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Effects of Chemical Reaction on the Unsteady Flow of an Incompressible Fluid over a Vertical Oscillating Plate

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Abstract. The first order uniform chemical reaction of a viscous incompressible unsteady flow of fluid passing on an oscillatory infinite long vertical plate is studied with modified temperature and uniform mass diffusivity. The temperature and concentration are raised linearly with respect to time around the plate . Laplace transformation technique is used to find the exact solution for the dimensionless governing equations, when plate moves about its mean position harmonically. The changing results of temperature, concentration and velocity is learned for distinct parameters like chemical reaction parameter, thermal Grashöf number, Schmidt number, time and phase angle are discussed through detailed graphical analysis.

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1. Introduction

Convection flows past a wall have a special interests for researchers in various industrial and engineering systems. This flow help us to understand the mechanism of various natural phenomena. Almost all home appliances involving heating or cooling process like deep freezers refrigerators, air conditioners are governed by this process. It also occurs in Geo-Hydrological systems like oceans on a large scale. It is observed that convection flow change incredibly in different processes in industry and nature. The motion in the convection flow may be extremely slow or it can also be extremely rapid on the other hand, as in hurricane. In cosmology, it is observed that convection of gases in black holes at speed almost approaching the speed of light.

Chambre and Young [5] studied about the first order chemical reaction in the neighboring region of a static horizontal plate. The flow with first-order chemical reaction nearby cylindrical catalyst pellet discussed by Ramanamurthy and Rao [18]. Hetnarski [12] studied about a simple algorithm for finding inverse Laplace transformations for complicated exponential form. Soundalgekar [23] analyzed the oscillating flow of a viscous, incompressible fluid moving on an in-finite iso-thermal vertical wall plane. He also analysed the effect due to the emulsion of concentration and temperature distinction of the flow moving on a vertical oscillating plate [22]. Apelblat [2, 3] discussed about the function of diffusion in the order of convection transport.

Singh [20] analyzed regarding the uniform cross magnetic field for the free convection flow of an electrically transmitting fluid passing through an exponentially accelerated infinite vertical wall. He considered that the magnetic lines of force remain unchanged with respect to the fluid and the plate. Das et al. [7] reviewed about the effects of homogeneous first order chemical reaction flow which passing through an impulsive movement on infinite vertical plate with uniform mass transfer and heat flux. Singh and Kumar [21] also analysed that the free-convection flow of a viscous and incompressible fluid passing through an exponentially accelerated infinite vertical wall. Muthucumarswamy and Ganesan [16] analyzed the flow which passing on infinite vertical plate with varying temperature and consistent mass diffusion. Ganesan and Loganathan [11] considered the incompressible, viscous unsteady flow that passing through a vertically moving cylinder and their relationship with free convection and thermal radiation, by taking a mass transfer and heat into account.

Muthucumarsawamy and Minakshisundaram [17] examined the reaction of chemical and their effects on vertical oscillating plate with varying mass diffusion and temperature. Lahurikar [15] discussed about the convection flow near an infinite vertical plate which is surrounded by a fluid and moving impulsively upward. In last decade, a great work dealing with such flows done by scientists, a few of them are [4, 6, 8, 9, 10, 13, 14]. Another fascinating problem by Rubbab et al. [19] in which shear stress is applied on the fluid forced by the vertical plate.

Motivated from the papers [17, 19] , we expressed the case, when vertical plate is oscillating and its vibratory motion caused the fluid to move around its surrounding. The rotation and heat of the plate are caused by the flow fluid in the presence of exponential heating. Initially, the plate is in rest position and after that it begins to oscillate in its own plane. Solutions corresponding to unsteady incompressible flow on a vertical oscillating plate with variable exponential temperature are derived by using Laplace transformation. Initially, the fluid and the plate have the same concentration and temperature. The concentration level and temperature of the plate is increasing linearly with respect to time. The Boussinesq approximation is used for the governing equations and the expressions for temperature, velocity and concentration are calculated by Laplace transformation. The results are shown graphically with correspondence of different parameters that are involved like Schmidt number, chemical reaction parameter, phase angle and time.

2. Governing Equations of the Problem

Let suppose an infinite vertical plate enclose by an in-compressible and viscous fluid. Initially, we are taking both the plate and the fluid at rest position with same concentration C'_∞ and temperature T_∞. We take x'-axis along the plate and y'-axis be at the horizontal direction perpendicular to the x'-axis. When time $t' = 0^+$, the plate begins moving about its initial position with velocity $u_0 cos\omega' t'$, where $\omega' t'$ is phase angle and u_0 is the velocity of plate. The temperature of the plate that changes exponentially, is increased to $T_\infty + T_\omega(1 - a' exp(-b't'))$, where T_ω is constant temperature. Here we assume that the effect of dissipation due to viscosity is trivial. The governing equations, by keeping it in mind that the boundary layer and Boussinesq approximations, for such flow are [17]

$$\frac{\partial u}{\partial t'} = g\beta(T - T_\infty) + g\beta^*(C' - C'_\infty) + \nu\frac{\partial^2 u}{\partial y'^2}, \tag{2. 1}$$

$$\rho C_p\frac{\partial T}{\partial t'} = k\frac{\partial^2 T}{\partial y'^2}, \tag{2. 2}$$

$$\frac{\partial C'}{\partial t'} = D\frac{\partial^2 C}{\partial y'^2} - K_l C', \tag{2. 3}$$

where u is the velocity component along the x'-axis, g is acceleration due to gravity, β^* is volumetric coefficient of expansion with concentration, β is volumetric coefficient of thermal expansion, ν is kinematic viscosity, ρ is density, C_p is specific heat at constant pressure, k is the coefficient of thermal conductivity, C' is species concentration in the fluid, T is the temperature, D is mass diffusion coefficient and K_l is the reaction of chemical parameter.

