Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 48(2)(2016) pp. 65-80

Fekete-Szegö Type Coefficient Inequalities for Certain Subclasses of Analytic Functions Involving Salagean Operator

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Received: 09 April, 2016 / Accepted: 25 July, 2016 / Published online: 19 August, 2016

Abstract. In this paper, we introduce and study certain subclasses of analytic functions involving Salagean operator. For these classes, the Fekete-Szegö type coefficient inequalities associated with the k-th root transformation $\left[f(z^k)\right]^{\frac{1}{k}}$ are derived. A similar problems are investigated for $\frac{z}{f(z)}$ when $f(z)$ belonging to these classes.

AMS (MOS) Subject Classification Codes: 30C45
Key Words: Univalent functions, starlike functions, convex functions, Sălăgean operator, subordination, Fekete-Szegö inequalities.

1. Introduction

Let $\wp$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{j\geq 2} a_j z^j \tag{1. 1}$$

which are analytic in the open unit disc $U = \{z : z \in C,\ |z| < 1\}$, Further, let S denote the class of functions which are univalent in U. Let Λ be the class of analytic functions p, normalized by $p(0) = 0$, satisfying the condition $|p(z)| < 1$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a Schwarz function $p \in \Lambda$ such that $f(z) = g(p(z))$. For a function $f(z)$ in $\wp$, we define

$$\mathcal{D}^0 f(z) = f(z),\ \mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z),$$
$$\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z) \quad (n \in N = \{1, 2, 3, ...\}).$$

Note that

$$\mathcal{D}^n f(z) = z + \sum_{j \geq 2} j^n a_j z^j, \quad (n \in N_0 = N \cup \{0\}). \tag{1. 2}$$

The differential operator $\mathcal{D}^n$ defined by (1. 2) was introduced by Sălăgean [17] and investigated recently by (for example) Ramachandran et al. [13]. In fact, several general families of operators such as those named as the Dziok-Srivastava operator [5, 6, 7], the Srivastava-Attiya operator [21] and the Srivastava-Wright operator [20] have been widely and extensively used in Geometric Function Theory in recent years. Let $\psi(z)$ be an analytic function with $\psi(0) = 1$, $\psi'(0) > 0$ and $Re\ \psi(z) > 0$ $(z \in U)$ which maps the unit disc U onto a starlike region with respect to 1 and is symmetric with respect to the real axis. Let $S^*(\psi)$ be the class of functions $f(z) \in S$ for which

$$\frac{z f'(z)}{f(z)} \prec \psi(z), \quad (z \in U)$$

and $C(\psi)$ be the class of functions $f(z) \in S$ for which

$$1 + \frac{z f''(z)}{f'(z)} \prec \psi(z), \quad (z \in U).$$

The classes $S^*(\psi)$ and $C(\psi)$ were introduced and studied by Ma and Minda [11]. They obtained the Fekete-Szegö inequality for functions in the class $S^*(\psi)$ and also for functions in the class $C(\psi)$. For a brief history of the Fekete-Szegö problem for class of starlike, convex, and close-to-convex functions, see the recent paper by Srivastava et al. [22]. It is well known that the $n-$th coefficient of a univalent function $f \in \wp$ is bounded by n. In 1933, Fekete and Szego [8] proved a noticeable result that the estimate

$$\left| a_3 - \eta a_2^2 \right| \leq 1 + 2 \exp\left(-\frac{2\eta}{1-\eta} \right),$$

holds for $f \in S$ and for $0 \leq \eta \leq 1$. This inequality is sharp for each η . Recently many authors have considered the Fekete-Szegö problem for typical classes of univalent functions (see, for instance [2], [3], [12], [14], [15], [16], [18], [19], [22], [23], [24]) . Here, we consider the following classes of functions,

$$R_n(b, \psi) := \left\{ f \in \wp : 1 + \frac{1}{b} \left[\frac{\mathcal{D}^{n+1} f(z)}{z} - 1 \right] \prec \psi(z) \right\},$$

$$S_n^*(\gamma, \psi) := \left\{ f \in \wp : (1-\gamma) \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} + \gamma \frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^n f(z)} \prec \psi(z) \right\},$$

$$L_n(\gamma, \psi) := \left\{ f \in \wp : \left(\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \right)^{\gamma} \left(\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} \right)^{1-\gamma} \prec \psi(z) \right\},$$

$$M_n(\gamma, \psi) := \left\{ f \in \wp : (1-\gamma) \left(\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \right) + \gamma \left(\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} \right) \prec \psi(z) \right\},$$

where b is non-zero complex number and $\gamma \geq 0$. By giving specific values to the parameters n, γ and ψ in the classes $R_n(b, \psi), S_n^*(\gamma, \psi), L_n(\gamma, \psi)$ and $M_n(\gamma, \psi)$, we obtain the following subclasses studied by earlier authors:

Remark 1.1.

(i) $R_0(b,\psi) = R_b(\psi)$(see Ali et al. [2]);
(ii) $S_0^*(\gamma,\psi) = S^*(\gamma,\psi)$(see Ali et al. [2]);
(iii) $S_0^*(0, \frac{1+Az}{1+Bz}) =: S^*(A,B)(-1 \le A < B \le 1)$ (see Janowski [9]);
(iv) $L_0(\gamma,\psi) = L(\gamma,\psi)$(see Ali et al. [2]);
(v) $L_0(0, \frac{1+Az}{1+Bz}) =: C(A,B)(-1 \le A < B \le 1)$ (see Janowski [9]);
(vi) $L_0(1, \frac{1+Az}{1+Bz}) =: S^*(A,B)(-1 \le A < B \le 1)$ (see Janowski [9]);
(vii)$L_0(0,\psi) =: C(\psi)$ and $L_0(1,\psi) =: S^*(\psi)$ (see Ma-Minda [11]);
(viii) $L_0(\gamma,\psi) =: M_{\gamma,\beta}(\psi)(\beta = 1-\gamma$ and $0 \le \gamma \le 1)$ (see Ravichadran et. al. [15]);
(ix) $M_0(\gamma,\psi) = \ M(\gamma,\psi)$(see Ali et al. [2]);
(x) $M_0(0, \frac{1+Az}{1+Bz}) =: S^*(A,B)(-1 \le A < B \le 1)$ (see Janowski [9]);
(xi) $M_0(1, \frac{1+Az}{1+Bz}) =: C(A,B)(-1 \le A < B \le 1)$ (see Janowski [9]).

