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Measurable Soft Mappings

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Abstract. The goal of this paper is to throw light on the novel concept of measurable soft mappings. The criteria for an extended real-valued soft mapping to be a Lebesgue measurable soft mapping would also be presented. The positive and negative parts of an extended real-valued soft mapping are also introduced therein. The measurability of soft mappings would also be the part of discussion. The definition of soft probability measure in connection with its applications to soft σ-algebra will also be briefly discussed. In the end, an application of soft sets would also be represented.

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1. Introduction

In 1999, Russian researcher Molodtsov [17] originated the idea of soft set theory as a mathematical device for dealing with uncertainty and decision making problems. The theory has many practical applications in a diversity of fields. Maji *et al.* [14, 15] employed soft sets theory in problems related to decision making and defined many operations on soft sets. Ali *et al.* [2] suggested some fruitful operations on soft sets. Chen *et al.* [7] laid foundation of parametrization reduction of soft sets and its applications. Shabir and Naz in [24], and Cagman *et al.* in [5] proposed the theory of soft topological spaces and accomplished various properties regarding soft topological spaces. In [22], Rong discussed the countabilities of soft topological spaces, soft separable spaces and soft Lindölof spaces and investigated some interesting results using these notions. Roy and Samanta[23] discussed some interesting results in the theory of soft topological spaces utilizing the ideas of soft base and soft sub-base. Zorlutuna and Çakir [28] worked on soft continuity, soft open-ness and soft closed-ness of soft mappings and also investigated the behavior of soft separation axioms and generalized the pasting lemma in view of soft set theory. Riaz and Fatima [21] used soft sets, soft elements and soft points to explore the notions of soft dense, nowhere soft dense sets, soft first category, soft second category and soft Baire space for soft metric spaces and established the Baire's category theorem for soft metric spaces. Pei and Miao [19] described remarkable relationship between the soft sets and information systems. In [18], Mukherjee *et al.* studied the notion of Measurable soft sets. Samanta and Das [8, 9, 10] proposed fundamental properties of soft real sets and soft real numbers. They also discussed soft elements and soft points in soft sets and proposed the idea of soft metric spaces. Samanta and Majumdar [16] proposed the notion of soft groups, and discussed the images and inverse images in the view of soft mappings on soft sets. Kharal and Ahmad [13] established mappings on soft classes, the images and inverse images of soft sets. Khameneh and Kilicman [12] discussed Soft σ-Algebras in connection with soft probability space. Riaz and Naeem [20] introduced different concepts of soft sets, including soft σ-ring, soft algebra, and soft σ-algebra. They presented different types of set functions, including soft finitely sub-additive, soft countably sub-additive, soft finitely additive, soft countably additive and soft monotone. They studied the concept of soft outer measure and soft Lebesgue outer measure. They also described interesting applications of soft mappings to decision-making.

In continuance to the significant work done by the aforementioned icons of Mathematics, we explore, in the following pages, the novel concept of measurable soft mappings. The criteria for an extended real-valued soft mapping to be a Lebesgue measurable soft mapping would also be presented. The positive and negative parts of an extended real-valued soft mapping would also be introduced therein. The measurability of soft mappings would be the part of discussion. In the sequel, the definition of soft probability measure in connection with its applications to soft σ-algebra will also be briefly discussed.

2. Preliminaries

Definition 2.1. [17] Let X be a universe and E a non-empty collection of decision variables. Suppose that 2^X is the aggregate of all subsets of X and $A(\neq \phi) \subseteq E$. The doublet (T, A), where $T : A \to 2^X$ is a mapping, is known as a *soft set* over X. Mathematically speaking, it may be expressed as

$$(T, A) = \{(\eta, T_A(\eta)) \,:\, \eta \in A, T_A(\eta) \in 2^X\}$$

T_A is another representation for (T, A).

Maji *et al.* [15] presented soft subsets as below:

Definition 2.2. [15] Let (T, A_1) and (G, A_2) be soft sets over X. If

(1) $A_1 \subseteq A_2$, and
(2) $T(\eta) \subseteq G(\eta), \forall \eta \in A_1$

then we write $(T, A_1) \,\tilde{\subseteq}\, (G, A_2)$ and call (T, A_1) a *soft subset* of (G, A_2). (G, A_2) is then called *soft superset* of (T, A_1) and is expressed as $(G, A_2) \,\tilde{\supseteq}\, (T, A_1)$.

Definition 2.3. [8] Let $\mathfrak{B}(\mathbb{R})$ be the collection of all non-void bounded subsets of the set $\mathbb{R}$ of real numbers. Assume that A is the collection of decision variates. The map $T : A \to \mathfrak{B}(\mathbb{R})$ is known as a *soft real set*, designated by (T, A). If (T, A) is a soft set comprising only one soft element , then after recognizing (T, A) with the corresponding soft element, it is termed as a *soft real number*.

We express a soft real number by $\tilde{r}$, whereas $\bar{r}$ will represent the particular type of soft real numbers such that $\bar{r}(\eta) = r$, for all $\eta \in A$. For example: $\bar{0}$ is the soft real number where $\bar{0}(\eta) = 0$, for every $\eta \in A$.

Definition 2.4. [8] Let $\mathcal{C}$ be the family of all closed bounded intervals of real numbers, then the mapping $\widetilde{I} : E \to \mathcal{C}$ is known as a *soft closed interval*. Each soft interval may be expressed as an ordered pair of soft real numbers. That is if $\widetilde{I} : E \to \mathcal{C}$ is defined by $\widetilde{I}(\lambda) = [a_\lambda, b_\lambda],\ \forall \lambda \in E$, then the soft interval $(\widetilde{I}, E)$ may be expressed as an ordered pair of soft real numbers (T_1, T_2), where $T_1(\lambda) = a_\lambda$ and $T_2(\lambda) = b_\lambda$, $\forall \lambda \in E$.

Similarly the mapping $\widetilde{I} : E \to \mathcal{C}$ is called a *soft open interval* if $\widetilde{I} : E \to \mathcal{C}$ is defined by $\widetilde{I}(\lambda) = (a_\lambda, b_\lambda)$, $\forall \lambda \in E$.

Definition 2.5. [26] Let $\widetilde{\leq}$ be an ordering of (T, A) and let $(T_1, A_1) \widetilde{\subseteq} (T, A)$. For $\eta \in A$, if $T(\eta) \widetilde{\leq} T_1(\lambda), \forall \lambda \in A_1$, then $T(\eta)$ is known as a *soft lower bound* of (T_1, A_1) in the ordered soft set $(T, A, \widetilde{\leq})$. $T(\mu)$ is termed as the *soft infimum* or *soft greatest lower bound* if it is greatest of all soft lower bounds of (T_1, A_1) in $(T, A, \widetilde{\leq})$.

Definition 2.6. [26] Let $\widetilde{\leq}$ be an ordering of (T, A) and let $(T_1, A_1) \widetilde{\subseteq} (T, A)$. For $\eta \in A$, if $T_1(\lambda) \widetilde{\leq} T(\eta), \forall \lambda \in A_1$, then $T(\eta)$ is known as a *soft upper bound* of (T_1, A_1) in the ordered soft set $(T, A, \widetilde{\leq})$. $T(\mu)$ is termed as the *soft supremum* or *soft least upper bound* if it is smallest of all soft upper bounds of (T_1, A_1) in $(T, A, \widetilde{\leq})$.

Definition 2.7. [13] Let $f : X \to Y$ and $u : E_1 \to E_2$ be mappings. Then a *soft mapping* $\psi_{fu} : (X, E_1) \to (Y, E_2)$, where (X, E_1) and (Y, E_2) are soft classes, is defined as:
For a soft set (F, A) in (X, E_1), $(\psi_{fu}(F, A), B)$, $B = u(A) \subseteq E_2$ is a soft set in (Y, E_2) given by

$$\psi_{fu}(F, A)(\eta_2) = \begin{cases} f(\bigcup_{\eta_1 \in u^{-1}(\eta_2) \cap A} F(\eta_1)), & \text{if} \quad u^{-1}(\eta_2) \cap A \neq \phi \\ \phi, & \text{otherwise} \end{cases}$$

for $\eta_2 \in B \subseteq E_2$. $(\psi_{fu}(F, A), B)$ is called *soft image* of a soft set (F, A).

The soft mapping ψ_{fu} is *soft injective* if both the mappings f and u are injective and is *soft surjective* if both of f and u are surjective.

Definition 2.8. [12, 20] An aggregate $\widetilde{\mathcal{A}}$ of soft subsets of $\check{X}$ is termed as a *soft σ-algebra* on $\check{X}$ if
1) $T_\phi \widetilde{\in} \widetilde{\mathcal{A}}$
2) If $T_A \widetilde{\in} \widetilde{\mathcal{A}}$ then $T_A^c \widetilde{\in} \widetilde{\mathcal{A}}$
3) If $\{\widetilde{\mathcal{A}}_i : i \in \mathbb{N}\} \widetilde{\in} \widetilde{\mathcal{A}}$, then $\widetilde{\cup}_{i=1}^{\infty} \widetilde{\mathcal{A}}_i \widetilde{\in} \widetilde{\mathcal{A}}$
The doublet $(\check{X}, \widetilde{\mathcal{A}})$ is known as a *soft measurable space*. Each $\widetilde{\mathcal{A}}_i \widetilde{\in} \widetilde{\mathcal{A}}$ is called a *measurable soft set*.