The initial and boundary conditions are given as

$$u = 0,\ \ T = T_\infty,\ \ C' = C'_\infty \ \texttt{for all}\ y'\ \texttt{at}\ t' \le 0, \tag{2. 4}$$

$$\begin{aligned} u &= u_0 H(t')\cos\omega' t',\ \ T = T_\infty + T_\omega(1 - a' exp(-b't')),\\ C' &= C'_\infty + C'_\omega(1 - a'\exp(-b't'))\ \texttt{at}\ y' = 0\ \texttt{for}\ t' > 0, \end{aligned}\tag{2. 5}$$

$$u = 0,\ \ T \to 0,\ \ C' \to C'_\infty\ \texttt{as}\ y' \to \infty, \tag{2. 6}$$

where $a' \ge 0$ and $b' > 0$ are constants and $H(t')$ is Heaviside step function which is defined as

$$H(t') = \begin{cases} 0 & \text{if } t' < 0 \\ 1 & \text{if } t' \ge 0\ . \end{cases}$$

We define here the dimensionless variables, for non-dimensionlizing our problem, are as follows:

$$\begin{aligned} U &= \frac{u}{u_0}, t = \frac{t' u_0^2}{\nu}, y = \frac{y' u_0}{\nu}, \theta = \frac{T - T_\infty}{T_\omega},\\ Gr &= \frac{g\beta\nu(T_\omega - T_\infty)}{u_0^3}, C = \frac{C' - C'_\infty}{C'_\omega - C'_\infty}, Gc = \frac{\nu g\beta^*(C'_\omega - C'_\infty)}{u_0^3},\\ Pr &= \frac{\mu C_p}{K}, Sc = \frac{\nu}{D}, K = \frac{\nu K_l}{u_0^2}, \omega = \frac{\omega'\nu}{u_0^2}, \end{aligned}\tag{2. 7}$$

where Pr is the prandtl number, Gr is the thermal Grashöf number, Gc is the mass Grashöf number, Sc is the Schmidt number, ω' is frequency and θ is the temperature. Eqs. (2.1)-(2.6) reduce to the following non-dimensional form

$$\frac{\partial U}{\partial t} = Gr\theta + GcC + \frac{\partial^2 U}{\partial t^2}, \tag{2. 8}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}, \tag{2. 9}$$

$$\frac{\partial C}{\partial t} = \frac{1}{Sc}\frac{\partial^2 C}{\partial y^2} - KC. \tag{2. 10}$$

Initial and boundary conditions in the non-dimensional form are

$$U = 0, \ \ \theta = 0, \ \ C = 0 \ \texttt{for all} \ y, \ \ \texttt{for} \ t \leq 0, \tag{2. 11}$$

$$U = H(t)\cos\omega t, \ \ \theta = 1 - a' exp(-b't), \ \texttt{at} \ y = 0 \ \texttt{for} \ t > 0, \tag{2. 12}$$

$$C = 1 - a' exp(-b't), \ \texttt{at} \ y = 0 \ \texttt{for} \ t > 0, \tag{2. 13}$$

$$U = 0, \ \ \theta \to 0, \ \ C \to 0, \ \ \texttt{as} \ y \to \infty. \tag{2. 14}$$

3. Solution of the problem

The problem defined by the Eqs. (2.8)-(2.10) along with initial and boundary conditions (2.11)-(2.14) is solved by using Laplace transformation. Taking Laplace transform of the Eq. (2.9) and initial and boundary conditions (2.11)-(2.12) for temperature, we obtain

$$Pr[s\bar{\theta}(y,s) - \theta(y,0)] = \frac{d^2\bar{\theta}}{dy^2}. \tag{3. 15}$$

Using initial condition $\theta(y,0) = 0$, Eq. (3.15) becomes

$$Prs\bar{\theta}(y,s) = \frac{d^2\bar{\theta}}{dy^2}, \tag{3. 16}$$

where $\bar{\theta}(y,s)$ is the Laplace transform of $\theta(y,t)$ and s is the Laplace transform parameter. The solution of the ordinary differential Eq. (3.16) is given by

$$\bar{\theta} = (\frac{1}{s} - \frac{a'}{s+b'})exp(-y\sqrt{Prs}). \tag{3. 17}$$

For the inverse Laplace transformation, we used the Hetnarski's algorithm [3],

$$\begin{aligned} L^{-1}\left[\frac{e^{(-c\sqrt{s+b})}}{s-a}\right] &= \frac{e^{at}}{2}\left[e^{(-c\sqrt{a+b})}erfc\left(\frac{c}{2\sqrt{t}} - \sqrt{(a+b)t}\right)\right] \\ &+ \frac{e^{at}}{2}\left[e^{(c\sqrt{a+b})}erfc\left(\frac{c}{2\sqrt{t}} + \sqrt{(a+b)t}\right)\right]. \end{aligned} \tag{3. 18}$$

Applying this above result to Eq. (3.17), we obtain the expression for temperature field

$$\begin{aligned} \theta &= erfc(\eta\sqrt{b'}) + \frac{a' exp(-b't)}{2}[exp(-2\eta i\sqrt{Prb't})erfc(\eta\sqrt{Pr} - \sqrt{b't}) \\ &+ exp(2\eta i\sqrt{Prb't})erfc(\eta\sqrt{Pr} + \sqrt{b't})], \end{aligned} \tag{3. 19}$$

where $\eta = \frac{y}{2\sqrt{t}}$.