For a function $f(z) \in S$ defined by (1. 1), the $k-$th root transform is given by:

$$F(z) = \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{j\ge 1} b_{kj+1} z^{kj+1}. \tag{1. 3}$$

Remark 1.2. *Set $k = 1$, the above expression reduce to the function f it self.*

In the present paper, we derive the Fekete-Szegö inequality for the classes $R_n(b,\psi)$, $S_n^*(\gamma,\psi),\ L_n(\gamma,\psi)$ and $M_n(\gamma,\psi)$ which we define a bove. In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.3. [1] *Let $p \in \Lambda$ and*

$$p(z) = c_1 z + c_2 z^2 + ... \quad (z \in U), \tag{1. 4}$$

then,

$$\left|c_2 - tc_1^2\right| \le \begin{cases} -t & \text{if } t \le -1 \\ 1 & \text{if } -1 \le t \le 1 \\ t & \text{if } t \ge 1. \end{cases}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $p(z)$ is z or one of its rotations. If $-1 < t < 1$, then the equality holds if and only if $p(z)$ is z^2 or one of its rotations. If $t = -1$, the equality holds if and only if $p(z) = z\frac{\lambda+z}{1+\lambda z}(0 \le \lambda \le 1)$ or one of its rotations. If $t = 1$, the equality holds if and only if $p(z) = -z\frac{\lambda+z}{1+\lambda z}(0 \le \lambda \le 1)$ or one of its rotations.

Lemma 1.4. [10] *(see also* [16]*) Let $p \in \Lambda$, then, for any complex number t,*

$$\left|c_2 - tc_1^2\right| \le max\{1;\ |t|\},$$

and the result is sharp for the functions given by $p(z) = z^2$ or $p(z) = z$.

2. Fekete-Szegö Problem

Theorem 2.1. *Let $f(z)$ be given by (1. 1). Assume that*

$$\psi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + ... \,. \tag{2. 5}$$

If $f \in S_n^*(\psi)$, and F is the $k-$th root transformation of f defined by (1. 3), then for any complex number η

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \leq \begin{cases} \frac{\left[k(2^{2n}-3^n)+3^n\right]A_1^2}{2^{2n+1}.3^n k^2}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)+\frac{A_2}{3^n.2k} & \text{if } \eta \leq \rho_1, \\ \frac{A_1}{3^n.2k} & \text{if } \rho_1 \leq \eta \leq \rho_2, \\ -\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1^2}{2^{2n+1}.3^n k^2}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{3^n.2k} & \text{if } \eta \geq \rho_1 \end{cases}$$

where

$$\rho_1 := \frac{2^{2n-1}}{3^n}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}-1\right)+k(1-3^n 4^{-n})+3^n 4^{-n}\right],$$
$$\rho_2 := \frac{2^{2n-1}}{3^n}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}+1\right)+k(1-3^n 4^{-n})+3^n 4^{-n}\right].$$

and

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \leq \frac{A_1}{3^n.2k}\max\left\{1;\ \left|\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}k}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)+\frac{A_2}{A_1}\right|\right\}.$$

Proof. If $f \in S_n^*(\psi)$, there exists a function $p(z) \in \Lambda$ given by (1. 4) such that

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} = \psi(p(z)). \tag{2. 6}$$

Since

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} = 1+2^n a_2 z+(2.3^n a_3-2^{2n}a_2^2)z^2+\left(3.4^n a_4-3.6^n a_2 a_{3-}2^{3n}a_2^3\right)z^3+...,$$

and

$$\psi(p(z)) = 1+A_1c_1z+\left(A_1c_2+A_2c_1^2\right)z^2+...\ .$$

From (2. 6), we obtain

$$2^n a_2 = A_1 c_1 \tag{2. 7}$$

and

$$3^n a_3 = \frac{1}{2}\left[A_1c_2+\left(A_2+A_1^2\right)c_1^2\right]. \tag{2. 8}$$

For $f \in \mathcal{A}$ given by (1. 1), it is easy to show that

$$\left[f(z^k)\right]^{\frac{1}{k}} = z+\frac{1}{k}a_2z^{k+1}+\left(\frac{1}{k}a_3-\frac{1}{2}\frac{(k-1)}{k^2}a_2^2\right)z^{2k+1}+...\ . \tag{2. 9}$$

By using (1. 3) and (2. 9), we have

$$b_{k+1} = \frac{1}{k}a_2, \tag{2. 10}$$

also

$$b_{2k+1} = \frac{1}{k}a_3-\frac{1}{2}\frac{(k-1)}{k^2}a_2^2. \tag{2. 11}$$

Using (2. 7) and (2. 8) in (2. 10) and (2. 11), we obtain

$$b_{k+1} = \frac{A_1c_1}{2^n k}$$

and

$$b_{2k+1} = \frac{1}{3^n 2k}\left[A_1c_2+A_2c_1^2+\frac{k(2^{2n}-3^n)+3^n}{2^{2n}k}A_1^2c_1^2\right],$$

and hence,

$$b_{2k+1}-\eta b_{k+1}^2=\frac{A_1}{3^n.2k}\left\{c_2-\left[-\tfrac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}k}\left(1-\tfrac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{A_1}\right]c_1^2\right\}.$$

The first part is obtained by applying Lemma 1.3.

If $-\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}k}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{A_1}\leq -1$, then,

$$\eta\leq\frac{2^{2n-1}}{3^n}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}-1\right)+k(1-3^n4^{-n})+3^n4^{-n}\right]\quad i.e.,\ \eta\leq\rho_1,$$

and Lemma 1.3 gives:

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1^2}{2^{2n+1}.3^nk^2}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)+\frac{A_2}{3^n.2k}.$$

For $-1\leq-\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}k}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{A_1}\leq 1$, we have,

$$\frac{2^{2n-1}}{3^n}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}-1\right)+k(1-3^n4^{-n})+3^n4^{-n}\right]\leq\eta$$

$$\leq\frac{2^{2n-1}}{3^n}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}+1\right)+k(1-3^n4^{-n})+3^n4^{-n}\right]$$

and Lemma 1.3 yields:

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq\frac{B_1}{3^n.2k}.$$

For $-\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}k}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{A_1}\geq 1$, we have,

$$\eta\geq\frac{2^{2n-1}}{3^n}\left[\frac{k}{A_1}\left(\frac{A_2}{A_1}+1\right)+k(1-3^n4^{-n})+3^n4^{-n}\right]\quad i.e.,\ \eta\geq\rho_2.$$

Applying Lemma 1.3, we have

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq-\frac{\left[k(2^{2n}-3^n)+3^n\right]A_1^2}{2^{2n+1}.3^nk^2}\left(1-\frac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{3^n.2k}.$$