Example 2.9. Let $X = \{g, r, s\}$ be the initial universe and $E = \{\eta_1, \eta_2\}$ be the set of parameters. Let

$T_{A_1} = T_\phi$,
$T_{A_2} = \{(\eta_1, \{g\}), (\eta_2, \{\})\}$,
$T_{A_3} = \{(\eta_1, \{r\}), (\eta_2, \{g, s\})\}$,
$T_{A_4} = \{(\eta_1, \{s\}), (\eta_2, \{r\})\}$,
$T_{A_5} = \{(\eta_1, \{g, r\}), (\eta_2, \{g, s\})\}$,
$T_{A_6} = \{(\eta_1, \{r, s\}), (\eta_2, \{g, r, s\})\}$,

$T_{A_7} = \{(\eta_1, \{g, s\}), (\eta_2, \{r\})\}$, and
$T_{A_8} = \breve{X}$.
Then $\widetilde{\mathcal{A}} = \{T_{A_i} : i = 1, 2, 3, ..., 8\}$ is a soft σ-algebra over X.

Definition 2.10. [20] Let $\widetilde{\mathcal{A}}$ be a soft σ-algebra of soft subsets over X and $\widetilde{\mu}$ be a soft real-valued mapping on $\widetilde{\mathcal{A}}$. Let $\{T_{A_i}\}$ be a sequence of soft sets in $\widetilde{\mathcal{A}}$. The soft mapping $\widetilde{\mu}$ is called
1) *finitely soft sub-additive* if $\widetilde{\mu}(\widetilde{\cup}_{i=1}^{n} T_{A_i}) \widetilde{\leq} \Sigma_{i=1}^{n} \widetilde{\mu}(T_{A_i})$.
2) *countably soft sub-additive* if $\widetilde{\mu}(\widetilde{\cup}_{i=1}^{\infty} T_{A_i}) \widetilde{\leq} \Sigma_{i=1}^{\infty} \widetilde{\mu}(T_{A_i})$.
3) *finitely soft additive* if $\widetilde{\mu}(\widetilde{\cup}_{i=1}^{n} T_{A_i}) = \Sigma_{i=1}^{n} \widetilde{\mu}(T_{A_i})$, where T_{A_i}'s are pairwise soft disjoint.
4) *countably soft additive* or *soft σ-additive* if $\widetilde{\mu}(\widetilde{\cup}_{i=1}^{\infty} T_{A_i}) = \Sigma_{i=1}^{\infty} \widetilde{\mu}(T_{A_i})$, where T_{A_i}'s are pairwise soft disjoint.
5) *soft monotone* if $T_A \widetilde{\subseteq} T_B \Rightarrow \widetilde{\mu}(T_A) \widetilde{\leq} \widetilde{\mu}(T_B),\ \forall\ T_A, T_B \widetilde{\in} \widetilde{\mathcal{A}}$.

Definition 2.11. [20] A non-negative soft extended real-valued set function $\widetilde{\mu}^*$ defined on 2^X is called a *soft outer measure* if
1) $\widetilde{\mu}^*(T_\phi) = \overline{0}$;
2) $\widetilde{\mu}^*$ is soft monotone; and
3) $\widetilde{\mu}^*$ is countably soft sub-additive i.e. $\widetilde{\mu}^*(\widetilde{\cup}_{i=1}^{\infty} T_{A_i}) \widetilde{\leq} \Sigma_{i=1}^{\infty} \widetilde{\mu}^*\ (T_{A_i})$.

Definition 2.12. [18] Let T_A be a soft set. A mapping $\widetilde{m}^* : 2^{\widetilde{\mathbb{R}}} \to [\overline{0}, \overline{\infty}]$ given as

$$\widetilde{m}^*(T_A) = \widetilde{\inf}\left\{\sum_n\ l(\widetilde{I}_n(\eta)) : T_A(\eta) \widetilde{\subseteq} \widetilde{\cup}_n\ \widetilde{I}_n(\eta), \eta \in E\right\}$$

where soft infimum is taken over soft finite or soft countable sequence $\{\widetilde{I}_n\}$ of soft open intervals and l stands for length of an interval, is called *soft Lebesgue outer measure*.
In other words

$$\widetilde{m}^*(T_A) = \sum_{\eta \in A} \widetilde{m}^*(T(\eta))$$

where $\widetilde{m}^*$ stands for the Lebesgue outer measure. Since $\widetilde{\mathbb{R}} \widetilde{\in} 2^{\widetilde{\mathbb{R}}}$, so for any $T_A \widetilde{\subseteq} \widetilde{\mathbb{R}}$, there must exist a soft sequence $\{\widetilde{I}_n\}$ of soft open intervals such that $T_A(\eta) \widetilde{\subseteq} \widetilde{\cup}_n\ \widetilde{I}_n(\eta)$ for all $\eta \in E$. One can take $\widetilde{I}_n = \widetilde{\mathbb{R}}$ for each n.

Remark. (1)Note that $T_A \widetilde{\subseteq} \widetilde{\cup}_n\ \widetilde{I}_n \Leftrightarrow T_A(\eta) \widetilde{\subseteq} \widetilde{\cup}_n\ \widetilde{I}_n(\eta), \forall \eta \in E$.
(2) Since the length of an interval is always non-negative, so $\widetilde{m}^*(T_A) \widetilde{\geq} \overline{0}$ for every $T_A \widetilde{\subseteq} \widetilde{\mathbb{R}}$.

Proposition 2.13. [18]
(i) The soft Lebesgue outer measure of null soft set is $\overline{0}$. i.e. $\widetilde{m}^*(T_\phi) = \overline{0}$.
(ii) The soft Lebesgue outer measure of a soft singleton set $\{P_\eta^x\}$, where $P_\eta^x \widetilde{\in} \widetilde{\mathbb{R}}$ is $\overline{0}$.
(iii) Soft Lebesgue outer measure of a soft countable set is $\overline{0}$.
(iv) The soft Lebesgue outer measure is soft monotone i.e. if $T_A \widetilde{\subseteq} T_B$, then $\widetilde{m}^*(T_A) \widetilde{\leq} \widetilde{m}^*(T_B)$.
(v) If $\{T_{A_n}\}$ is any sequence of soft sets of soft real numbers, then $\widetilde{m}^*(\widetilde{\cup}_n\ T_{A_n}) \widetilde{\leq} \Sigma_n\ \widetilde{m}^*(T_{A_n})$ i.e. the soft Lebesgue outer measure $\widetilde{m}^*$ is countably soft sub-additive.

Definition 2.14. [18] A soft set $T_E \widetilde{\in} \widetilde{\mathbb{R}}$ is called *Lebesgue measurable soft set* or simply *measurable soft set* if for each $T_A \widetilde{\subseteq} \widetilde{\mathbb{R}}$ we have

$$\widetilde{m}^*(T_A) = \widetilde{m}^*(T_A \widetilde{\cap} T_E) + \widetilde{m}^*(T_A \widetilde{\cap} T_E^c)$$

3. Measurable Soft Mappings

Definition 3.1. Let $(X, \widetilde{\mathcal{A}}, \widetilde{\mu})$ be a soft measure space and $T_A \widetilde{\in} \widetilde{\mathcal{A}}$. An extended real-valued soft mapping ψ_{fu} defined on $\widetilde{\mathcal{A}}$ is said to be a *measurable soft mapping* if for each $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$, $\{P_\eta^x \widetilde{\in} \widetilde{\mathcal{A}} : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\in} \widetilde{\mathcal{A}}$. In particular, if $\widetilde{\mathcal{A}}$ is the class $\widetilde{\mathbf{m}}$ of Lebesgue measurable soft subsets of $\widetilde{\mathbb{R}}$, then the measurable soft mapping ψ_{fu} is called a *Lebesgue measurable soft mapping*.

Stated differently, $\psi_{fu} : \widetilde{\mathcal{A}} \to \widetilde{\mathbb{R}}_\infty$ is *Lebesgue measurable soft mapping* if and only if $\{P_\eta^x \widetilde{\in} \widetilde{\mathcal{A}} : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$ is measurable soft set for each $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$.

Theorem 3.2. *Let ψ_{fu} be an extended real-valued soft mapping defined on a measurable soft set T_A. Then the following statements are equivalent:*
1) ψ_{fu} is soft measurable.
2) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$ is measurable for all $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$.
3) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{<} \overline{\alpha}\}$ is measurable for all $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$.
4) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\leq} \overline{\alpha}\}$ is measurable for all $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$.

Proof: $(1) \Rightarrow (2)$:

We prove first that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \geq \overline{\alpha}\} = \widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$. For this, let $P_\eta^{x_1} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$. Then, $\psi_{fu}(P_\eta^{x_1}) \widetilde{\geq} \overline{\alpha}$, $\forall \overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$. In particular, $\psi_{fu}(P_\eta^{x_1}) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}$ for every $\overline{n} \widetilde{\in} \widetilde{\mathbb{N}}$. This implies in turn that $P_\eta^{x_1} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$ for each $\overline{n} \widetilde{\in} \widetilde{\mathbb{N}}$, and hence $P_\eta^{x_1} \widetilde{\in} \widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$. Thus, $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\} \widetilde{\subseteq} \widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$.