Taking Laplace transform of the Eq. (2.10) and initial and boundary conditions (2.11)-(2.14) for concentration, we obtain

$$\frac{d^2\bar{C}}{dy^2} = Sc\left[s\bar{C}(y,s) - \bar{C}(y,0) + K\bar{C}(y,s)\right]. \tag{3. 20}$$

Using initial $\bar{C}(y,0) = 0$, The Eq. (3.20) becomes

$$\frac{d^2\bar{C}}{dy^2} = Sc(s+K)\bar{C}(y,s), \tag{3. 21}$$

where $\bar{C}(y,s)$ is the Laplace transform of $C(y,t)$.
The solution of the ordinary differential Eq. (3.21) is given by

$$\bar{C} = (\frac{1}{s} - \frac{a'}{s+b'})exp(-y\sqrt{Sc(s+k)}). \tag{3. 22}$$

Taking Laplace inverse of Eq. (3.22) by using Eq. (3.18), we get the solution for concentration

$$\begin{aligned} C &= \frac{1}{2}[exp(2\eta\sqrt{KtSc})erfc(\eta\sqrt{Sc} + \sqrt{Kt}) + exp(-2\eta\sqrt{KtSc})erfc(\eta\sqrt{Sc} \\ &- \sqrt{Kt})] - \frac{a'exp(-b't)}{2}[exp(2\eta i\sqrt{Scb't})erfc(\eta\sqrt{Sc} + i\sqrt{b't}) \\ &+ exp(-2\eta i\sqrt{Scb't})erfc(\eta\sqrt{Sc} - i\sqrt{b't})]. \end{aligned} \tag{3. 23}$$

Now we apply the Laplace transformation to the Eq. (2.8), we obtain

$$\frac{d^2\bar{U}}{dy^2} = s\ \bar{U}(y,s) - Gr\ \bar{\theta}(y,s) - Gc\ \bar{C}(y,s), \tag{3. 24}$$

where $\bar{\theta}$ and $\bar{C}$ is given from Eq. (3.17) and Eq. (3.22). Substituting the values of $\bar{\theta}(y,s)$ and $\bar{C}(y,s)$ on right hand side of Eq. (3.24), we have

$$\begin{aligned} \frac{d^2\bar{U}}{dy^2} &= s\ \bar{U}(y,s) - Gr\left[\left(\frac{1}{s} - \frac{a'}{s+b'}\right)exp\left(-y\sqrt{Prs}\right)\right] \\ &- Gc\left[\left(\frac{1}{s} - \frac{a'}{s+b'}\right)exp\left(-y\sqrt{Prs}\right)\right]. \end{aligned} \tag{3. 25}$$

Taking the Laplace transform of boundary condition $(2.12)_1$, we have

$$\bar{U}(0,s) = L[cos\omega t] = \frac{s}{s^2+\omega^2} = \frac{1}{2(s+i\omega)} + \frac{1}{2(s-i\omega)}. \tag{3. 26}$$

Using undetermined coefficient method, we obtain the solution for Eq. (3.25) as

$$\begin{aligned} \bar{U} &= exp(-y\sqrt{s})\left[\frac{1}{2(s+i\omega)} + \frac{1}{2(s-i\omega)} + \frac{c}{s^2} - \frac{a'c}{sb'} + \frac{a'c}{b'(s+b')} + \frac{ad}{s} - \frac{ad}{s-a}\right. \\ &+ \left.\frac{a^2a'd}{(s+b')(b'-a)} + \frac{a^2a'd}{(s-a)(a+b')}\right] - exp(-y\sqrt{Prs})\left[\frac{c}{s^2} + \frac{a'c}{sb'} - \frac{a'c}{b'(s+b')}\right] \\ &- exp(-y\sqrt{Sc(s+K)})\left[\frac{ad}{s} - \frac{ad}{s-a} - \frac{a^2a'd}{(s+b')(b'-a)}\right], \end{aligned} \tag{3. 27}$$

where

$$\eta = \frac{y}{2\sqrt{t}}, \quad a = \frac{KSc}{1-Sc}, \quad c = \frac{Gr}{Pr-1} \text{ and } d = \frac{Gc}{a^2(1-Sc)}.$$

Taking inverse Laplace transform of Eq. (3.27) and using the Hetnarski algorithm [12], we obtain the expression for velocity

$$
\begin{aligned}
U \;=\;& \frac{e^{i\omega t}}{4}\Big[exp(2\eta\sqrt{i\omega t})erfc(\eta+\sqrt{i\omega t})+exp(-2\eta\sqrt{i\omega t})erfc(\eta-\sqrt{i\omega t})\Big]\\
&+\frac{e^{-i\omega t}}{4}\Big[exp(2\eta\sqrt{-i\omega t})erfc(\eta+\sqrt{-i\omega t})+exp(-2\eta\sqrt{-i\omega t})erfc(\eta-\sqrt{-i\omega t})\Big]\\
&+\; ct\left[(1+2\eta^2)erfc(\eta)-2\eta\frac{1}{\pi}exp(-\eta^2)\right]-ct\Big[(1+2\eta^2 Pr)erfc(\eta\sqrt{Pr})\\
&-\; 2\eta\sqrt{\frac{Pr}{\pi}}exp(-\eta\sqrt{(Pr)})\Big]-\frac{ca'}{b'}erfc(\eta)+\frac{a'c}{b'}\frac{exp(-b')}{2}\Big[exp(-2\eta i\sqrt{b't})\\
&\times\; erfc(\eta-i\sqrt{b't})+exp(2\eta i\sqrt{b't})erfc(\eta+i\sqrt{b't})\Big]+ad\,erfc(\eta)\\
&-\;\frac{ad\,e^{at}}{2}\Big[exp(-2\eta\sqrt{at})erfc(\eta-\sqrt{at})+exp(2\eta\sqrt{at})erfc(\eta+\sqrt{at})\Big]\\
&+\;\frac{a^2a'd\,e^{-b't}}{2(b'-a)}\Big[exp(-2\eta i\sqrt{b't})erfc(\eta-i\sqrt{b't})+exp(2\eta i\sqrt{b't})erfc(\eta+i\sqrt{b't})\Big]\\
&+\;\frac{a^2a'd\,exp^{-b't}}{2(a+b')}\Big[exp(-2\eta\sqrt{at})erfc(\eta-\sqrt{at})+exp(2\eta\sqrt{at})erfc(\eta+\sqrt{at})\Big]\\
&+\;\frac{a'c\,e^{-b't}}{2b'}\Big[exp(-2\eta i\sqrt{Prb't})erfc(\eta\sqrt{Pr}-\sqrt{b't})+exp(2\eta i\sqrt{Prb't})\\
&\times\; erfc(\eta\sqrt{Pr}+\sqrt{b't})\Big]-\frac{a\,d}{2}\Big[exp(-2\eta\sqrt{ScKt})erfc(\eta\sqrt{Sc}-\sqrt{Kt})\\
&+\; exp(2\eta\sqrt{ScKt})erfc(\eta\sqrt{Sc}+\sqrt{Kt})\Big]+\frac{a\,d\,exp(at)}{2}\Big[exp(-2\eta\sqrt{Sc(S+K)t})\\
&\times erfc(\eta\sqrt{Sc}-\sqrt{(a+K)t})+exp(2\eta\sqrt{Sc(S+K)t})erfc(\eta\sqrt{Sc}+\sqrt{(a+K)t})\Big]\\
&-\;\frac{a^2a'd\,e^{-b'}}{2(b'-a)}\Big[exp(-2\eta i\sqrt{b't})erfc(\eta-i\sqrt{b't})+exp(2\eta i\sqrt{b't})erfc(\eta+i\sqrt{b't})\Big]\\
&+\;\frac{a^2a'd}{(a+b')}\Big[exp(-2\eta\sqrt{Sc(a+K)t})erfc(\eta\sqrt{Sc}-\sqrt{(a+K)t})\\
&+\; exp(2\eta\sqrt{Sc(a+K)t})erfc(\eta\sqrt{Sc}+\sqrt{(a+K)t})\Big]. \qquad (3.\ 28)
\end{aligned}
$$