The second part follows by applying Lemma 1.4

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|=\frac{A_1}{3^n2k}\left|c_2-\left[-\tfrac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}.k}\left(1-\tfrac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)-\frac{A_2}{A_1}\right]c_1^2\right|$$

$$\leq\frac{A_1}{3^n2k}\max\left\{1;\ \left|\tfrac{\left[k(2^{2n}-3^n)+3^n\right]A_1}{2^{2n}k}\left(1-\tfrac{2.3^n\eta}{k(2^{2n}-3^n)+3^n}\right)+\frac{A_2}{A_1}\right|\right\}.$$

□

Theorem 2.2. *Let $f(z)$ be given by (1. 1). Assume that b is non zero complex number and $\psi(z)$ is given by (2. 5). If $f\in R_n(b,\psi)$, and F is given by (1. 3), then for any complex number η*

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|\leq\frac{|b|\,A_1}{3^{n+1}k}\max\left\{1;\ \left|\frac{3^{n+1}bA_1}{2^{2n+3}}\left(1-\frac{1-2\eta}{k}\right)-\frac{A_2}{A_1}\right|\right\}.$$

Proof. Let $f \in R_n(b,\psi)$, there exists a function $p(z) \in \Lambda$ defined by (1. 4) such that

$$1+\frac{1}{b}\left[\frac{\mathcal{D}^{n+1}f(z)}{z}-1\right]=\psi(p(z)). \tag{2. 12}$$

Since

$$1+\frac{1}{b}\left[\frac{\mathcal{D}^{n+1}f(z)}{z}-1\right]=1+\frac{2^{n+1}}{b}a_2z+\frac{3^{n+1}}{b}a_3z^2+\frac{4^{n+1}}{b}a_4z^3+...,$$

and

$$\psi(p(z))=1+A_1c_1z+\left(A_1c_2+A_2c_1^2\right)z^2+...,$$

by using (2. 12), we have

$$a_2=\frac{bA_1c_1}{2^{n+1}} \tag{2. 13}$$

and

$$a_3=\frac{b}{3^{n+1}}\left[A_1c_2+A_2c_1^2\right]. \tag{2. 14}$$

Using (2. 13) and (2. 14) in (2. 10) and (2. 11), it follows:

$$b_{k+1}=\frac{bA_1c_1}{2^{n+1}k},$$

and

$$b_{2k+1}=\frac{bA_1c_2}{k.3^{n+1}}+\frac{bA_2c_1^2}{k.3^{n+1}}-\frac{b^2A_1^2c_1^2}{2^{2n+3}k}+\frac{b^2A_1^2c_1^2}{2^{2n+3}k^2},$$

and hence

$$b_{2k+1}-\eta b_{k+1}^2=\frac{bA_1}{3^{n+1}.k}\left\{c_2-\left[\frac{3^{n+1}bA_1}{2^{2n+3}}\left(1-\frac{1-2\eta}{k}\right)-\frac{A_2}{A_1}\right]c_1^2\right\} \tag{2. 15}$$

Applying Lemma 1.4 yields:

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|=\frac{|b|\,A_1}{3^{n+1}.k}\left|c_2-\left[\frac{3^{n+1}bA_1}{2^{2n+3}}\left(1-\frac{1-2\eta}{k}\right)-\frac{A_2}{A_1}\right]c_1^2\right|$$

$$\leq\frac{|b|\,A_1}{3^{n+1}k}\max\left\{1;\ \left|\frac{3^{n+1}bA_1}{2^{2n+3}}\left(1-\frac{1-2\eta}{k}\right)\ -\frac{A_2}{A_1}\right|\right\}.$$

□

Remark 2.3. *Let* $k=1,\ n=0$ *and*

$$\psi(z)=(1+Bz)\,/\,(1+Az)\ \ (-1\leq A\leq B\leq 1),$$

in Theorem 2.2, we obtain the result in [4]*, Theorem 4, p. 894.*

Putting $b=1$, in Theorem 2.2, equation (2. 15) becomes:

$$b_{2k+1}-\eta b_{k+1}^2=\frac{A_1}{3^{n+1}.k}\left\{c_2-\left[\frac{3^{n+1}A}{2^{2n+3}}\left(1-\frac{1-2\eta}{k}\right)-\frac{A_2}{A_1}\right]c_1^2\right\}.$$

Remark 2.4. *If we set* $b=1$, *in Theorem 2.2, we obtain the following Corollary.*

Corollary 2.5. *If $f \in \wp$ satisfies $\frac{\mathcal{D}^{n+1}f}{z} \prec \psi(z)$, then,*

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq \begin{cases} -\frac{A_1^2}{2^{2n+3}k}\left(1 - \frac{1-2\eta}{k}\right) + \frac{A_2}{3^{n+1}k} & \text{if } \eta \leq \rho_1, \\ \frac{A_1}{3^{n+1}k} & \text{if } \rho_1 \leq \eta \leq \rho_2, \\ \frac{A_1^2}{2^{2n+3}k}\left(1 - \frac{1-2\eta}{k}\right) - \frac{A_2}{3^{n+1}k} & \text{if } \eta \geq \rho_1 \end{cases}$$

where

$$\rho_1 = \frac{2^{2n+2}kA_2}{3^{n+1}A_1^2} - \frac{2^{2n+2}k}{3^{n+1}A_1} - \frac{k}{2} + \frac{1}{2} \text{ and } \rho_2 = \frac{2^{2n+2}kA_2}{3^{n+1}A_1^2} + \frac{2^{2n+2}k}{3^{n+1}A_1} - \frac{k}{2} + \frac{1}{2}.$$

Theorem 2.6. *Let $f(z)$ be given by (1. 1). Assume that $\gamma \geq 0$ and $\psi(z)$ is given by (2. 5). If $f \in S_n^*(\gamma, \psi)$, and F is given by (1. 3), then for any complex number η*

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq \begin{cases} -\frac{A_1}{3^n 2k(1+3\gamma)}\upsilon, & \text{if } \eta \leq \rho_1, \\ \frac{A_1}{3^n 2k(1+3\gamma)} & \text{if } \rho_1 \leq \eta \leq \rho_2, \\ \frac{A_1}{3^n 2k(1+3\gamma)}\upsilon, & \text{if } \eta \geq \rho_1 \end{cases}$$

where

$$\rho_1 := \frac{k(1+2\gamma)^2 2^{2n-1}}{3^n A_1(1+3\gamma)}\left[\frac{A_1}{(1+2\gamma)} + \frac{A_2}{A_1} - \frac{3^n A_1(k-1)(1+3\gamma)}{2^{2n}k(1+2\gamma)^2} - 1\right],$$

$$\rho_2 := \frac{k(1+2\gamma)^2 2^{2n-1}}{3^n A_1(1+3\gamma)}\left[\frac{A_1}{(1+2\gamma)} + \frac{A_2}{A_1} - \frac{3^n A_1(k-1)(1+3\gamma)}{2^{2n}k(1+2\gamma)^2} + 1\right],$$

$$\upsilon := \frac{A_1}{(1+2\gamma)}\left[\frac{3^n(k-1)(1+3\gamma)}{2^{2n}k(1+2\gamma)} + \frac{3^n\eta(1+3\gamma)}{2^{2n-1}k(1+2\gamma)} - 1\right] - \frac{A_2}{A_1}.$$

and

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq \frac{A_1}{3^n 2k(1+3\gamma)} \max\{1;\ |\upsilon|\}.$$