Conversely, suppose that $P_\eta^{x_2} \widetilde{\in} \widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$ so that $P_\eta^{x_2} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$, $\forall \overline{n} \widetilde{\in} \widetilde{\mathbb{N}}$ and hence $\psi_{fu}(P_\eta^{x_2}) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}$, $\forall \overline{n} \widetilde{\in} \widetilde{\mathbb{N}}$. Thus, $\psi_{fu}(P_\eta^{x_2}) \widetilde{\geq} \overline{\alpha}$, $\forall \overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$. Therefore, $P_\eta^{x_2} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$. So, $\widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\} \widetilde{\subseteq} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$.

Hence,

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\} = \widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \frac{1}{\overline{n}}\}$$

Since soft intersection of countable number of measurable soft sets is soft measurable, so $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$ is soft measurable.

$(2) \Rightarrow (3)$:

We prove that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{<} \overline{\alpha}\} = T_A \widetilde{\setminus} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$. For this, let $P_\eta^{x_1} \widetilde{\in} \{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{<} \overline{\alpha}\}$. It means that $\psi_{fu}(P_\eta^{x_1}) \widetilde{<} \overline{\alpha}$. Thus, $P_\eta^{x_1} \widetilde{\in} T_A$ but $P_\eta^{x_1} \widetilde{\notin} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$ i.e. $P_\eta^{x_1} \widetilde{\in} T_A \widetilde{\setminus} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$. Therefore, $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{<} \overline{\alpha}\} \widetilde{\subseteq} T_A \widetilde{\setminus} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$.

The converse follows by reverse steps.

Since both T_A and $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$ are soft measurable and soft difference of two measurable soft sets is again soft measurable, so it follows that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{<} \overline{\alpha}\}$ is soft measurable.

$(3) \Rightarrow (4)$:

Since $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\leq} \overline{\alpha}\} = \widetilde{\bigcap}_{n=1}^{\infty} \{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{<} \overline{\alpha} + \frac{1}{\overline{n}}\}$, so it follows that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\leq} \overline{\alpha}\}$ is soft measurable.

$(4) \Rightarrow (1)$:

We know that $\{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = T_A \widetilde{\setminus} \{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{\leq} \overline{\alpha}\}$. Since both of T_A and

$\{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{\leq} \overline{\alpha}\}$ are soft measurable and soft difference of two measurable soft sets is also soft measurable, so $\{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$ should be a measurable soft set.

Corollary 3.3. *The soft set* $\{P_\eta^x : \psi_{fu}(P_\eta^x) = \overline{\alpha}\}$ *is soft measurable for each extended soft real number* $\overline{\alpha}$*.*

Proof: Let $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$. Then $\{P_\eta^x : \psi_{fu}(P_\eta^x) = \overline{\alpha}\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\leq} \overline{\alpha}\} \widetilde{\cap} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{\geq} \overline{\alpha}\}$, being the soft intersection of two measurable soft sets, is soft measurable.

- If $\overline{\alpha} = \overline{\infty}$, then

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) = \overline{\infty}\} = \widetilde{\cap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{n}\}$$

- If $\overline{\alpha} = -\overline{\infty}$, then

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) = -\overline{\infty}\} = \widetilde{\cap}_{n=1}^{\infty} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{<} -\overline{n}\}$$

Hence, $\{P_\eta^x : \psi_{fu}(P_\eta^x) = \overline{\alpha}\}$ is a measurable soft set for each extended soft real number $\overline{\alpha}$.

Example 3.4. Let T_P be a non-measurable soft subset of $\widetilde{\mathbb{R}}$. Suppose that $T_A = \{P_\eta^x \widetilde{\in} T_D : P_\eta^x \widetilde{>} \overline{0}\}$ and $T_B = \{P_\eta^x \widetilde{\in} T_D : P_\eta^x \widetilde{\leq} \overline{0}\}$. Assume that $\psi_{gu} : T_A \to T_P$ and $\phi_{hv} : T_B \to T_P^c$ are any bijective soft mappings. Define a soft mapping $\chi_{fw} : T_D \to \widetilde{\mathbb{R}}$ as

$$\chi_{fw}(P_\eta^x) = \begin{cases} \psi_{gu}(P_\eta^x), & \text{if} \quad P_\eta^x \widetilde{\in} T_A \\ \phi_{hv}(P_\eta^x), & \text{if} \quad P_\eta^x \widetilde{\in} T_B \end{cases}$$

Clearly χ_{fw} is soft bijective and assumes at most one value at each soft point. Thus for any soft real number $\overline{\alpha}$, the soft set $\{P_\eta^x \widetilde{\in} T_D : \chi_{fw}(P_\eta^x) = \overline{\alpha}\}$ contains exactly one soft point and hence it is a measurable soft set. However, the soft set $\{P_\eta^x \widetilde{\in} T_D : \chi_{fw}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$ being the same as T_P is non-measurable soft set. Hence χ_{fw} is a non-measurable soft mapping.

Definition 3.5. Let ψ_{fu} be an extended real-valued soft mapping defined on some soft set T_A. The *positive* and *negative parts* of ψ_{fu} are defined, respectively, as

$$\psi_{fu}^{+}(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \psi_{fu} \widetilde{\vee} \overline{0}$$

$$\psi_{fu}^{-}(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = -\psi_{fu} \widetilde{\vee} \overline{0}$$

for all $P_\eta^x \widetilde{\in} T_A$.

Obviously ψ_{fu}^{+} and ψ_{fu}^{-} are extended real-valued soft mappings and $\psi_{fu}^{+}, \psi_{fu}^{-} \widetilde{\geq} \overline{0}$.

Lemma 3.6. *Let* ψ_{fu}^{+} *and* ψ_{fu}^{-} *be the positive and negative parts of an extended real-valued soft mapping* ψ_{fu}*, respectively. Then,* $\psi_{fu} = \psi_{fu}^{+} - \psi_{fu}^{-}$*.*

Proof: Let $P_\eta^x \widetilde{\in} T_A$. Then there are three possibilities for the values of ψ_{fu} at P_η^x:

- If $\psi_{fu}(P_\eta^x) = \overline{0}$, then $\psi_{fu}^{+}(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$ and $\psi_{fu}^{-}(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$, so that $(\psi_{fu}^{+} - \psi_{fu}^{-})(P_\eta^x) = \psi_{fu}^{+}(P_\eta^x) - \psi_{fu}^{-}(P_\eta^x) = \overline{0} - \overline{0} = \overline{0} = \psi_{fu}(P_\eta^x)$, $\forall P_\eta^x \widetilde{\in} T_A$.
- If $\psi_{fu}(P_\eta^x) \widetilde{>} \overline{0}$, then $\psi_{fu}^{+}(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \psi_{fu}(P_\eta^x)$ and $\psi_{fu}^{-}(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$, so that $(\psi_{fu}^{+} - \psi_{fu}^{-})(P_\eta^x) = \psi_{fu}^{+}(P_\eta^x) - \psi_{fu}^{-}(P_\eta^x) = \psi_{fu}(P_\eta^x) - \overline{0} = \psi_{fu}(P_\eta^x)$, $\forall\, P_\eta^x \widetilde{\in} T_A$.
- If $\psi_{fu}(P_\eta^x) \widetilde{<} \overline{0}$, then $\widetilde{f}^{+}(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$ and $\psi_{fu}^{-}(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = -\psi_{fu}(P_\eta^x)$, so that $(\psi_{fu}^{+} - \psi_{fu}^{-})(P_\eta^x) = \psi_{fu}^{+}(P_\eta^x) - \psi_{fu}^{-}(P_\eta^x) = \overline{0} - (-\psi_{fu}(P_\eta^x)) = \psi_{fu}(P_\eta^x)$, $\forall P_\eta^x \widetilde{\in} T_A$.

Lemma 3.7. *Let* ψ_{fu}^{+} *and* ψ_{fu}^{-} *be the positive and negative parts of an extended real-valued soft mapping* ψ_{fu}*, respectively. Then,* $|\psi_{fu}| = \psi_{fu}^{+} + \psi_{fu}^{-}$*.*

Proof: Let $P_\eta^x \widetilde{\in} T_A$. Then there are three possibilities for the values of ψ_{fu} at P_η^x:

• If $\psi_{fu}(P_\eta^x) = \overline{0}$, then $\psi_{fu}^+(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$ and $\psi_{fu}^-(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$, so that $(\psi_{fu}^+ + \psi_{fu}^-)(P_\eta^x) = \psi_{fu}^+(P_\eta^x) + \psi_{fu}^-(P_\eta^x) = \overline{0} + \overline{0} = \overline{0} = |\overline{0}| = |\psi_{fu}(P_\eta^x)| = |\psi_{fu}|(P_\eta^x), \forall P_\eta^x \widetilde{\in} T_A$.

• If $\psi_{fu}(P_\eta^x) > 0$, then $\psi_{fu}^+(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \psi_{fu}(P_\eta^x)$ and $\psi_{fu}^-(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$, so that $(\psi_{fu}^+ + \psi_{fu}^-)(P_\eta^x) = \psi_{fu}^+(P_\eta^x) + \psi_{fu}^-(P_\eta^x) = \psi_{fu}(P_\eta^x) + \overline{0} = \psi_{fu}(P_\eta^x) = |\psi_{fu}(P_\eta^x)| = |\psi_{fu}|(P_\eta^x), \forall\, P_\eta^x \widetilde{\in} T_A$.