In order to get the physical insight into the problem, the numerical values of temperature, concentration and velocity have been computed from (3.19), (3.23) and (3.28). While evaluating these expressions which observed that argument of the error function is complicated for which the following formula [1] is used:

$$
\begin{aligned}
erf(a+ib) \;=\;& erf(a)+\frac{e^{-a^2}}{2\pi aS}\left[1-cos(2ab)+isin(2ab)\right]\\
&+\;\frac{2e^{-a^2}}{\pi}\sum_{n=1}^{\infty}\frac{-n^2/4}{n^2+4n^2}\left[f_n(a,b)+ig_n(a,b)+\epsilon(a,b)\right]. \qquad (3.\ 29)
\end{aligned}
$$

where

$f_n(a,b) = \ 2a - 2acosh(nb)cos(2ab) + nsinh(nb)sin(2ab)$ and
$g_n(a,b) = 2acos(nb)sin(2ab) + nsinh(nb)cos(2ab), \mid \epsilon(a,b) \mid \approx 10^{-16} \mid erf(a+ib) \mid .$

Using the above formula (3.29), separating the real and imaginary parts of Eq. (3.19), the real part θ_1 is

$$
\begin{aligned}
\theta_1 &= 1 - erf(\eta\sqrt{Pr}) + \frac{a'e^{-b't}}{2}\Big[cos(2\eta\sqrt{Prb'}t)(1 - erf(x_1) \\
&- \frac{e^{-x_1^2}}{2\pi x_1}(1 - cos(2x_1y_1)) - \frac{2e^{-x_1^2-1/4}}{\pi(1+4x_1^2)}f_1(x_1,y_1) - \frac{2e^{-x_1^2-1}}{\pi(4+4x_1)}f_2(x_1,y_1)\Big] \\
&- sin(2\eta\sqrt{Prb'}t)\Big[\frac{e^{-x_1^2}}{2\pi x_1}sin(2x_1y_1) + \frac{2e^{-x_1^2-1/4}}{\pi(1+4x_1^2)}g_1(x_1,y_1) \\
&+ \frac{2e^{-x_1^2-1}}{\pi(4+4x_1)}g_2(x_1,y_1)\Big] + cos(2\eta\sqrt{Prb'}t)\Big[1 - erf(x_2) \\
&- \frac{e^{-x_2^2}}{2\pi x_2}(1 - cos(2x_2y_2)) - \frac{2e^{-x_2^2-1/4}}{\pi(1+4x_2^2)}f_1(x_2,y_2) - \frac{2e^{-x_2^2-1}}{\pi(4+4x_2)}f_2(x_2,y_2)\Big] \\
&+ sin(2\eta\sqrt{Prb'}t)\Big[\frac{e^{-x_2^2}}{2\pi x_2}sin(2x_2y_2) + \frac{2e^{-x_1^2-1/4}}{\pi(1+4x_1^2)}g_1(x_2,y_2) \\
&+ \frac{2e^{-x_2^2-1}}{\pi(4+4x_2)}g_2(x_2,y_2))\Big],
\end{aligned}
\tag{3. 30}
$$

and the imaginary part θ_2 is

$$
\begin{aligned}
\theta_2 &= \frac{a'e^{-b't}}{2}\Big[- sin(2\eta\sqrt{Prb'}t)(1 - erf(x_1) - \frac{e^{-x_1^2}}{2\pi x_1}(1 - cos(2x_1y_1)) \\
&- \frac{2e^{-x_1^2-1/4}}{\pi(1+4x_1^2)}f_1(x_1,y_1) - \frac{2e^{-x_1^2-1}}{\pi(4+4x_1)}f_2(x_1,y_1))\Big] - cos(2\eta\sqrt{Prb'}t) \\
&\times \Big[\frac{e^{-x_1^2}}{2\pi x_1}sin(2x_1y_1) + \frac{2e^{-x_1^2-1/4}}{\pi(1+4x_1^2)}g_1(x_1,y_1) + \frac{2e^{-x_1^2-1}}{\pi(4+4x_1)}g_2(x_1,y_1)\Big] \\
&+ sin(2\eta\sqrt{Prb'}t)\Big[1 - erf(x_2) - \frac{e^{-x_2^2}}{2\pi x_2}(1 - cos(2x_2y_2)) - \frac{2e^{-x_2^2-1/4}}{\pi(1+4x_2^2)} \\
&\times f_1(x_2,y_2) - \frac{2e^{-x_1^2-1}}{\pi(4+4x_1)}f_2(x_2,y_2)\Big] - cos(2\eta\sqrt{Prb'}t)\Big[\frac{e^{-x_2^2}}{2\pi x_2}sin(2x_2y_2) \\
&+ \frac{2e^{-x_2^2-1/4}}{\pi(1+4x_2^2)}g_1(x_2,y_2) + \frac{2e^{-x_2^2-1}}{\pi(4+4x_2)}g_2(x_2,y_2))\Big].
\end{aligned}
\tag{3. 31}
$$

