\mathbf{I}_x \otimes \mathbf{I}_y \otimes \mathbf{T}_{zz}))\mathbf{U} + f(\mathbf{U}) - \mathbf{p} = \mathbf{0}\\
&\mathbf{K}'(\mathbf{U}) = ((\mathbf{T}_{xx} \otimes \mathbf{I}_y \otimes \mathbf{I}_z) + (\mathbf{I}_x \otimes \mathbf{T}_{yy} \otimes \mathbf{I}_z) + (\mathbf{I}_x \otimes \mathbf{I}_y \otimes \mathbf{T}_{zz})) + \mathrm{diag}\,(g'(\mathbf{U}))\\
&\mathbf{K}''(\mathbf{U})\mathbf{v}_1\mathbf{v}_2 = g''(\mathbf{U}) \odot \mathbf{v}_1 \odot \mathbf{v}_2\\
&\mathbf{K}'''(\mathbf{U})\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 = g'''(\mathbf{U}) \odot \mathbf{v}_1 \odot \mathbf{v}_2 \odot \mathbf{v}_3\,,
\end{aligned} \tag{4. 10}$$

where $\mathbf{I}$ denotes the identity matrix, and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are vectors, $\otimes$ is a Kronecker product and $\mathbf{T}_{..}$ is the discretization of the second order derivative. In Tables 3 and 4, We uses different grid sizes to compute the absolute error in the numerical solution. We observe that as we refine the grid, we get more accurate results. We achieve almost 15-digit accuracy in the computed solution. It is important to note that we perform only one iteration and many multi-steps. It means, we compute only once the LU factors of the Jacobian at the initial guess and use these LU factors repeatedly in the multi-step part to solve the system of linear equations to achieve the high order of convergence.

$s \setminus N$		$8 \times 8 \times 8$	$10 \times 10 \times 10$	$12 \times 12 \times 12$
2	$\|\|\mathbf{U}_k - \mathbf{U}_{\text{analytical}}\|\|_\infty$	5.00e-06	5.40e-06	5.62e-06
3	-	7.82e-09	8.03e-09	8.37e-09
4	-	5.63e-10	1.17e-11	1.22e-11
5	-	5.62e-10	3.20e-13	1.51e-14
6	-	5.62e-10	3.22e-13	9.33e-15

TABLE 6. IZFZA: absolute error (ABE) in the solution of (4. 10) versus different grid sizes, number of iterations = 1, initial guess $\mathbf{U} = \mathbf{0}$, $g(u) = u^3$.

$s \setminus N$		$8 \times 8 \times 8$	$10 \times 10 \times 10$	$12 \times 12 \times 12$
2	$\|\|\mathbf{U}_k - \mathbf{U}_{\text{analytical}}\|\|_\infty$	3.70e-04	3.99e-04	4.14e-04
3	-	5.53e-06	6.03e-06	6.30e-06
4	-	8.21e-08	8.93e-08	9.34e-08
5	-	1.65e-09	1.31e-09	1.38e-09
6	-	5.63e-10	1.93e-11	2.02e-11
7	-	5.63e-10	3.19e-13	2.94e-13
8	-	5.63e-10	3.24e-13	7.77e-15

TABLE 7. IZFZA: ABE in the solution of (4. 10) versus different grid sizes, number of iterations = 1, initial guess $\mathbf{U} = \mathbf{0}$, $g(u) = u^4$.