• If $\psi_{fu}(P_\eta^x) \widetilde{<} \overline{0}$, then $\psi_{fu}^+(P_\eta^x) = \max\{\psi_{fu}(P_\eta^x), \overline{0}\} = \overline{0}$ and $\psi_{fu}^-(P_\eta^x) = \max\{-\psi_{fu}(P_\eta^x), \overline{0}\} = -\psi_{fu}(P_\eta^x)$, so that $(\psi_{fu}^+ + \psi_{fu}^-)(P_\eta^x) = \psi_{fu}^+(P_\eta^x) + \psi_{fu}^-(P_\eta^x) = 0 + (-\psi_{fu}(P_\eta^x)) = -\psi_{fu}(P_\eta^x) = |\psi_{fu}(P_\eta^x)| = |\psi_{fu}|(P_\eta^x)$, $\forall\, P_\eta^x \widetilde{\in} T_A$.

Theorem 3.8. *Let ψ_{fu} and ϕ_{hv} be two measurable soft mappings defined on the same soft measurable domain T_D and $\overline{c}$ be some soft real number. Then (1) $\psi_{fu} + \overline{c}$ (2) $\overline{c}\psi_{fu}$ (3) $\psi_{fu} + \phi_{hv}$ (4) $\psi_{fu} - \phi_{hv}$ (5) ψ_{fu}^2 (6) $\psi_{fu}\phi_{hv}$ (7) $\frac{\psi_{fu}}{\phi_{hv}}, \phi_{hv} \neq \overline{0}$ (8) $\psi_{fu} \widetilde{\vee} \phi_{hv}$ (9) $\psi_{fu} \widetilde{\wedge} \phi_{hv}$ (10) $|\psi_{fu}|$ are measurable soft mappings.*

Proof: (1) Consider $\{P_\eta^x : (\psi_{fu} + \overline{c})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) + \overline{c} \widetilde{>} \overline{\alpha}\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \overline{c}\}$ for each $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$. Since ψ_{fu} is given to be soft measurable, so $\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \overline{c}\}$ is soft measurable. Thus, $\psi_{fu} + \overline{c}$ is a measurable soft mapping.

(2) There arise three cases depending upon whether $\overline{c} = \overline{0}$, $\overline{c} \widetilde{>} \overline{0}$ or $\overline{c} \widetilde{<} \overline{0}$.

<u>Case I</u>: When $\overline{c} = \overline{0}$

• If $\overline{\alpha} \widetilde{>} \overline{0}$, then $\{P_\eta^x \widetilde{\in} T_D : (\overline{c}\psi_{fu})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_D : \overline{c}\psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = T_\phi$, which is a measurable soft set.

• If $\overline{\alpha} \widetilde{\leq} \overline{0}$, we have $\{P_\eta^x \widetilde{\in} T_D : (\overline{c}\psi_{fu})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_D : \overline{c}\psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = T_D$, which is a measurable soft set.

<u>Case II</u>: When $\overline{c} \widetilde{>} \overline{0}$

Here $\{P_\eta^x \widetilde{\in} T_D : (\overline{c}\psi_{fu})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_D : \overline{c}\psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_D : \psi_{fu}(P_\eta^x) \widetilde{>} \frac{\overline{\alpha}}{\overline{c}}\}$, which is a measurable soft set for ψ_{fu} is soft measurable.

<u>Case III</u>: When $\overline{c} \widetilde{<} \overline{0}$

Here $\{P_\eta^x \widetilde{\in} T_D : (\overline{c}\psi_{fu})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_D : \overline{c}\psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_D : \psi_{fu}(P_\eta^x) \widetilde{<} \frac{\overline{\alpha}}{\overline{c}}\}$, which is a measurable soft set by Theorem 3.2.

(3) If $\psi_{fu}(P_\eta^x) + \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}$ then $\psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha} - \phi_{hv}(P_\eta^x), \forall\, \overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$. Thus we can find a soft rational number $\overline{r}$ such that $\psi_{fu}(P_\eta^x) \widetilde{>} \overline{r} \widetilde{>} \overline{\alpha} - \phi_{hv}(P_\eta^x)$. We show that $\{P_\eta^x : (\psi_{fu} + \phi_{hv})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \widetilde{\bigcup}_{\overline{r} \widetilde{\in} \widetilde{\mathbb{Q}}} [\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\} \widetilde{\cap} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}]$. For this, let $P_\eta^{x_1} \widetilde{\in} \{P_\eta^x : (\psi_{fu} + \phi_{hv})(P_\eta^x) \widetilde{>} \overline{\alpha}\}$. Then, $(\psi_{fu} + \phi_{hv})(P_\eta^{x_1}) \widetilde{>} \overline{\alpha}$ i.e. $\psi_{fu}(P_\eta^{x_1}) + \phi_{hv}(P_\eta^{x_1}) \widetilde{>} \overline{\alpha}$ and hence $\psi_{fu}(P_\eta^{x_1}) \widetilde{>} \overline{\alpha} - \phi_{hv}(P_\eta^{x_1})$. Thus, there exists $\overline{r} \widetilde{\in} \widetilde{\mathbb{Q}}$ such that $\psi_{fu}(P_\eta^{x_1}) \widetilde{>} \overline{r} \widetilde{>} \overline{\alpha} - \phi_{hv}(P_\eta^{x_1})$. This implies that

$P_\eta^{x_1} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\}$ and $P_\eta^{x_1} \widetilde{\in} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}$

$\Rightarrow P_\eta^{x_1} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\} \widetilde{\cap} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}$

$\Rightarrow P_\eta^{x_1} \widetilde{\in} \widetilde{\bigcup}_{\overline{r} \widetilde{\in} \widetilde{\mathbb{Q}}} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\} \widetilde{\cap} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}$.

Conversely, suppose that $P_\eta^{x_2} \widetilde{\in} \widetilde{\bigcup}_{\overline{r} \widetilde{\in} \widetilde{\mathbb{Q}}} [\{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\} \widetilde{\cap} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}]$. It means that there exists $\overline{r} \widetilde{\in} \widetilde{\mathbb{Q}}$ such that $P_\eta^{x_2} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\} \widetilde{\cap} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}$. So $P_\eta^{x_2} \widetilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{r}\}$ and $P_\eta^{x_2} \widetilde{\in} \{P_\eta^x : \overline{\alpha} - \phi_{hv}(P_\eta^x) \widetilde{<} \overline{r}\}$. This yields $\psi_{fu}(P_\eta^{x_2}) \widetilde{>} \overline{r}$ and $\overline{\alpha} - \phi_{hv}(P_\eta^{x_2}) \widetilde{<} \overline{r}$. Therefore, we get $\psi_{fu}(P_\eta^{x_2}) \widetilde{>} \overline{r} \widetilde{>} \overline{\alpha} - \phi_{hv}(P_\eta^{x_2})$. In particular, $\psi_{fu}(P_\eta^{x_2}) \widetilde{>} \overline{\alpha} - \phi_{hv}(P_\eta^{x_2})$ i.e. $\psi_{fu}(P_\eta^{x_2}) + \phi_{hv}(P_\eta^{x_2}) \widetilde{>} \overline{\alpha}$. Thus, $(\psi_{fu} + \phi_{hv})(P_\eta^{x_2}) \widetilde{>} \overline{\alpha}$ and hence $P_\eta^{x_2} \widetilde{\in} \{P_\eta^x : (\psi_{fu} + \phi_{hv})(P_\eta^x) \widetilde{>} \overline{\alpha}\}$.

Since countable soft union of measurable soft sets is soft measurable, so the desired result follows.

(4) The result follows quickly from (2) and (3).

(5) If $\overline{\alpha} \widetilde{<} \overline{0}$, then $\{x : \psi_{fu}^2(P_\eta^x) \widetilde{<} \overline{\alpha}\} = T_D$, which is a measurable soft set. If $\overline{\alpha} \widetilde{\geq} \overline{0}$, then we may write $\{x : \psi_{fu}^2(P_\eta^x) \widetilde{<} \overline{\alpha}\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{<} \sqrt{\overline{\alpha}}\} \widetilde{\cup} \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} -\sqrt{\overline{\alpha}}\}$. Both the soft sets on the RHS are soft measurable and hence their soft union should also be soft measurable.

(6) Since $\psi_{fu}\phi_{hv} = \frac{1}{4}\{(\psi_{fu}+\phi_{hv})^2 - (\psi_{fu}-\phi_{hv})^2\}$, so it follows from above results that $\psi_{fu}\phi_{hv}$ is soft measurable.

(7) For $\phi_{hv}(P_\eta^x) \neq \overline{0}$, we have

$$\{P_\eta^x \widetilde{\in} T_D : \frac{\overline{1}}{\phi_{hv}(P_\eta^x)} \widetilde{>} \overline{\alpha}\} = \begin{cases} \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{0}\} & \text{if } \overline{\alpha} = \overline{0} \\ \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{0}\} \widetilde{\cap} \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{>} \frac{\overline{1}}{\overline{\alpha}} & \text{if } \overline{\alpha} \widetilde{>} \overline{0} \\ \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{0}\} \widetilde{\cup} [\{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{<} \overline{0}\} \widetilde{\cap} \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{<} \frac{\overline{1}}{\overline{\alpha}}\}] & \text{if } \overline{\alpha} \widetilde{<} \overline{0} \end{cases}$$

This implies that $\frac{1}{\phi_{hv}}$ is a measurable mapping. Since $\frac{\psi_{fu}}{\phi_{hv}} = \psi_{fu} . \frac{1}{\phi_{hv}}, \phi_{hv} \neq \overline{0}$, so by (6), $\frac{\psi_{fu}}{\phi_{hv}}$ is a measurable soft mapping.