Similarly from Eq. (3.23), the real part C_1 is

$$
\begin{aligned}
C_1 &= \frac{1}{2}\Bigg[exp(-2\eta\sqrt{ScKt})erfc(\eta\sqrt{Sc}-\sqrt{Kt})+exp(2\eta\sqrt{ScKt})erfc(\eta\sqrt{Sc}\\
&+ \sqrt{Kt})\Bigg]-\frac{a'e^{-b't}}{2}\Bigg[cos(2\eta\sqrt{Scb't})[(1-erf(x_3)-\frac{e^{-x_3^2}}{2\pi x_3}(1-cos(2x_3y_3))\\
&- \frac{2e^{-x_3^2-1/4}}{\pi(1+4x_3^2)}f_1(x_3,y_3)-\frac{2e^{-x_3^2-1}}{\pi(4+4x_3)}f_2(x_3,y_3))\Bigg]-sin(2\eta\sqrt{Scb't})\\
&\times \Bigg[\frac{e^{-x_3^2}}{2\pi x_3}sin(2x_3y_3)+\frac{2e^{-x_3^2-1/4}}{\pi(1+4x_3^2)}g_1(x_3,y_3)+\frac{2e^{-x_3^2-1}}{\pi(4+4x_3)}g_2(x_3,y_3)\Bigg]\\
&+ cos(2\eta\sqrt{Scb't})\Bigg[(1-erf(x_4)-\frac{e^{-x_4^2}}{2\pi x_4}(1-cos(2x_4y_4))\\
&- \frac{2e^{-x_4^2-1/4}}{\pi(1+4x_4^2)}f_1(x_4,y_4)-\frac{2e^{-x_4^2-1}}{\pi(4+4x_4)}f_2(x_4,y_4))\Bigg]+sin(2\eta\sqrt{Scb't})\\
&\times \Bigg[\frac{e^{-x_4^2}}{2\pi x_4}sin(2x_4y_4)+\frac{2e^{-x_4^2-1/4}}{\pi(1+4x_4^2)}g_1(x_4,y_4)+\frac{2e^{-x_4^2-1}}{\pi(4+4x_4)}g_2(x_4,y_4)\Bigg], \qquad (3.\ 32)
\end{aligned}
$$

and the imaginary part C_2 is

$$
\begin{aligned}
C_2 &= \frac{a'e^{-b't}}{2}\Bigg[-sin(2\eta\sqrt{Scb't})(1-erf(x_3)-\frac{e^{-x_3^2}}{2\pi x_3}(1-cos(2x_3y_3))\\
&- \frac{2e^{-x_3^2-1/4}}{\pi(1+4x_3^2)}f_1(x_3,y_3)-\frac{2e^{-x_3^2-1}}{\pi(4+4x_3)}f_2(x_3,y_3))\Bigg]-cos(\eta\sqrt{Scb't})\Bigg[\frac{e^{-x_3^2}}{2\pi x_3}\\
&\times sin(2x_3y_3)+\frac{2e^{-x_3^2-1/4}}{\pi(1+4x_3^2)}g_1(x_3,y_3)+\frac{2e^{-x_3^2-1}}{\pi(4+4x_3)}g_2(x_3,y_3)\Bigg]\\
&+ sin(2\eta\sqrt{Scb't})\Bigg[1-erf(x_4)\frac{e^{-x_4^2}}{2\pi x_4}(1-cos(2x_4y_4))-\frac{2e^{-x_4^2-1/4}}{\pi(1+4x_4^2)}\\
&\times f_1(x_4,y_4)-\frac{2e^{-x_4^2-1}}{\pi(4+4x_4)}f_2(x_4,y_4)\Bigg]+cos(2\eta\sqrt{Scb't})\Bigg[\frac{e^{-x_4^2}}{2\pi x_4}sin(2x_4y_4)\\
&+ \frac{2e^{-x_4^2-1/4}}{\pi(1+4x_4^2)}g_1(x_4,y_4)+\frac{2e^{-x_4^2-1}}{\pi(4+4x_4)}g_2(x_4,y_4)\Bigg]. \qquad (3.\ 33)
\end{aligned}
$$