Proof. If f belongs to $S_n^*(\gamma, \psi)$, there exists a function $p(z) \in \Lambda$ given by (1. 4) such that

$$(1-\gamma)\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} + \gamma\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^n f(z)} = \psi(p(z)).$$

Since

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} = 1 + 2^n a_2 z + (3^n.2a_3 - 2^{2n}a_2^2)z^2 + ... \tag{2. 16}$$

and

$$\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^n f(z)} = 1 + 3.2^n a_2 z + (8.3^n a_3 - 3.2^{2n}a_2^2)z^2 + ..., \tag{2. 17}$$

then equations (2. 16) and (2. 17) yield:

$$(1-\gamma)\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} + \gamma\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^n f(z)} = 1 + (1+2\gamma)2^n a_2 z \tag{2. 18}$$

$$+ \left[2(1+3\gamma)3^n a_3 - (1+2\gamma)2^{2n}a_2^2\right]z^2 + ...$$

Since

$$\psi(p(z)) = 1 + A_1c_1z + \left(A_1c_2 + A_2c_1^2\right)z^2 + ...,$$

It follows from (2. 18) that

$$a_2 = \frac{A_1 c_1}{(1+2\gamma)2^n} \tag{2. 19}$$

and

$$a_3 = \frac{1}{2(1+2\gamma)3^n}\left[A_1 c_2 + A_2 c_1^2 + \frac{A_1^2 c_1^2}{(1+2\gamma)}\right]. \tag{2. 20}$$

Using (2. 19) and (2. 20) in (2. 10) and (2. 11), we get

$$b_{k+1} = \frac{A_1 c_1}{k(1+2\gamma)2^n},$$

and

$$b_{2k+1} = \frac{1}{3^n 2k(1+3\gamma)}\left[A_1 c_2 + A_2 c_1^2 + \frac{A_1^2 c_1^2}{(1+2\gamma)}\right] - \frac{A_1^2 c_1^2\,(k-1)}{2^{n+1}k^2(1+2\gamma)^2},$$

and hence,

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| = \frac{A_1}{3^n.2k(1+3\gamma)}\left\{c_2 - \upsilon c_1^2\right\},$$

where

$$\upsilon := \frac{A_1}{(1+2\gamma)}\left[\frac{3^n(k-1)(1+3\gamma)}{2^{2n}k(1+2\gamma)} + \frac{3^n\eta(1+3\gamma)}{2^{2n-1}k(1+2\gamma)} - 1\right] - \frac{A_2}{A_1}.$$

Now an application of Lemma 1.3 gives the first part of the result. If $\eta \le \rho_1$, then by applying Lemma 1.3, we have

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \le -\frac{A_1}{3^n.2k(1+3\gamma)}\upsilon.$$

For $\rho_1 \le \eta \le \rho_2$, then by applying Lemma 1.3, we get

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \le \frac{A_1}{3^n.2k(1+3\gamma)}.$$

Next, if $\mu \ge \rho_2$, by applying Lemma 1.3, we write

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \le \frac{A_1}{3^n.2k(1+3\gamma)}\upsilon.$$

Now by applying Lemma 1.4, we get the second part of the result. □

Remark 2.7. *When $\gamma = 0$ the above Theorem 2.6 reduces to Theorem 2.1.*

Theorem 2.8. *Let $f(z)$ be given by (1. 1). Assume that $\xi = (1-\gamma), \gamma \ge 0$ and $\psi(z)$ is given by (2. 5). If $f \in L_n(\gamma, \psi)$, and F is given by (1. 3), then for any complex number η*

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \le \begin{cases} -\frac{A_1}{2.3^n k(\gamma+3\xi)}\upsilon, & \text{if } \eta \le \rho_1, \\ \frac{A_1}{2.3^n k(\gamma+3\xi)} & \text{if } \rho_1 \le \eta \le \rho_2, \\ \frac{A_1}{2.3^n k(\gamma+3\xi)}\upsilon, & \text{if } \eta \ge \rho_2 \end{cases} \tag{2. 21}$$

where

$$\rho_1 := \frac{k}{3^n 2^{1-2n}(\gamma+3\xi)}\left[\frac{(\gamma+2\xi)^2}{A_1}\left(\frac{A_2}{A_1} - 1\right) - \frac{(\gamma+2\xi)^2 - 3(\gamma+4\xi)}{2} - \frac{3^n(k-1)(\gamma+3\xi)}{2^{2n}k}\right],$$

$$\rho_2 := \frac{k}{3^n 2^{1-2n}(\gamma+3\xi)}\left[\frac{(\gamma+2\xi)^2}{A_1}\left(\frac{A_2}{A_1} + 1\right) - \frac{(\gamma+2\xi)^2 - 3(\gamma+4\xi)}{2} - \frac{3^n(k-1)(\gamma+3\xi)}{2^{2n}k}\right],$$

$$\upsilon := \frac{A_1}{(\gamma+2\xi)^2}\left[\tfrac{(\gamma+2\xi)^2-3(\gamma+4\xi)}{2}+\tfrac{3^n(k-1)(\gamma+3\xi)}{2^{2n}k}+\tfrac{\eta 3^n 2^{1-2n}(\gamma+3\xi)}{k}\right]-\tfrac{A_2}{A_1}.$$

and

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \le \frac{A_1}{2.3^n k(\gamma+3\xi)}\max\{1;\ |\upsilon|\}.$$

Proof. If $f \in L_n(\gamma,\psi)$, there exists a function $p(z) \in \Lambda$ given by (1. 4) such that

$$\left(\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}\right)^{\gamma}\left(\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)^{\xi}=\psi(p(z)). \tag{2. 22}$$

Since

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}=1+2^n a_2 z+(3^n.2a_3-2^{2n}a_2^2)z^2+...$$

and therefore,

$$\left(\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}\right)^{\gamma}=1+\gamma 2^n a_2 z+\left[2\gamma 3^n a_3+\frac{\gamma^2-3\gamma}{2}2^{2n}a_2^2)\right]z^2+... \tag{2. 23}$$