(8) For any $\overline{\alpha} \widetilde{\in} \mathbb{R}$, we have

$$\{P_\eta^x : (\psi_{fu} \widetilde{\vee} \phi_{hv})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\cup} \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$$

Since soft union of two measurable soft mappings is a measurable soft mapping, so the result follows.

(9) By definition, $(\psi_{fu} \widetilde{\wedge} \phi_{hv})(P_\eta^x) = \min\{\psi_{fu}(P_\eta^x), \phi_{hv}(P_\eta^x)\}$. Thus, For any $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$, we have

$$\{P_\eta^x : (\psi_{fu} \widetilde{\wedge} \phi_{hv})(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\cap} \{P_\eta^x : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$$

Since soft intersection of two measurable soft mappings is a measurable soft mapping, so the result follows.

(10) From (2) and (8), we conclude that $\psi_{fu}^+ = \psi_{fu} \widetilde{\vee} \overline{0}$ and $\psi_{fu}^- = (-\psi_{fu}) \widetilde{\vee} \overline{0}$ are measurable soft mappings. Moreover, since $|\psi_{fu}| = \psi_{fu}^+ + \psi_{fu}^-$, so by (3), the measurability of $|\psi_{fu}|$ follows.

Remark. If ψ_{fu}^+ and ψ_{fu}^- are soft measurable, then by (4) of above theorem, $\psi_{fu} = \psi_{fu}^+ - \psi_{fu}^-$ is also soft measurable. Thus, ψ_{fu} is soft measurable if and only if ψ_{fu}^+ and ψ_{fu}^- are soft measurable.

If, however, $|\psi_{fu}|$ is soft measurable then it is not necessary for ψ_{fu} to be soft measurable. For an illustration, assume that T_A is a non-measurable soft set. Define a soft mapping ψ_{fu} by $\psi_{fu} = \chi_{T_A} - \frac{\overline{1}}{2}$, where the characteristic soft mapping χ_{T_A} is defined as

$$\chi_{T_A}(P_\eta^x) = \begin{cases} \overline{1} & \text{if } P_\eta^x \widetilde{\in} T_A \\ \overline{0} & \text{if } P_\eta^x \widetilde{\notin} T_A \end{cases}$$

Here ψ_{fu} is not soft measurable but $|\psi_{fu}| = \frac{\overline{1}}{2}$ is soft measurable.

Theorem 3.9. *Let ψ_{fu} be an extended real-valued measurable soft mapping defined on T_D and T_A be a measurable soft subset of T_D. Then the soft restriction of ψ_{fu} to T_A is also soft measurable.*

Proof: Since $\{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\subseteq} \{P_\eta^x \widetilde{\in} T_D : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\}, \forall\, \overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$; so $\{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = T_A \widetilde{\cap} \{P_\eta^x \widetilde{\in} T_D : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$.

Since soft intersection of two measurable soft sets is also soft measurable, so the result follows.

Corollary 3.10. *Let T_A and T_B be measurable soft sets. Suppose that ψ_{fu} is a soft mapping with domain $T_A \widetilde{\cup} T_B$. Then ψ_{fu} is soft measurable if and only if its soft restrictions to T_A and T_B are soft measurable.*

Proof: Let ψ_{fu} be soft measurable on T_A and T_B. Then clearly

$$\{P_\eta^x \widetilde{\in} T_A \widetilde{\cup} T_B : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\cup} \{P_\eta^x \widetilde{\in} T_B : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$$

By assumptions, the soft sets on the RHS are soft measurable, so ψ_{fu} is soft measurable.

Conversely suppose that ψ_{fu} is measurable on $T_A \widetilde{\cup} T_B$. Then by Theorem 3.9, the soft restrictions of ψ_{fu} to T_A and T_B must be soft measurable.

Theorem 3.11. *Let ψ_{fu} be a mapping with soft measurable domain T_D. Then ψ_{fu} is soft measurable if and only if the soft mapping*

$$\phi_{hv}(P_\eta^x) = \begin{cases} \psi_{fu}(P_\eta^x) & \text{if} \quad P_\eta^x \widetilde{\in} T_D \\ \overline{0} & \text{if} \quad P_\eta^x \widetilde{\notin} T_D \end{cases}$$

is soft measurable.

Proof: If $P_\eta^x \widetilde{\in} T_D$, then $\phi_{hv}(P_\eta^x) = \psi_{fu}(P_\eta^x)$ and ψ_{fu} is given to be soft measurable on T_D, so ϕ_{hv} is soft measurable on T_D. In case $P_\eta^x \widetilde{\notin} T_D$, we have $\phi_{hv}(P_\eta^x) = \overline{0}$ which is soft measurable, being a constant mapping.

Conversely suppose that ϕ_{hv} is a measurable soft mapping on $T_D \widetilde{\cup} T_D^c$. Then the soft restriction $\phi_{hv}|_{T_D} = \psi_{fu}$ is a measurable soft mapping by Theorem 3.9.

Definition 3.12. A property is said to *hold almost everywhere* if the set of soft points where this property fails to hold has soft measure $\overline{0}$. Thus, two soft mappings ψ_{fu} and ϕ_{hv} with same soft domain T_D are *soft equal almost everywhere* if $\widetilde{m}(\{P_\eta^x \widetilde{\in} T_D : \psi_{fu}(P_\eta^x) \neq \phi_{hv}(P_\eta^x)\}) = \overline{0}$.

Example 3.13. Define $\psi_{fu} : \widetilde{\mathbb{R}} \to \{\overline{1}, \overline{2}\}$ by

$$\psi_{fu}(P_\eta^x) = \begin{cases} \overline{1} & \text{if} \quad P_\eta^x \widetilde{\notin} \widetilde{\mathbb{Q}} \\ \overline{2} & \text{if} \quad P_\eta^x \widetilde{\in} \widetilde{\mathbb{Q}} \end{cases}$$

then $\psi_{fu} = \overline{1}$ almost everywhere because $\widetilde{m}(\widetilde{\mathbb{Q}}) = \overline{0}$.

Example 3.14. Define $\psi_{fu} : \widetilde{\mathbb{R}} \to \{\overline{1}, \overline{2}\}$ by

$$\psi_{fu}(P_\eta^x) = \begin{cases} \overline{1} & \text{if} \quad P_\eta^x \widetilde{\notin} \widetilde{\mathbb{Q}} \\ \overline{2} & \text{if} \quad P_\eta^x \widetilde{\in} \widetilde{\mathbb{Q}} \end{cases}$$

and $\phi_{hv} : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}$ by

$$\phi_{hv}(P_\eta^x) = \begin{cases} \overline{1} & \text{if} \quad P_\eta^x \widetilde{\notin} \widetilde{\mathbb{Q}} \\ P_\eta^x & \text{if} \quad P_\eta^x \widetilde{\in} \widetilde{\mathbb{Q}} \end{cases}$$

then $\psi_{fu} = \phi_{hv}$ almost everywhere because $\widetilde{m}(\{P_\eta^x \widetilde{\in} \widetilde{\mathbb{R}} : \psi_{fu}(P_\eta^x) \neq \phi_{hv}(P_\eta^x)\}) = \widetilde{m}(\widetilde{\mathbb{Q}}) = \overline{0}$.

Example 3.15. Define $\psi_{fu} : \widetilde{\mathbb{R}} \to \widetilde{\mathbb{R}}_{\overline{\infty}}$, where $\widetilde{\mathbb{R}}_{\overline{\infty}}$ denotes the soft set of extended soft real numbers, by

$$\psi_{fu}(P_\eta^x) = \begin{cases} \overline{3} & \text{if} \quad P_\eta^x \widetilde{\notin} \widetilde{\mathbb{Q}} \\ \overline{\infty} & \text{if} \quad P_\eta^x \widetilde{\in} \widetilde{\mathbb{Q}} \end{cases}$$

then ψ_{fu} is soft finite almost everywhere because $\widetilde{m}(\{P_\eta^x : \psi_{fu}(x) = \overline{\infty}\}) = \widetilde{m}(\widetilde{\mathbb{Q}}) = \overline{0}$.

Definition 3.16. A sequence $\{(\psi_{fu})_n\}$ of soft mappings defined on T_A is said to be *soft convergent almost everywhere* to a soft mapping ψ_{fu} if the soft set of soft points where $\{(\psi_{fu})_n\}$ fails to be soft convergent to ψ_{fu} has soft measure $\overline{0}$.

Theorem 3.17. *Let ψ_{fu} be a measurable soft mapping with $\psi_{fu} = \phi_{hv}$ almost everywhere on T_A. Then ϕ_{hv} is also soft measurable.*

Proof: Let $T_B = \{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \neq \phi_{hv}(P_\eta^x)\}$. Then, by hypothesis, T_B is soft measurable with $\widetilde{m}(T_B) = \overline{0}$. Moreover, $T_A \widetilde{\setminus} T_B$, being the soft difference of two measurable soft sets, is also soft measurable.