4.4. **2-D nonlinear wave equation.** The 2-D nonlinear wave equation can be written as

$$u_{tt} - c^2 (u_{xx} + u_{yy}) + g(u) = p(x, y), \qquad (x, y, t) \in (-1, 1)^2 \times (0, 2] \qquad (4.\ 11)$$

where nonlinear function $g(u) = u^s$ and c, s are constants. By assuming the solution $u = \exp(-t)\sin(x+y)$, we compute the source term $p(x, y)$. The 2-D nonlinear wave equation is solved by imposing Dirichlet boundary conditions. By the application of Chebyshev pseudo-spectral collocation method. we discretize (4. 11) and get the following system of nonlinear equations

$$\begin{aligned}
\mathbf{K}(\mathbf{U}) &= ((\mathbf{T}_{tt}\otimes\mathbf{I}_x\otimes\mathbf{I}_y)-c^2((\mathbf{I}_t\otimes\mathbf{T}_{xx}\otimes\mathbf{I}_y)+(\mathbf{I}_t\otimes\mathbf{I}_x\otimes\mathbf{T}_{yy})))\mathbf{U}\\
&\quad + f(\mathbf{U})-\mathbf{p}=\mathbf{0}\\
\mathbf{K}'(\mathbf{U}) &= (\mathbf{T}_{tt}\otimes\mathbf{I}_x\otimes\mathbf{I}_y)-c^2((\mathbf{I}_t\otimes\mathbf{T}_{xx}\otimes\mathbf{I}_y)+(\mathbf{I}_t\otimes\mathbf{I}_x\otimes\mathbf{T}_{yy}))+\mathrm{diag}\,(g'(\mathbf{U}))\\
\mathbf{K}''(\mathbf{U})\mathbf{v}_1\mathbf{v}_2 &= g''(\mathbf{U})\odot\mathbf{v}_1\odot\mathbf{v}_2\\
\mathbf{K}'''(\mathbf{U})\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3 &= g'''(\mathbf{U})\odot\mathbf{v}_1\odot\mathbf{v}_2\odot\mathbf{v}_3\,,
\end{aligned} \tag{4. 12}$$

Three different size grids are used to solve the 2-D nonlinear wave equation. Table 5 depicts that we achieve 11-digit accuracy by performing a single iteration of our proposed iterative scheme IZFZA. In all tables, iterations are carried out until we do not find any further improvement in the numerical accuracy of the solution.

$s\ \backslash N$		$10\times10\times20$	$20\times20\times20$	$20\times20\times30$
2	$\lvert\lvert\mathbf{U}_k-\mathbf{U}_{\text{analytical}}\rvert\rvert_\infty$	1.86e-03	3.61e-03	1.95e-03
3	-	1.87e-04	1.75e-03	1.11e-04
6	-	8.65e-06	1.50e-04	6.47e-06
8	-	1.20e-06	2.96e-05	1.01e-06
10	-	1.66e-07	5.82e-06	1.57e-07
12	-	2.28e-08	1.14e-06	2.44e-08
14	-	3.21e-09	2.25e-07	3.80e-09
16	-	9.88e-10	4.43e-08	5.91e-10
18	-	7.26e-10	8.72e-09	9.21e-11

Table 8. IZFZA: ABE in the solution of (4. 12) versus different grid sizes, Number of iterations = 1, $g(u)=u^5$, $c=1$, initial guess $\mathbf{U}=\mathbf{0}$.

5. Conclusions

It is true that the use of Fréchet derivatives of higher orders for solving a system of nonlinear equations is not of practical interest and the reason is the high computational cost. We have shown that it is not the case when we deal with the system of nonlinear equations associated with IVPs and BVPs. The applicability of higher order Fréchet derivatives is their low computational cost in the case of IVPs and BVPs. We exploit this benefit to constructing the multi-step frozen Jacobian higher order iterative scheme. The frozen Jacobian provides us the LU factors that we repeatedly use in the multi-step part to achieve high CO. In two problems, we compute the CCO that confirms the theoretical CO but in many BVPs, it is not possible to verify the theoretical CO. We employ a single iteration in all simulations to achieve good numerical accuracy.

6. Author's contributions

The idea of the developed iterative method is proposed by the second and third author and other authors contributed equally to the article.

7. Acknowledgments

We are thankful to reviewers for their suggestions to improve the quality of the article.

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