Separating the velocity parts from Eq. (3.28), the real part U_1 is

$$
\begin{aligned}
U_1 = &\ \frac{1}{4}[e^{\eta\sqrt{2wt}}cos(wt+\eta\sqrt{2wt})(1-erf(x_5)-\frac{e^{-x_5^2}}{2\pi x_5}(1-cos(2x_5y_5))\\
- &\ \frac{2e^{-x_5^2-1/4}}{\pi(1+4x_5^2)}f_1(x_5,y_5)-\frac{2e^{-x_5^2-1}}{\pi(4+4x_5)}f_2(x_5,y_5))+e^{\eta\sqrt{2wt}}sin(wt-\eta\sqrt{2wt})\\
\times &\ (\frac{e^{-x_5^2}}{2\pi x_5}sin(2x_5y_5)+\frac{2e^{-x_5^2-1/4}}{\pi(1+4x_5^2)}g_1(x_5,y_5)+\frac{2e^{-x_5^2-1}}{\pi(4+4x_5)}g_2(x_5,y_5))]\\
+ &\ \frac{1}{4}[e^{-\eta\sqrt{2wt}}cos(wt-\eta\sqrt{2wt})(1-erf(x_6)-\frac{e^{-x_6^2}}{2\pi x_6}(1-cos(2x_6y_6))\\
- &\ \frac{2e^{-x_6^2-1/4}}{\pi(1+4x_6^2)}f_1(x_6,y_6)-\frac{2e^{-x_6^2-1}}{\pi(4+4x_6)}f_2(x_6,y_6))+e^{-\eta\sqrt{2wt}}sin(wt-\eta\sqrt{2wt})
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad \frac{e^{-x_6^2}}{2\pi x_6} sin(2x_6y_6) + \frac{2e^{-x_5^2-1/4}}{\pi(1+4x_5^2)} g_1(x_6,y_6) + \frac{2e^{-x_6^2-1}}{\pi(4+4x_6)} g_2(x_6,y_6))] \\
+ &\quad \frac{1}{4}[e^{\eta\sqrt{2wt}} cos(wt+\eta\sqrt{2wt})(1-erf(x_7) - \frac{e^{-x_7^2}}{2\pi x_7}(1-cos(2x_7y_7)) \\
- &\quad \frac{2e^{-x_7^2-1/4}}{\pi(1+4x_7^2)} f_1(x_7,y_7) - \frac{2e^{-x_7^2-1}}{\pi(4+4x_7)} f_2(x_7,y_7)) - e^{\eta\sqrt{2wt}} sin(wt+\eta\sqrt{2wt}) \\
\times &\quad (\frac{e^{-x_7^2}}{2\pi x_7} sin(2x_7y_7) + \frac{2e^{-x_7^2-1/4}}{\pi(1+4x_7^2)} g_1(x_7,y_7) + \frac{2e^{-x_7^2-1}}{\pi(4+4x_7)} g_2(x_7,y_7))] \\
+ &\quad \frac{1}{4}[e^{\eta\sqrt{2wt}} cos(wt-\eta\sqrt{2wt})(1-erf(x_8) - \frac{e^{-x_8^2}}{2\pi x_8}(1-cos(2x_8y_8)) \\
- &\quad \frac{2e^{-x_8^2-1/4}}{\pi(1+4x_8^2)} f_1(x_8,y_8) - \frac{2e^{-x_8^2-1}}{\pi(4+4x_8)} f_2(x_8,y_8)) - e^{-\eta\sqrt{2wt}} sin(wt-\eta\sqrt{2wt}) \\
\times &\quad (\frac{e^{-x_8^2}}{2\pi x_8} sin(2x_8y_8) + \frac{2e^{-x_8^2-1/4}}{\pi(1+4x_8^2)} g_1(x_8,y_8) + \frac{2e^{-x_8^2-1}}{\pi(4+4x_8)} g_2(x_8,y_8))] \\
+ &\quad \frac{a'c}{b'}\frac{e^{-b'}}{2}[cos(2\eta\sqrt{b't})(1-erf(x_9) - \frac{e^{-x_9^2}}{2\pi x_9}(1-cos(2x_9y_9)) \\
- &\, \frac{2e^{-x_9^2-1/4}}{\pi(1+4x_9^2)} f_1(x_9,y_9) - \frac{2e^{-x_9^2-1}}{\pi(4+4x_9)} f_2(x_9,y_9))) - sin(2\eta\sqrt{b't})(\frac{e^{-x_9^2}}{2\pi x_9} sin(2x_9y_9) \\
+ &\quad \frac{2e^{-x_9^2-1/4}}{\pi(1+4x_9^2)} g_1(x_9,y_9) + \frac{2e^{-x_9^2-1}}{\pi(4+4x_9)} g_2(x_9,y_9)) + cos(2\eta\sqrt{b't})(1-erf(x_{10}) \\
- &\, \frac{e^{-x_{10}^2}}{2\pi x_{10}}(1-cos(2x_{10}y_{10})) - \frac{2e^{-x_{10}^2-1/4}}{\pi(1+4x_{10}^2)} f_1(x_{10},y_{10}) - \frac{2e^{-x_{10}^2-1}}{\pi(4+4x_{10})} f_2(x_{10},y_{10})) \\
+ &\, sin(2\eta\sqrt{b't})\frac{e^{-x_{10}^2}}{2\pi x_{10}} sin(2x_{10}y_{10}) + (\frac{e^{-x_{10}^2}}{2\pi x_{10}} sin(2x_{10}y_{10}) + \frac{2e^{-x_{10}^2-1/4}}{\pi(1+4x_{10}^2)} g_1(x_{10},y_{10}) \\
+ &\, \frac{2e^{-x_{10}^2-1}}{\pi(4+4x_{10})} g_2(x_{10},y_{10}))] + \frac{2e^{-x_{10}^2-1/4}}{\pi(1+4x_{10}^2)} g_1(x_{10},y_{10}) + \frac{2e^{-x_{10}^2-1}}{\pi(4+4x_{10})} g_2(x_{10},y_{10}))] \\
+ &\quad \frac{a^2a'd}{b'-a}\frac{e^{-b'}}{2}[cos(2\eta\sqrt{b't})(1-erf(x_{11}) - \frac{e^{-x_{11}^2}}{2\pi x_{11}})(1-cos(2x_{11}y_{11})) \\
- &\quad \frac{2e^{-x_{11}^2-1/4}}{\pi(1+4x_{11}^2)} f_1(x_{11},y_{11}) - \frac{2e^{-x_{11}^2-1}}{\pi(4+4x_{11})} f_2(x_{11},y_{11}))) - \frac{e^{-x_{11}^2}}{2\pi x_{11}} sin(2\eta\sqrt{b't}) \\
\times &\quad sin(2x_{11}y_{11}) + \frac{2e^{-x_{11}^2-1/4}}{\pi(1+4x_{11}^2)} g_1(x_{11},y_{11}) + \frac{2e^{-x_{11}^2-1}}{\pi(4+4x_{11})} g_2(x_{11},y_{11})) \\
+ &\, cos(2\eta\sqrt{b't})(1-erf(x_{12}) - \frac{e^{-x_{12}^2}}{2\pi x_{12}}(1-cos(2x_{12}y_{12})) - \frac{2e^{-x_{12}^2-1/4}}{\pi(1+4x_{12}^2)} f_1(x_{12},y_{12}) \\
- &\quad \frac{2e^{-x_{12}^2-1}}{\pi(4+4x_{12})} f_2(x_{12},y_{12}))) + sin(2\eta\sqrt{b't})\frac{e^{-x_{12}^2}}{2\pi x_{12}} sin(2x_{12}y_{12}) + \frac{2e^{-x_{12}^2-1/4}}{\pi(1+4x_{12}^2)} \\
\times &\quad g_1(x_{12},y_{12}) + \frac{2e^{-x_{12}^2-1}}{\pi(4+4x_{12})} g_2(x_{12},y_{12}))] - \frac{a'c\,e^{-b'}}{2b'}[cos(2\eta\sqrt{b't})(1-erf(x_{13}) \\
- &\, \frac{e^{-x_{13}^2}}{2\pi x_{13}}(1-cos(2x_{13}y_{13})) - \frac{2e^{-x_{13}^2-1/4}}{\pi(1+4x_{13}^2)} f_1(x_{13},y_{13}) - \frac{2e^{-x_{13}^2-1}}{\pi(4+4x_{13})} f_2(x_{13},y_{13}))
\end{aligned}
$$