Since $T_A = (T_A \widetilde{\setminus} T_B) \widetilde{\cup} T_B$, so for any $\overline{\alpha} \widetilde{\in} \widetilde{\mathbb{R}}$ we have

$$\{P_\eta^x \widetilde{\in} T_A : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_A \widetilde{\setminus} T_B : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\cup} \{P_\eta^x \widetilde{\in} T_B : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$$

Since $\psi_{fu} = \phi_{hv}$ on $T_A \widetilde{\setminus} T_B$, so we obtain

$$\{P_\eta^x \widetilde{\in} T_A : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\} = \{P_\eta^x \widetilde{\in} T_A \widetilde{\setminus} T_B : \psi_{fu}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\cup} \{P_\eta^x \widetilde{\in} T_B : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$$

- Since ψ_{fu} is soft measurable on $T_A \widetilde{\setminus} T_B$, so the first soft set on RHS is soft measurable.
- Since $\{P_\eta^x \widetilde{\in} T_B : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\} \widetilde{\subseteq} T_B$ and $\widetilde{m}(T_B) = \overline{0}$, so the second soft set on RHS is also soft measurable.

Hence the set $\{P_\eta^x \widetilde{\in} T_A : \phi_{hv}(P_\eta^x) \widetilde{>} \overline{\alpha}\}$ is soft measurable, as required.

Definition 3.18. Let $\widetilde{L}$ and $\widetilde{M}$, respectively, be the soft sets of soft rational and soft irrational numbers in $[\overline{0}, \overline{1}]$. Then,

$$\widetilde{m}(\widetilde{M}) = \widetilde{m}(\widetilde{L}) + \widetilde{m}(\widetilde{M}) = \widetilde{m}(\widetilde{L} \widetilde{\cup} \widetilde{M}) = \widetilde{m}([\overline{0}, \overline{1}]) = \overline{1}$$

Define $\psi_{fu} : [\overline{0}, \overline{1}] \to \{\overline{0}, \overline{1}\}$ by

$$\psi_{fu}(P_\eta^x) = \begin{cases} \overline{1} & \text{if } P_\eta^x \widetilde{\in} \widetilde{L} \\ \overline{0} & \text{if } P_\eta^x \widetilde{\in} \widetilde{M} \end{cases}$$

then $\psi_{fu} = \overline{0}$ almost everywhere. Furthermore, the constant soft mapping having value $\overline{0}$ is a continuous soft mapping but ψ_{fu} is not soft continuous. Thus, we conclude that if ψ_{fu} is soft continuous and $\psi_{fu} = \phi_{hv}$ almost everywhere, then ϕ_{hv} is not necessarily soft continuous. The soft mapping ψ_{fu} is called *Dirichlet's soft mapping*.

The mapping $\psi_{fu} : \widetilde{\mathbb{R}} \to \{\overline{0}, \overline{1}\}$ defined as

$$\psi_{fu}(P_\eta^x) = \begin{cases} \overline{1} & \text{if } P_\eta^x \widetilde{\notin} \widetilde{\mathbb{Q}} \\ \overline{0} & \text{if } P_\eta^x \widetilde{\in} \widetilde{\mathbb{Q}} \end{cases}$$

then $\psi_{fu} = \overline{1}$ almost everywhere on $\widetilde{\mathbb{R}}$. Since a constant soft mapping is soft measurable, so by Theorem 3.17, ψ_{fu} must be soft measurable.

Define $\phi_{hv} : \widetilde{\mathbb{R}} \to \{\overline{0}, \overline{1}\}$ as

$$\phi_{hv}(P_\eta^x) = \begin{cases} \overline{0} & \text{if } P_\eta^x \widetilde{\notin} \widetilde{\mathbb{Q}} \\ \overline{1} & \text{if } P_\eta^x \widetilde{\in} \widetilde{\mathbb{Q}} \end{cases}$$

then $\phi_{hv} = \overline{0}$ almost everywhere on $\widetilde{\mathbb{R}}$. Since a constant soft mapping is soft measurable, so by Theorem 3.17, ϕ_{hv} should be soft measurable.

Theorem 3.19. *Let $\{(\psi_{fu})_n\}$ be a sequence of extended real-valued measurable soft mappings with same soft domain T_A. Then*
1) $\max_{1 \leq i \leq n} (\psi_{fu})_i$ *is soft measurable for each n.*
2) $\min_{1 \leq i \leq n} (\psi_{fu})_i$ *is soft measurable for each n.*
3) $\inf_{n \in \mathbb{N}} (\psi_{fu})_n$ *is soft measurable.*
4) $\sup_{n \in \mathbb{N}} (\psi_{fu})_n$ *is soft measurable.*
5) $\overline{\lim} (\psi_{fu})_n = \limsup (\psi_{fu})_n$ *is soft measurable.*
6) $\underline{\lim} (\psi_{fu})_n = \liminf (\psi_{fu})_n$ *is soft measurable.*
7) If $\lim_{n \to \infty} (\psi_{fu})_n(P_\eta^x) = \psi_{fu}(P_\eta^x)$ *exists, then ψ_{fu} is soft measurable.*

Proof:

1) Let $\psi_{fu} = \max_{1 \le i \le n} (\psi_{fu})_i$. We show that $\{P_\eta^x : \psi_{fu}(x) \tilde{>} \overline{\alpha}\} = \tilde{\cup}_{i=1}^n \{P_\eta^x : (\psi_{fu})_i(P_\eta^x) \tilde{>} \overline{\alpha}\}$, for any $\overline{\alpha}$. For this, let $P_\eta^{x_1} \tilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\}$, then $\psi_{fu}(P_\eta^{x_1}) \tilde{>} \overline{\alpha}$. But, by our assumption, $\psi_{fu}(P_\eta^x) = \max_{1 \le i \le n} (\psi_{fu})_i(P_\eta^x)$. So there is $j \in \{1, 2, 3, ..., n\}$ such that $\psi_{fu}(P_\eta^x) = (\psi_{fu})_j(P_\eta^x)$ i.e. $\psi_{fu}(P_\eta^{x_1}) = (\psi_{fu})_j(P_\eta^{x_1}) \tilde{>} \overline{\alpha}$ which implies that $P_\eta^{x_1} \tilde{\in} \{P_\eta^x : (\psi_{fu})_j(P_\eta^x) \tilde{>} \overline{\alpha}\}$ for some $j \in \{1, 2, 3, ..., n\}$. Hence, $P_\eta^{x_1} \tilde{\in} \tilde{\cup}_{i=1}^n \{P_\eta^x : (\psi_{fu})_i(P_\eta^x) \tilde{>} \overline{\alpha}\}$.

Conversely suppose that $P_\eta^{x_2} \tilde{\in} \tilde{\cup}_{i=1}^n \{P_\eta^x : (\psi_{fu})_i(P_\eta^x) \tilde{>} \overline{\alpha}\}$, then $P_\eta^{x_2} \tilde{\in} \{P_\eta^x : (\psi_{fu})_k(P_\eta^x) \tilde{>} \overline{\alpha}\}$ for some $k \in \{1, 2, 3, ..., n\}$ i.e. $(\psi_{fu})_k(P_\eta^{x_2}) \tilde{>} \overline{\alpha}$. Also, by definition, $(\psi_{fu})_i(P_\eta^x) \tilde{\le} \max_{1 \le i \le n} (\psi_{fu})_i(P_\eta^x) = \psi_{fu}(P_\eta^x), \forall i = 1, 2, 3, ..., n$ and so $(\psi_{fu})_k(P_\eta^{x_2}) \tilde{\le} \psi_{fu}(P_\eta^{x_2})$. This means that $\psi_{fu}(P_\eta^{x_2}) \tilde{>} \overline{\alpha}$. Thus, $P_\eta^{x_2} \tilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\}$.

Since finite soft union of measurable soft sets is soft measurable, so this concludes the proof.

2) Let $\psi_{fu} = \min_{1 \le i \le n} (\psi_{fu})_i$. It can be easily shown that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\} = \tilde{\cap}_{i=1}^n \{P_\eta^x : (\psi_{fu})_i(P_\eta^x) \tilde{>} \overline{\alpha}\}, \forall \overline{\alpha}$.

3) Let $\psi_{fu} = \inf_{n \in \mathbb{N}} (\psi_{fu})_n$. If $P_\eta^{x_1} \tilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\}$, then $\psi_{fu}(P_\eta^{x_1}) \tilde{>} \overline{\alpha}$. Clearly, $\psi_{fu}(P_\eta^{x_1}) \tilde{\le} (\psi_{fu})_n(P_\eta^{x_1})$ for each $n \in \mathbb{N}$. Thus,

$$\overline{\alpha} \tilde{<} \psi_{fu}(P_\eta^{x_1}) \tilde{\le} (\psi_{fu})_n(P_\eta^{x_1}),\ n \in \mathbb{N}$$

Therefore, $P_\eta^{x_1} \in \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\}$ for each $n \in \mathbb{N}$ and hence $P_\eta^{x_1} \tilde{\in} \tilde{\cap}_{n=1}^\infty \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\}$. Thus, $\{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\} \tilde{\subseteq} \tilde{\cap}_{n=1}^\infty \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\}$.