Similarly,

$$\left(\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)=1+2^{n+1}a_2 z+(3^{n+1}.2a_3-2^{2n+2}a_2^2)z^2+...$$

and therefore,

$$\left(\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)^{\xi}=1+2^{n+1}\xi a_2 z+\left[2\xi.3^{n+1}a_3+\left(\xi^2-3\xi\right)2^{2n+1}a_2^2\right]z^2+... \tag{2. 24}$$

Thus, from (2. 23) and (2. 24),

$$\left(\tfrac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}\right)^{\gamma}\left(\tfrac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)^{\xi}=1+2^n\left(\gamma+2\xi\right)a_2 z$$
$$+\left[2.3^n\left(\gamma+3\xi\right)a_3+2^{2n}\tfrac{\left[(\gamma+2\xi)^2-3(\gamma+4\xi)\right]}{2}a_2^2\right]z^2+...\ .$$

Since

$$\psi(p(z))=1+A_1c_1z+\left(A_1c_2+A_2c_1^2\right)z^2+...\ .$$

From (2. 22), we get

$$a_2=\frac{A_1c_1}{(\gamma+2\xi)2^n} \tag{2. 25}$$

and

$$a_3=\frac{A_1c_2+A_2c_1^2}{2\left(\gamma+3\xi\right)3^n}-\frac{\left[(\gamma+2\xi)^2-3\left(\gamma+4\xi\right)\right]A_1^2c_1^2}{4.3^n(\gamma+2\xi)^2\left(\gamma+3\xi\right)} \tag{2. 26}$$

Using (2. 25) and (2. 26) in (2. 10) and (2. 11), we get

$$b_{k+1}=\frac{A_1c_1}{k.2^n(\gamma+2\xi)},$$

and

$$b_{2k+1}=\frac{A_1c_2+A_2c_1^2}{3^n2k\left(\gamma+3\xi\right)}-\frac{\left[(\gamma+2\xi)^2-3\left(\gamma+4\xi\right)\right]A_1^2c_1^2}{4k3^n(\gamma+2\xi)^2\left(\gamma+3\xi\right)}-\frac{A_1^2c_1^2\left(k-1\right)}{2^{2n+1}k^2(\gamma+2\xi)^2},$$

and hence,

$$\left|b_{2k+1}-\eta b_{k+1}^2\right|=\frac{A_1}{2k.3^n\left(\gamma+3\xi\right)}\left\{c_2-\upsilon c_1^2\right\}.$$

where,

$$\upsilon = \frac{A_1}{(\gamma+2\xi)^2}\left[\frac{(\gamma+2\xi)^2-3(\gamma+4\xi)}{2}\right.$$
$$\left.+\frac{3^n(k-1)(\gamma+3\xi)}{2^{2n}k}+\frac{\eta.3^n 2^{1-2n}(\gamma+3\xi)}{k}\right]-\frac{A_2}{A_1}.$$

Our results follow now by an application of Lemmas 1.3 and 1.4. □

Remark 2.9. *If we set $k=1$ and $n=0$, then inequality (2. 21) is the result obtained in* [15], *Theorem 2.1, p.3.*

Theorem 2.10. *Let $f(z)$ be given by (1. 1). Assume that $\gamma \geq 0$ and $\psi(z)$ is given by (2. 5). If f belongs to $M_n(\alpha,\psi)$, and F is given by (1. 3), then for any complex number η*

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \leq \begin{cases} -\varsigma\upsilon, & \text{if } \eta \leq \rho_1, \\ \varsigma & \text{if } \rho_1 \leq \eta \leq \rho_2, \\ \varsigma\upsilon, & \text{if } \eta \geq \rho_2 \end{cases}$$

where

$$\rho_1 := \frac{k}{2^{1-2n}.3^n(1+2\gamma)}\left[\frac{(1+\gamma)^2}{A_1}\left(\frac{A_2}{A_1}-1\right)+(1+3\gamma)-\frac{3^n(k-1)(1+2\gamma)}{2^{2n}k}\right],$$
$$\rho_2 := \frac{k}{2^{1-2n}.3^n(1+2\gamma)}\left[\frac{(1+\gamma)^2}{A_1}\left(\frac{A_2}{A_1}+1\right)+(1+3\gamma)-\frac{3^n(k-1)(1+2\gamma)}{2^{2n}k}\right],$$
$$\upsilon := \frac{A_1}{(1+\gamma)^2}\left[\frac{3^n(k-1)(1+2\gamma)}{2^{2n}k}+\frac{2^{1-2n}.3^n\eta}{k}(1+2\gamma)-(1+3\gamma)\right]-\frac{A_2}{A_1}. \quad \text{and } \varsigma = \frac{A_1}{2.3^n k(1+2\gamma)}$$

and

$$\left|b_{2k+1}-\eta b_{k+1}^2\right| \leq \frac{A_1}{2.3^n k(1+2\gamma)}\max\{1;\ |\upsilon|\}.$$

Proof. If $f \in M_n(\gamma,\psi)$, there exists a function $p(z) \in \Lambda$ given by (1. 4) such that

$$(1-\gamma)\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}+\gamma\left(\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)=\psi(p(z)). \tag{2. 27}$$

Since

$$(1-\gamma)\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}=1-\gamma+(1-\gamma)2^n a_2 z+(1-\gamma)\left[3^n.2a_3-2^{2n}a_2^2\right]z^2+\ldots \tag{2. 28}$$

and

$$\gamma\left(\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)=\gamma+\gamma 2^{n+1}a_2 z+\left[2\gamma\left(3^{n+1}a_3-2^{2n+1}a_2^2\right)\right]z^2+\ldots\ . \tag{2. 29}$$

Then from equations (2. 28) and (2. 29), it folows:

$$(1-\gamma)\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}+\gamma\left(\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)}\right)=1+2^n(1+\gamma)a_2 z$$
$$+\left[2.3^n(1+2\gamma)a_3-2^{2n}(1+3\gamma)a_2^2\right]z^2+\ldots$$

Since

$$\psi(p(z))=1+A_1c_1z+\left(A_1c_2+A_2c_1^2\right)z^2+\ldots\ .$$

From (2. 27), we get

$$a_2=\frac{A_1c_1}{(1+\gamma)2^n} \tag{2. 30}$$

and

$$a_3 = \frac{1}{2\left(1+2\gamma\right)3^n}\left[A_1 c_2 + A_2 c_1^2 + \frac{(1+3\gamma)A_1^2 c_1^2}{\left(1+\gamma\right)^2}\right]. \tag{2. 31}$$