$$
\begin{aligned}
&- \quad sin(2\eta\sqrt{b't})(\frac{e^{-x_{13}^2}}{2\pi x_{13}} sin(2x_{13}y_{13}) + \frac{2e^{-x_{13}^2-1/4}}{\pi(1+4x_{13}^2)} g_1(x_{13},y_{13}) + \frac{2e^{-x_{13}^2-1}}{\pi(4+4x_{13})}\\
&\times \quad g_2(x_{13},y_{13})) + cos(2\eta\sqrt{b't})(1-erf(x_{14}) - \frac{e^{-x_{14}^2}}{2\pi x_{14}}(1-cos(2x_{14}y_{14}))\\
&- \quad \frac{2e^{-x_{14}^2-1/4}}{\pi(1+4x_{14}^2)} f_1(x_{14},y_{14}) - \frac{2e^{-x_{14}^2-1}}{\pi(4+4x_{14})} f_2(x_{14},y_{14})) + sin(2\eta\sqrt{b't})(\frac{e^{-x_{14}^2}}{2\pi x_{14}}\\
&\times \quad sin(2x_{14}y_{14}) + \frac{2e^{-x_{14}^2-1/4}}{\pi(1+4x_{14}^2)} g_1(x_{14},y_{14}) + \frac{2e^{-x_{14}^2-1}}{\pi(4+4x_{14})} g_2(x_{14},y_{14}))]\\
&- \frac{a^2a'd\,e^{-b'}}{2(b'-a)}[cos(2\eta\sqrt{b't})(1-erf(x_{15}) - \frac{e^{-x_{15}^2}}{2\pi x_{15}}(1-cos(2x_{15}y_{15})) - \frac{2e^{-x_{15}^2-1/4}}{\pi(1+4x_{15}^2)}\\
&\times \quad f_1(x_{15},y_{15}) - \frac{2e^{-x_{15}^2-1}}{\pi(4+4x_{15})} f_2(x_{15},y_{15})) - sin(2\eta\sqrt{b't})(\frac{e^{-x_{15}^2}}{2\pi x_{15}} sin(2x_{15}y_{15})\\
&+ \quad \frac{2e^{-x_{15}^2-1/4}}{\pi(1+4x_{15}^2)} g_1(x_{15},y_{15}) + \frac{2e^{-x_{15}^2-1}}{\pi(4+4x_{15})} g_2(x_{15},y_{15})) + cos(2\eta\sqrt{b't})(1-erf(x_{12})\\
&- \frac{e^{-x_{16}^2}}{2\pi x_{16}}(1-cos(2x_{16}y_{16})) - \frac{2e^{-x_{16}^2-1/4}}{\pi(1+4x_{16}^2)} f_1(x_{16},y_{16}) - \frac{2e^{-x_{16}^2-1}}{\pi(4+4x_{16})} f_2(x_{16},y_{16}))\\
&+ sin(2\eta\sqrt{b't})(\frac{e^{-x_{16}^2}}{2\pi x_{16}} sin(2x_{16}y_{16}) + \frac{2e^{-x_{16}^2-1/4}}{\pi(1+4x_{16}^2)} g_1(x_{16},y_{16}) + \frac{2e^{-x_{16}^2-1}}{\pi(4+4x_{16})} g_2(x_{16},y_{16}))],
\end{aligned} \tag{3. 34}
$$