Conversely, suppose that $P_\eta^{x_2} \tilde{\in} \tilde{\cap}_{n=1}^\infty \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\}$, so that $(\psi_{fu})_n(P_\eta^{x_2}) \tilde{>} \overline{\alpha}, \forall\, n \in \mathbb{N}$. This shows that $\overline{\alpha}$ is a soft lower bound of $\{(\psi_{fu})_1(P_\eta^{x_2}), (\psi_{fu})_2(P_\eta^{x_2}), (\psi_{fu})_3(P_\eta^{x_2}), ...\}$. But

$$\psi_{fu}(P_\eta^{x_2}) = \inf\{(\psi_{fu})_1(P_\eta^{x_2}), (\psi_{fu})_2(P_\eta^{x_2}), (\psi_{fu})_3(P_\eta^{x_2}), ...\}$$

A soft lower bound is always less or equal to the soft greatest lower bound. So $\overline{\alpha} \tilde{<} \psi_{fu}(P_\eta^{x_2})$. Therefore, $P_\eta^{x_2} \tilde{\in} \{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\}$ and hence $\tilde{\cap}_{n=1}^\infty \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\} \tilde{\subseteq} \{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\}$.

Thus, $\{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\} = \tilde{\cap}_{n=1}^\infty \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\}$.

Since soft intersection of a countable number of measurable soft sets is soft measurable, so $\psi_{fu} = \min_{1 \le i \le n} (\psi_{fu})_i$ must be soft measurable.

4) Let $\psi_{fu} = \sup_{n \in \mathbb{N}} (\psi_{fu})_n$. Since the set $\{P_\eta^x : \psi_{fu}(P_\eta^x) \tilde{>} \overline{\alpha}\} = \tilde{\cup}_{n=1}^\infty \{P_\eta^x : (\psi_{fu})_n(P_\eta^x) \tilde{>} \overline{\alpha}\}$ is soft measurable, so $\psi_{fu} = \sup_{n \in \mathbb{N}} (\psi_{fu})_n$ is a measurable soft mapping, as desired.

5) We know that $\overline{\lim}\, (\psi_{fu})_n = \inf_n(\sup_{i \ge n}(\psi_{fu})_i) = \overline{\lambda}$, say. Suppose that

$$(\psi_{FU})_n = \sup\{(\psi_{fu})_n, (\psi_{fu})_{n+1}, (\psi_{fu})_{n+2}, ..., (\psi_{fu})_{n+i}, ...\},\ n = 1, 2, 3, ...$$

Since each $(\psi_{fu})_n$ is given to be soft measurable, so by part (4), each $(\psi_{FU})_n$ is soft measurable. Now $\{(\psi_{fu})_n\}$ is a sequence of measurable soft mappings and $\overline{\lambda} = \inf_n\, (\psi_{FU})_n$ so that, by (3), $\overline{\lambda} = \overline{\lim}\, (\psi_{fu})_n$ is a measurable soft mapping.

6) We know that $\underline{\lim}\, (\psi_{fu})_n = \sup_n(\inf_{i \ge n}(\psi_{fu})_i) = \overline{\lambda}$, say. Suppose that

$$(\psi_{FU})_n = \inf\{(\psi_{fu})_n, (\psi_{fu})_{n+1}, (\psi_{fu})_{n+2}, ..., (\psi_{fu})_{n+i}, ...\},\ n = 1, 2, 3, ...$$

Each $(\psi_{FU})_n$ is soft measurable due to the soft measurability of $(\psi_{fu})_n$ for each n and part (3). Consequently, $\overline{\lambda} = \underline{\lim}\, (\psi_{fu})_n$ is a measurable soft mapping, by part (4)

7) By hypothesis, $\psi_{fu} = \overline{\lim}\, (\psi_{fu})_n = \underline{\lim}\, (\psi_{fu})_n$, which are soft measurable by parts (5) and (6) respectively. So ψ_{fu} should be soft measurable.

Remark. The results in parts (3) to (7) of above theorem cannot be extended to the case of soft uncountable operations. For example, if I is any indexing set and each $(\psi_{fu})_i$ is soft measurable for $i \in I$, then $\sup_{i\in I}\ (\psi_{fu})_i$ need not be soft measurable as can be seen in forthcoming example.

Example 3.20. Let $\widetilde{E} \,\widetilde{\subseteq}\, [\overline{0}, \overline{1}]$ be a non-measurable soft set. Define a soft mapping

$$(\psi_{fu})_i(P_\eta^x) = \begin{cases} \overline{0} & \text{if} \quad P_\eta^x \neq i \\ \overline{1} & \text{if} \quad P_\eta^x = i \end{cases}$$

For each $i \in \widetilde{E}$, the mapping $(\psi_{fu})_i$ is soft measurable but $\sup_{i\in\widetilde{E}}(\psi_{fu})_i = \chi_{\widetilde{E}}$, the characteristic soft mapping, which is not soft measurable.

Theorem 3.21. *A continuous soft mapping defined on a measurable soft set is soft measurable.*

Proof:

Let ψ_{fu} be a continuous soft mapping defined on $T_D \,\widetilde{\subseteq}\, \widetilde{\mathbb{R}}$. For soft measurability of ψ_{fu}, it suffices to show that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{\alpha}\}$ is soft measurable for each $\overline{\alpha} \,\widetilde{\in}\, \widetilde{\mathbb{R}}$.

Notice that $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{\alpha}\} = \psi_{fu}^{-1}(\overline{\alpha}, \overline{\infty})$. Since ψ_{fu} is soft continuous, so $\psi_{fu}^{-1}(\overline{\alpha}, \overline{\infty})$ is a soft open subset of T_D. By definition of soft relative topology on T_D, there exists a soft open set $T_G \,\widetilde{\subseteq}\, \widetilde{\mathbb{R}}$ such that $\psi_{fu}^{-1}(\overline{\alpha}, \overline{\infty}) = T_D \,\widetilde{\cap}\, T_G$. The soft set T_G, being a soft open set, is soft measurable. Thus, $\psi_{fu}^{-1}(\overline{\alpha}, \overline{\infty})$ is a measurable soft set and hence ψ_{fu} is a measurable soft mapping.

Theorem 3.22. *Let ψ_{fu} be a measurable soft mapping and T_G a soft open set, then $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{\in}\, T_G\}$ is a measurable soft set.*

Proof:

We know that every non-empty soft set T_G in $\widetilde{\mathbb{R}}$ is the soft union of a countable soft collection of soft open intervals, so $T_G = \widetilde{\cup}_{k=1}^{\infty} \widetilde{I}_k$, where $\widetilde{I}_k = (\overline{a}_k, \overline{b}_k)$ are pairwise soft disjoint soft open intervals. Thus,

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{\in}\, T_G\} = \widetilde{\cup}_{k=1}^{\infty} \left[\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{a}_k\} \,\widetilde{\cap}\, \{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{<}\, \overline{b}_k\}\right]$$

and hence the result.

Theorem 3.23. *Let ψ_{fu} and ϕ_{hv} be measurable soft mappings defined on a same soft set T_D. Then the soft sets*
1) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \phi_{hv}(P_\eta^x)\}$,
2) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{<}\, \phi_{hv}(P_\eta^x)\}$,
3) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{\leq}\, \phi_{hv}(P_\eta^x)\}$, and
4) $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{\geq}\, \phi_{hv}(P_\eta^x)\}$
are soft measurable.

Proof:

1) We know that between any two soft real numbers, there is a soft rational number, so $\psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{r} \,\widetilde{>}\, \phi_{hv}(P_\eta^x)$. Thus, we have the desired result from

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \phi_{hv}(P_\eta^x)\} = \widetilde{\cup}_{\overline{r}\,\widetilde{\in}\,\widetilde{\mathbb{Q}}} \left[\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{r}\} \,\widetilde{\cap}\, \{P_\eta^x : \phi_{hv}(P_\eta^x) \,\widetilde{<}\, \overline{r}\}\right]$$

2) We know that between any two soft real numbers, there is a soft rational number, so $\psi_{fu}(P_\eta^x) \,\widetilde{<}\, \overline{r} \,\widetilde{<}\, \phi_{hv}(P_\eta^x)$. Thus, we have the desired result from

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{<}\, \phi_{hv}(P_\eta^x)\} = \widetilde{\bigcup}_{\overline{r}\,\widetilde{\in}\,\widetilde{\mathbb{Q}}} [\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{<}\, \overline{r}\} \,\widetilde{\cap}\, \{P_\eta^x : \phi_{hv}(P_\eta^x) \,\widetilde{>}\, \overline{r}\}]$$

3) Since $\{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{\leq}\, \phi_{hv}(P_\eta^x)\} = T_D \,\widetilde{\setminus}\, \{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \phi_{hv}(P_\eta^x)\}$, so the result follows from (1).
4) The result follows from

$$\{P_\eta^x : \psi_{fu}(P_\eta^x) = \phi_{hv}(P_\eta^x)\} = \{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{\leq}\, \phi_{hv}(P_\eta^x)\} \,\widetilde{\setminus}\, \{P_\eta^x : \psi_{fu}(P_\eta^x) \,\widetilde{<}\, \phi_{hv}(P_\eta^x)\}$$