By using (2. 30) and (2. 31) in (2. 10) and (2. 11), it folows:

$$b_{k+1} = \frac{A_1 c_1}{k(1+\gamma)2^n},$$

and

$$b_{2k+1} = \frac{1}{2.3^n k\left(1+2\gamma\right)}\left[A_1 c_2 + A_2 c_1^2 + \frac{(1+3\gamma)A_1^2 c_1^2}{\left(1+\gamma\right)^2}\right] - \frac{3^n\left(k-1\right)\left(1+2\gamma\right)}{2^{2n}k}\frac{A_1^2 c_1^2}{\left(1+\gamma\right)^2},$$

and hence,

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| = \frac{A_1}{2.3^n k\left(1+2\gamma\right)}\left\{c_2 - \upsilon c_1^2\right\}.$$

where,

$$\upsilon = \frac{A_1}{(1+\gamma)^2}\left[\frac{3^n(k-1)(1+2\gamma)}{2^{2n}k} + \frac{2^{1-2n}3^n\eta}{k}(1+2\gamma) - (1+3\gamma)\right] - \frac{A_2}{A_1}.$$

From Lemma 1.3 and Lemma 1.4, the results are obtained. If $\eta \leq \rho_1$, by applying Lemma 1.3, we have

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq -\varsigma\upsilon.$$

Similarly, if $\rho_1 \leq \eta \leq \rho_2$, then by applying Lemma 1.3, we get

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq \varsigma.$$

Next, if $\eta \geq \rho_2$, by applying Lemma 1.3, we write

$$\left|b_{2k+1} - \eta b_{k+1}^2\right| \leq \varsigma\upsilon.$$

□

Remark 2.11. *Putting $k = 1$, $\gamma = 1$ and $n = 0$, in Theorem 2.10, we obtain the result in* [11], *Theorem 3, p. 164.*

3. Fekete-Szegö problem for z/f

In this section, the Fekete-Szegö type coefficient inequalities associated with the rational function Ψ of the form

$$\Psi(z) = z/f(z) = 1 + \sum_{j\geq 1} d_j z^j, \tag{3. 32}$$

where f belongs to the classes $S_n^*(\psi),\ R_n(b,\psi),\ S_n^*(\gamma,\psi), L_n(\gamma,\psi)$ and $M_n(\gamma,\psi)$ are derived.

Theorem 3.1. *Let $f(z)$ be given by (1. 1). Assume that $\psi(z)$ is given by (2. 5). If $f \in S_n^*(\psi)$, and Ψ is given by (3. 32), then for any complex number η*

$$\left|d_2 - \eta d_1^2\right| \leq \begin{cases} \left[\frac{1}{2^{2n}}\left(1-\eta\right) - \frac{1}{2.3^n}\right]A_1^2 - \frac{A_2}{2.3^n}, & \text{if } \eta \leq \rho_1, \\ -\frac{A_1}{2.3^n}, & \text{if } \rho_1 \leq \eta \leq \rho_2, \\ \left[-\frac{1}{2^{2n}}\left(1-\eta\right) + \frac{1}{2.3^n}\right]A_1^2 + \frac{A_2}{2.3^n}, & \text{if } \eta \geq \rho_2 \end{cases}$$

where

$$\rho_1 := \left(1 - \frac{2^{2n-1}}{3^n}\right) + \frac{2^{2n-1}}{3^n A_1} - \frac{2^{2n-1} A_2}{3^n A_1^2}, \quad \rho_2 := \left(1 - \frac{2^{2n-1}}{3^n}\right) - \frac{2^{2n-1}}{3^n A_1} - \frac{2^{2n-1} A_2}{3^n A_1^2}.$$

and

$$\left|d_2 - \eta d_1^2\right| \leq \frac{A_1}{2.3^n} \max\left\{1; \left|\left[\frac{3^n}{2^{2n-1}}(1-\eta) - 1\right] A_1 - \frac{A_2}{A_1}\right|\right\}.$$

Proof. It is easy to see that

$$z/f(z) = 1 - a_2 z + \left(a_2^2 - a_3\right) z^2 + \dots . \tag{3. 33}$$

From (3. 32) and (3. 33), we have

$$d_1 = -a_2 \tag{3. 34}$$

and

$$d_2 = a_2^2 - a_3 \tag{3. 35}$$

Using (2. 7) and (2. 8) in (3. 34) and (3. 35), we obtain

$$d_1 = \frac{-A_1 c_1}{2^n}$$

and

$$d_2 = \frac{A_1^2 c_1^2}{2^{2n}} - \frac{1}{2.3^n}\left[A_1 c_2 + \left(A_2 + A_1^2\right) c_1^2\right],$$

and hence,

$$d_2 - \eta d_1^2 = -\frac{A_1}{2.3^n}\left\{c_2 - \left[\left(\frac{3^n}{2^{2n-1}}(1-\eta) - 1\right) A_1 - \frac{A_2}{A_1}\right] c_1^2\right\}.$$

By applying Lemma 1.3, we have the first part of the result and by Lemma 1.4, we get the second result:

$$\left|d_2 - \eta d_1^2\right| = \frac{A_1}{2.3^n}\left|c_2 - \left[\left(\frac{3^n}{2^{2n-1}}(1-\eta) - 1\right) A_1 - \frac{A_2}{A_1}\right] c_1^2\right|$$

$$\left|d_2 - \eta d_1^2\right| \leq \frac{A_1}{2.3^n} \max\left\{1; \left|\left(\frac{3^n}{2^{2n-1}}(1-\eta) - 1\right) A_1 - \frac{A_2}{A_1}\right|\right\}.$$

□

Theorem 3.2. *Let $f(z)$ be given by (1. 1). Assume that $\psi(z)$ is given by (2. 5). If $f \in R_n(b,\psi)$, and Ψ is given by (3. 32), then for any complex number η*

$$\left|d_2 - \eta d_1^2\right| \leq \frac{|b|\, A_1}{3^{n+1}} \max\left\{1; \left|\frac{3^{n+1}}{2^{2n+2}}(1-\eta)\, b A_1 - \frac{A_2}{A_1}\right|\right\}.$$

Proof. Using (2. 13) and (2. 14) in (3. 34) and (3. 35), we have

$$d_1 = \frac{-b A_1 c_1}{2^{n+1}}$$

and

$$d_2 = \frac{1}{2^{2n+2}} b^2 A_1^2 c_1^2 - \frac{b}{3^{n+1}}\left(A_1 c_2 + A_2 c_1^2\right),$$

and hence,

$$d_2 - \eta d_1^2 = -\frac{b A_1}{3^{n+1}}\left\{c_2 - \left[\frac{3^{n+1}}{2^{2n+2}}(1-\eta)\, b A_1 - \frac{A_2}{A_1}\right] c_1^2\right\}.$$

Lemma 1.4 gives:

$$\left|d_2-\eta d_1^2\right|=\frac{|b|\,A_1}{3^{n+1}}\left|c_2-\left[\frac{3^{n+1}}{2^{2n+2}}\,(1-\eta)\,bA_1-\frac{A_2}{A_1}\right]c_1^2\right|.$$

$$\left|d_2-\eta d_1^2\right|\leq\frac{|b|\,A_1}{3^{n+1}}\max\left\{1;\;\left|\frac{3^{n+1}}{2^{2n+2}}\,(1-\eta)\,bA_1-\frac{A_2}{A_1}\right|\right\}.$$

For functions with non-negative derivative. □

Remark 3.3. *Putting $b=1$, in Theorem 3.2, we get the following corollary.*

Corollary 3.4. *Let $f(z)$ be given by (1. 1). If $\frac{\mathcal{D}^{n+1}f(z)}{z}\prec\psi(z)$, then*

$$\left|d_2-\eta d_1^2\right|\leq\begin{cases}\frac{1}{2^{2n+2}}\,(1-\eta)\,A_1^2-\frac{A_2}{3^{n+1}}, & \text{if } \eta\leq\rho_1,\\[2ex] \frac{A_1}{3^{n+1}}, & \text{if } \rho_1\leq\eta\leq\rho_2,\\[2ex] -\frac{1}{2^{2n+2}}\,(1-\eta)\,A_1^2+\frac{A_2}{3^{n+1}}, & \text{if } \eta\geq\rho_2\end{cases}$$

where,

$$\rho_1=1+\frac{2^{2n+2}}{3^{n+1}A_1}-\frac{2^{2n+2}A_2}{3^{n+1}A_1^2}\quad \text{and}\quad \rho_2=1-\frac{2^{2n+2}}{3^{n+1}A_1}-\frac{2^{2n+2}A_2}{3^{n+1}A_1^2}$$

Theorem 3.5. *Let $f(z)$ be given by (1. 1). Assume that $\gamma\geq 0$ and $\psi(z)$ is given by (2. 5). If $f\in S_n^*(\gamma,\psi)$ and Ψ is given by (3. 32), then for any complex number η*

$$\left|d_2-\eta d_1^2\right|\leq\begin{cases}\varsigma\upsilon, & \text{if } \eta\leq\rho_1,\\[2ex] \varsigma, & \text{if } \rho_1\leq\eta\leq\rho_2,\\[2ex] -\varsigma\upsilon, & \text{if } \eta\geq\rho_2\end{cases}$$

where

$$\rho_1:=\frac{2^{2n-1}\,(1+2\gamma)^2}{3^nA_1(1+3\gamma)}-\frac{2^{2n-1}A_2\,(1+2\gamma)^2}{3^n(1+3\gamma)A_1^2}-\frac{2^{2n-1}\,(1+2\gamma)}{3^n(1+3\gamma)}+1,$$

$$\rho_2:=-\frac{2^{2n-1}\,(1+2\gamma)^2}{3^nA_1(1+3\gamma)}-\frac{2^{2n-1}A_2\,(1+2\gamma)^2}{3^n(1+3\gamma)A_1^2}-\frac{2^{2n-1}\,(1+2\gamma)}{3^n(1+3\gamma)}+1,$$

$$\upsilon:=\frac{3^nA_1(1+3\gamma)}{2^{2n-1}\,(1+2\gamma)^2}-\frac{3^n\eta(1+3\gamma)A_1}{2^{2n-1}\,(1+2\gamma)^2}-\frac{A_1}{(1+2\gamma)}-\frac{A_2}{A_1},\ \varsigma:=-\frac{A_1}{2.3^n\,(1+3\gamma)}$$

and

$$\left|d_2-\eta d_1^2\right|\leq\frac{A_1}{2.3^n\,(1+3\gamma)}\max\left\{1;\;|\upsilon|\right\}.$$

Proof. The equations (2. 19), (2. 20), (3. 34) and (3. 35), yield:

$$d_1=\frac{-A_1c_1}{2^n\,(1+2\gamma)},$$

and

$$d_2=\frac{A_1^2c_1^2}{2^{2n}\,(1+2\gamma)^2}-\frac{1}{3^n2\,(1+3\gamma)}\left(A_1c_2+A_2c_1^2+\frac{A_1^2c_1^2}{(1+2\gamma)}\right),$$

and hence,

$$d_2-\eta d_1^2=-\frac{A_1}{3^n2(1+3\gamma)}\left\{c_2-\left[\frac{3^nA_1(1+3\gamma)}{2^{2n-1}(1+2\gamma)^2}-\frac{3^n\eta(1+3\gamma)A_1}{2^{2n-1}(1+2\gamma)^2}-\frac{A_1}{(1+2\gamma)}-\frac{A_2}{A_1}\right]c_1^2\right\}.$$

By applying Lemma 1.3, we have the first part of the result and by Lemma 1.4, we get the second result:

$$\left|d_2-\eta d_1^2\right|=\tfrac{A_1}{3^n.2(1+3\gamma)}\left|c_2-\left[\tfrac{3^nA_1(1+3\gamma)}{2^{2n-1}(1+2\gamma)^2}-\tfrac{3^n\eta(1+3\gamma)A_1}{2^{2n-1}(1+2\gamma)^2}-\tfrac{A_1}{(1+2\gamma)}-\tfrac{A_2}{A_1}\right]c_1^2\right|$$
$$\leq\frac{A_1}{3^n.2\left(1+3\gamma\right)}\max\left\{1;\ |\upsilon|\right\}.$$

□

Theorem 3.6. *Let $f(z)$ be given by (1. 1). Assume that $\xi=(1-\gamma)\,,\ \gamma\geq 0$ and $\psi(z)$ is given by (2. 5). If $f\in L_n(\gamma,\psi)$, and Ψ is given by (3. 32), then for any complex number η*

$$\left|d_2-\eta d_1^2\right|\leq\begin{cases}\frac{A_1}{3^n2(\gamma+3\xi)}\upsilon, & \text{if } \eta\leq\rho_1,\\ \frac{A_1}{3^n2(\gamma+3\xi)}, & \text{if } \rho_1\leq\eta\leq\rho_2,\\ -\frac{A_1}{3^n2(\gamma+3\xi)}\upsilon, & \text{if } \eta\geq\rho_2\end{cases}$$