Theorem 3.24. *Let ψ_{fu} be a real-valued soft mapping defined on a soft measurable domain T_D and T_G be a soft open set in $\widetilde{\mathbb{R}}$. Then ψ_{fu} is soft measurable if and only if $\psi_{fu}^{-1}(T_G)$ is soft measurable.*

Proof:

Let ψ_{fu} be soft measurable and T_G be a soft open set in $\widetilde{\mathbb{R}}$. Then $T_G = \widetilde{\bigcup}_{k=1}^{\infty} \widetilde{I}_k$, where $\widetilde{I}_k = (\overline{a}_k, \overline{b}_k)$ are pairwise soft disjoint soft open intervals. Since $\widetilde{I}_k = (\overline{a}_k, \overline{b}_k) = (-\overline{\infty}, \overline{b}_k) \,\widetilde{\cap}\, (\overline{a}_k, \overline{\infty})$, so

$\psi_{fu}^{-1}(\widetilde{I}_k) = \psi_{fu}^{-1}(\overline{a}_k, \overline{b}_k) = \psi_{fu}^{-1}(-\overline{\infty}, \overline{b}_k) \,\widetilde{\cap}\, \psi_{fu}^{-1}(\overline{a}_k, \overline{\infty})$
$= \{P_\eta^x \,\widetilde{\in}\, T_D : \psi_{fu}(P_\eta^x) \,\widetilde{<}\, \overline{b}_k\} \,\widetilde{\cap}\, \{P_\eta^x \,\widetilde{\in}\, T_D : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{a}_k\}$

Thus, $\psi_{fu}^{-1}(\widetilde{I}_k)$ is soft measurable for each k, so $\psi_{fu}^{-1}(T_G) = \psi_{fu}^{-1}(\widetilde{\bigcup}_{k=1}^{\infty} \widetilde{I}_k) = \widetilde{\bigcup}_{k=1}^{\infty} \psi_{fu}^{-1}(\widetilde{I}_k)$ is soft measurable.

Conversely, suppose that $\psi_{fu}^{-1}(T_G)$ is a measurable soft set for any soft open set T_G in $\widetilde{\mathbb{R}}$. In particular, for $T_G = (\overline{\alpha}, \overline{\infty}),\ \overline{\alpha} \,\widetilde{\in}\, \widetilde{\mathbb{R}}$, $\psi_{fu}^{-1}(T_G) = \psi_{fu}^{-1}(\overline{\alpha}, \overline{\infty}) = \{P_\eta^x \,\widetilde{\in}\, T_D : \psi_{fu}(P_\eta^x) \,\widetilde{>}\, \overline{\alpha}\}$ is a measurable soft set. Hence ψ_{fu} is soft measurable.

4. Soft Probability Measure

In this section, we first discuss some basic definitions related to elementary probability theory and then focus on our main topic viz. Soft Probability Measure. Soft probability space was inroduced by Khameneh and Kilicman (See [12]).

Definition 4.1. A process by which we obtain some information, is called an *experiment* i.e. any observation or action whose outcome is uncertain is known as an experiment. For an illustration, suppose that someone wishes to buy a car. Assume that the collection of choices is $X = \{c_1, c_2, ..., c_7\}$ and the set of specifications is $E = \{\eta_1, \eta_2, ..., \eta_9\}$. Then the selection of a car amongst the choices available in accordance with the specifications desired by the person is an experiment.

Definition 4.2. The aggregate of all possible outcomes of an experiment is called *soft sample space* and is designated by T_A.

Definition 4.3. Any particular soft subset of the soft set T_A is termed as an *event*.

Definition 4.4. Two events T_{A_1} and T_{A_2} of soft sample space T_A are called *soft mutually exclusive* or *soft disjoint* if $T_{A_1} \,\widetilde{\cap}\, T_{A_2} = T_\phi$.

Definition 4.5. Two events T_{A_1} and T_{A_2} of soft sample space T_A are called *soft exhaustive* if $T_{A_1} \,\widetilde{\cup}\, T_{A_2} = T_A$.

Definition 4.6. [12] Let $\widetilde{\mathcal{A}}$ be a soft σ-algebra on X. The mapping $P : \widetilde{\mathcal{A}} \to [0, 1]$ is called *soft probability measure* on $\widetilde{\mathcal{A}}$ if

i) $P(\breve{X}) = 1$
ii) $P(\cup_i\ (T_i, E)) = \Sigma_i\ P(T_i, E)$, where $(T_i, E)\ \widetilde{\cap}\ (T_j, E) = T_\phi,\ \forall i \neq j$.

A soft probability space over X is denoted by the triple $(X, \widetilde{\mathcal{A}}, P)$ where $\widetilde{\mathcal{A}}$ is a soft σ-algebra over X and P is the soft probability measure over $\widetilde{\mathcal{A}}$. The pair $((F, E), P(F, E))$ is used to represent a description of objects of X as well as the probability of such description.

Example 4.7. Consider Example 2.9. We may re-represent it in the following way:
$(T_{A_1}, P(T_{A_1})) = (T_\phi, 0)$,
$(T_{A_2}, P(T_{A_2})) = (\{(\eta_1, \{g\}), (\eta_2, \{\})\}, 0.15)$,
$(T_{A_3}, P(T_{A_3})) = (\{(\eta_1, \{r\}), (\eta_2, \{g, s\})\}, 0.1)$,
$(T_{A_4}, P(T_{A_4})) = (\{(\eta_1, \{s\}), (\eta_2, \{r\})\}, 0.4)$,
$(T_{A_5}, P(T_{A_5})) = (\{(\eta_1, \{g, r\}), (\eta_2, \{g, s\})\}, 0.6)$,
$(T_{A_6}, P(T_{A_6})) = (\{(\eta_1, \{r, s\}), (\eta_2, \{g, r, s\})\}, 0.85)$,
$(T_{A_7}, P(T_{A_7})) = (\{(\eta_1, \{g, s\}), (\eta_2, \{r\})\}, 0.9)$, and
$(T_{A_8}, P(T_{A_8})) = (\breve{X}, 1)$.

This representation yields descriptions of elements of X as well as the probability of such descriptions (See [12]).

5. AN APPLICATION OF SOFT SET THEORY

Assume that a person wants to travel with his pregnant wife from some destination S_1 to another destination S_2 via road. Suppose that the collection of buses to travel is $X = \{v_1, ..., v_7\}$. The facilities that these vehicles provide may be expressed as $E = \{\eta_1, ..., \eta_9\}$, the set of decision variables; where

v_1 = Daewoo Express
v_2 = Faisal Movers
v_3 = Bilal Travels
v_4 = Rajput Travels
v_5 = Skyways
v_6 = Sandhu Transport Company
v_7 = Niazi Express

and

η_1 = Comfortable Seats
η_2 = Cooperative Staff
η_3 = Wifi equipped
η_4 = DVD Player with headphone for each passenger
η_5 = Comfortable route
η_6 = Refreshment
η_7 = Non-stop
η_8 = Security guard
η_9 = Bus hostess

Keeping in view the condition of his wife, he has to choose the vehicle that possesses the qualities amongst the members of the set $A = \{\eta_1, \eta_2, \eta_5, \eta_6, \eta_9\}$ with corresponding weights $w_1 = 0.9$, $w_2 = 0.6$, $w_5 = 0.4$, $w_6 = 0.3$, and $w_9 = 0.7$. Suppose that

$$T_A = \{(\eta_1, \{v_1, v_2, v_4\}), (\eta_2, \{v_1, v_3\}), (\eta_5, \{v_1\}), (\eta_6, \{v_1, v_2, v_7\}), (\eta_9, \{v_1, v_2, v_3, v_4, v_7\})\}$$

is a soft set. We represent this soft set in the form of a membership table along with the corresponding weights and the choice values as below:

X	$\eta_1, w_1 = 0.9$	$\eta_2, w_2 = 0.6$	$\eta_5, w_5 = 0.4$	$\eta_6, w_6 = 0.3$	$\eta_9, w_9 = 0.7$	Weighted Choice Value
v_1	1	1	1	1	1	2.9
v_2	1	0	0	1	1	1.9
v_3	0	1	0	0	1	1.3
v_4	1	0	0	0	1	1.6
v_5	0	0	0	0	0	0
v_6	0	0	0	0	0	0
v_7	0	0	0	1	1	1.0

where the weighted choice values are computed using the formula $\Sigma_j\ (w_j \times v_{ij})$ (See [14]).

It is vivid from above table that the person should prefer v_1 i.e. Daewoo Express. His 2^{nd} priority should be v_2 i.e. Faisal Movers. v_4 i.e. Rajput Travels stands on the 3^{rd} priority.

6. Conclusion

We introduced the notion of measurable soft mappings and the criteria for an extended real-valued soft mapping to be a Lebesgue measurable soft mapping. The positive and negative parts of an extended real-valued soft mapping were also introduced. We also discussed the measurability of soft mappings. A large number of results was also given to elaborate different notions. The definition of soft probability measure in connection with its application to soft σ-algebra is briefly discussed at the end. To make the ideas presented more digestible, the aid of appropriate examples where needed is taken. We hope that the results investigated in this paper make a significant and technically sound contribution to the field and will be beneficial for the researchers for further advancement and enhancement of the research work in the field of soft set theory, especially in soft measure theory.

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