where

$$\rho_1:=1-\frac{2^{2n-1}\left(\gamma+2\xi\right)^2A_2}{3^n(\gamma+3\xi)A_1^2}+\frac{2^{2n-2}\left[\left(\gamma+2\xi\right)^2-3\left(\gamma+4\xi\right)\right]}{3^n(\gamma+3\xi)}+\frac{2^{2n-1}\left(\gamma+2\xi\right)^2}{3^n(\gamma+3\xi)A_1},$$

$$\rho_2:=1-\frac{2^{2n-1}\left(\gamma+2\xi\right)^2A_2}{3^n(\gamma+3\xi)A_1^2}+\frac{2^{2n-2}\left[\left(\gamma+2\xi\right)^2-3\left(\gamma+4\xi\right)\right]}{3^n(\gamma+3\xi)}-\frac{2^{2n-1}\left(\gamma+2\xi\right)^2}{3^n(\gamma+3\xi)A_1},$$

$$\upsilon:=\frac{\left[\left(\gamma+2\xi\right)^2-3\left(\gamma+4\xi\right)\right]A_1}{2(\gamma+2\xi)^2}+\frac{3^n(1-\eta)(\gamma+3\xi)A_1}{2^{2n-1}(\gamma+2\xi)^2}-\frac{A_2}{A_1},$$

and

$$\left|d_2-\eta d_1^2\right|\leq\frac{A_1}{3^n.2(\gamma+3\xi)}\max\left\{1;\ |\upsilon|\right\}.$$

Proof. Using (2. 25) and (2. 26) in (3. 34) and (3. 35), we obtain

$$d_1=\frac{-A_1c_1}{2^n\left(\gamma+2\xi\right)},$$

and

$$d_2=\frac{A_1^2c_1^2}{2^{2n}\left(\gamma+2\xi\right)^2}-\frac{A_1c_2+A_2c_1^2}{3^n.2(\gamma+3\xi)}+\frac{\left[\left(\gamma+2\xi\right)^2-3\left(\gamma+4\xi\right)\right]A_1^2c_1^2}{3^n.4(\gamma+3\xi)\left(\gamma+2\xi\right)^2},$$

and hence,

$$d_2-\mu d_1^2=-\frac{A_1}{3^n.2\left(\gamma+3\xi\right)}\left\{c_2-\upsilon c_1^2\right\}.$$

Where,

$$\upsilon:=\frac{\left[\left(\gamma+2\xi\right)^2-3\left(\gamma+4\xi\right)\right]A_1}{2(\gamma+2\xi)^2}+\frac{3^n(1-\mu)(\gamma+3\xi)A_1}{2^{2n-1}(\gamma+2\xi)^2}-\frac{A_2}{A_1}.$$

By applying Lemma 1.3, we have the first part of the result and by Lemma 1.4, we get The second result:

$$\left|d_2-\eta d_1^2\right|=\frac{A_1}{2.3^n\left(\gamma+3\xi\right)}\left|c_2-\upsilon c_1^2\right|$$

$$\leq \frac{A_1}{2.3^n\left(\gamma+3\xi\right)} \max\left\{1;\ |\upsilon|\right\}.$$

□

Theorem 3.7. *Let $f(z)$ be given by (1. 1). Assume that $\gamma \geq 0$ and $\psi(z)$ is given by (2. 5). If f belongs to $M_n(\gamma, \psi)$ and Ψ is given by (3. 32), then for any complex number η*

$$\left|d_2 - \eta d_1^2\right| \leq \begin{cases} \frac{A_1}{2.3^n(1+2\gamma)}\upsilon, & \text{if } \eta \leq \rho_1, \\ \frac{A_1}{2.3^n(1+2\gamma)}, & \text{if } \rho_1 \leq \eta \leq \rho_2, \\ -\frac{A_1}{2.3^n(1+2\gamma)}\upsilon, & \text{if } \eta \geq \rho_2 \end{cases}$$

where

$$\rho_1 := 1 - \frac{2^{2n-1}\left(1+\gamma\right)^2 A_2}{3^n(1+2\gamma)A_1^2} - \frac{2^{2n-1}(1+3\gamma)}{3^n(1+2\gamma)} + \frac{2^{2n-1}\left(1+\gamma\right)^2}{3^n(1+2\gamma)A_1},$$

$$\rho_2 := 1 - \frac{2^{2n-1}\left(1+\gamma\right)^2 A_2}{3^n(1+2\gamma)A_1^2} - \frac{2^{2n-1}(1+3\gamma)}{3^n(1+2\gamma)} - \frac{2^{2n-1}\left(1+\gamma\right)^2}{3^n(1+2\gamma)A_1},$$

$$\upsilon := \frac{\left[3^n(1+2\gamma) - 2^{2n-1}(1+3\gamma)\right] A_1}{2^{2n-1}\left(1+\gamma\right)^2} - \frac{3^n \eta B_1(1+2\gamma)}{2^{2n-1}\left(1+\gamma\right)^2} - \frac{A_2}{A_1}.$$

and

$$\left|d_2 - \eta d_1^2\right| \leq \frac{A_1}{2.3^n(1+2\alpha)} \max\left\{1;\ |\upsilon|\right\}.$$

Proof. Using (2. 30) and (2. 31) in (3. 34) and (3. 35), we have

$$d_1 = \frac{-A_1 c_1}{2^n\left(1+\gamma\right)},$$

and

$$d_2 = \frac{A_1^2 c_1^2}{2^{2n}\left(1+\gamma\right)^2} - \frac{1}{2.3^n(1+2\gamma)}\left[A_1 c_2 + A_2 c_1^2 + \frac{(1+3\gamma)A_1^2 c_1^2}{\left(1+\gamma\right)^2}\right],$$

and hence,

$$d_2 - \eta d_1^2 = -\frac{A_1}{2.3^n(1+2\gamma)}\left\{c_2 - \upsilon c_1^2\right\},$$

where,

$$\upsilon := \frac{\left[3^n(1+2\gamma) - 2^{2n-1}(1+3\gamma)\right] A_1}{2^{2n-1}\left(1+\gamma\right)^2} - \frac{3^n \eta A_1(1+2\gamma)}{2^{2n-1}\left(1+\gamma\right)^2} - \frac{A_2}{A_1}.$$

By applying Lemma 1.3, we have the first part of the result and by Lemma 1.4, we get the second result:

$$\left|d_2 - \eta d_1^2\right| = \frac{A_1}{2.3^n(1+2\gamma)}\left|c_2 - \rho c_1^2\right|$$

$$\leq \frac{A_1}{2.3^n(1+2\gamma)} \max\left\{1;\ |\rho|\right\}.$$

□

Remark 3.8. *For $n = 0$, we get the result obtained by Ali et al.* [2].

4. ACKNOWLEDGMENTS

The authors thank the referees for their valuable suggestions which led to the improvement of this paper.

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