and the imaginary part U_2 is

$$
\begin{aligned}
U_2 = &\frac{1}{4}[e^{\eta\sqrt{2wt}} sin(wt+\eta\sqrt{2wt})(1-erf(x_5) - \frac{e^{-x_5^2}}{2\pi x_5}(1-cos(2x_5y_5)) - \frac{2e^{-x_5^2-1/4}}{\pi(1+4x_5^2)}\\
&\times f_1(x_5,y_5) - \frac{2e^{-x_5^2-1}}{\pi(4+4x_5)} f_2(x_5,y_5)) - e^{\eta\sqrt{2wt}} cos(wt+\eta\sqrt{2wt}) \frac{e^{-x_5^2}}{2\pi x_5} sin(2x_5y_5)\\
&+ \frac{2e^{-x_5^2-1/4}}{\pi(1+4x_5^2)} g_1(x_5,y_5) + \frac{2e^{-x_5^2-1}}{\pi(4+4x_5)} g_2(x_5,y_5))] + \frac{1}{4}[e^{-\eta\sqrt{2wt}} sin(wt-\eta\sqrt{2wt})\\
&\times (1-erf(x_6) - \frac{e^{-x_6^2}}{2\pi x_6}(1-cos(2x_6y_6)) - \frac{2e^{-x_6^2-1/4}}{\pi(1+4x_6^2)} f_1(x_6,y_6) - \frac{2e^{-x_6^2-1}}{\pi(4+4x_6)}\\
&\times f_2(x_6,y_6)) - e^{-\eta\sqrt{2wt}} cos(wt-\eta\sqrt{2wt})(\frac{exp(-x_6^2)}{2\pi x_6} sin(2x_6y_6) + \frac{2e^{-x_6^2-1/4}}{\pi(1+4x_6^2)}\\
&\times g_1(x_6,y_6) + \frac{2e^{-x_6^2-1}}{\pi(4+4x_6)} g_2(x_6,y_6))] + \frac{1}{4}[e^{\eta\sqrt{2wt}} sin(wt+\eta\sqrt{2wt})(1-erf(x_7)\\
&- \frac{e^{-x_7^2}}{2\pi x_7}(1-cos(2x_7y_7)) - \frac{2e^{-x_7^2-1/4}}{\pi(1+4x_7^2)} f_1(x_7,y_7) - \frac{2e^{-x_7^2-1}}{\pi(4+4x_7)} f_2(x_7,y_7))\\
&- e^{\eta\sqrt{2wt}} cos(wt+\eta\sqrt{2wt}) \frac{e^{-x_7^2}}{2\pi x_7} sin(2x_7y_7) + \frac{2e^{-x_7^2-1/4}}{\pi(1+4x_7^2)} g_1(x_7,y_7) + \frac{2e^{-x_7^2-1}}{\pi(4+4x_7)}\\
&\times g_2(x_7,y_7))] - \frac{1}{4}[e^{-\eta\sqrt{2wt}} sin(wt-\eta\sqrt{2wt})(1-erf(x_8) - \frac{e^{-x_8^2}}{2\pi x_8}(1-cos(2x_8y_8))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2e^{-x_8^2-1/4}}{\pi(1+4x_8^2)} f_1(x_8,y_8) - \frac{2e^{-x_8^2-1}}{\pi(4+4x_8)} f_2(x_8,y_8)) + e^{-\eta\sqrt{2wt}} cos(wt-\eta\sqrt{2wt}) \frac{e^{-x_8^2}}{2\pi x_8} \\
&\times sin(2x_8y_8) + \frac{2e^{-x_8^2-1/4}}{\pi(1+4x_8^2)} g_1(x_8,y_8) + \frac{2e^{-x_8^2-1}}{\pi(4+4x_8)} g_2(x_8,y_8))] + \frac{a'c\,e^{-b'}}{2b'} \\
&\times [-sin(2\eta\sqrt{b't})(1-erf(x_9) - \frac{e^{-x_9^2}}{2\pi x_9}(1-cos(2x_9y_9)) - \frac{2e^{-x_9^2-1/4}}{\pi(1+4x_9^2)} f_1(x_9,y_9) \\
&- \frac{2e^{-x_9^2-1}}{\pi(4+4x_9)} f_2(x_9,y_9)) - cos(2\eta\sqrt{b't})(\frac{e^{-x_9^2}}{2\pi x_9} sin(2x_9y_9) + \frac{2e^{-x_9^2-1/4}}{\pi(1+4x_9^2)} g_1(x_9,y_9) \\
&+ \frac{2e^{-x_9^2-1}}{\pi(4+4x_9)} g_2(x_9,y_9)) + sin(2\eta\sqrt{b't})(1-erf(x_{10}) - \frac{e^{-x_{10}^2}}{2\pi x_{10}})(1-cos(2x_{10}y_{10})) \\
&- \frac{2e^{-x_{10}^2-1/4}}{\pi(1+4x_{10}^2)} f_1(x_{10},y_{10}) - \frac{2e^{-x_{10}^2-1}}{\pi(4+4x_{10})} f_2(x_{10},y_{10}))) - cos(2\eta\sqrt{b't})(\frac{e^{-x_{10}^2}}{2\pi x_{10}} \\
&\times sin(2x_{10}y_{10}) + \frac{2e^{-x_{10}^2-1/4}}{\pi(1+4x_{10}^2)} g_1(x_{10},y_{10}) + \frac{2e^{-x_{10}^2-1}}{\pi(4+4x_{10})} g_2(x_{10},y_{10}))] \\
&+ \frac{a^2a'd}{b'-a}\frac{e^{-b'}}{2}[-sin(2\eta\sqrt{b't})(1-erf(x_{11}) - \frac{e^{-x_{11}^2}}{2\pi x_{11}})(1-cos(2x_{11}y_{11})) \\
&- \frac{2e^{-x_{11}^2-1/4}}{\pi(1+4x_{11}^2)} f_1(x_{11},y_{11}) - \frac{2e^{-x_{11}^2-1}}{\pi(4+4x_{11})} f_2(x_{11},y_{11}) - cos(2\eta\sqrt{b't})(\frac{e^{-x_{11}^2}}{2\pi x_{11}} \\
&\times sin(2x_{11}y_{11}) + \frac{2e^{-x_{11}^2-1/4}}{\pi(1+4x_{11}^2)} g_1(x_{11},y_{11}) + \frac{2e^{-x_{11}^2-1}}{\pi(4+4x_{11})} g_2(x_{11},y_{11})) + sin(2\eta\sqrt{b't}) \\
&\times (1-erf(x_{12}) \frac{e^{-x_{12}^2}}{2\pi x_{12}}(1-cos(2x_{12}y_{12})) - \frac{2e^{-x_{12}^2-1/4}}{\pi(1+4x_{12}^2)} f_1(x_{12},y_{12}) - \frac{2e^{-x_{12}^2-1}}{\pi(4+4x_{12})} \\
&\times f_2(x_{12},y_{12})) - cos(2\eta\sqrt{b't})(\frac{e^{-x_{12}^2}}{2\pi x_{12}} sin(2x_{12}y_{12}) + \frac{2e^{-x_{12}^2-1/4}}{\pi(1+4x_{12}^2)} g_1(x_{12},y_{12}) \\
&+ \quad \frac{2e^{-x_{12}^2-1}}{\pi(4+4x_{12})} g_2(x_{12},y_{12}))] - \frac{a'c\,e^{-b'}}{2b'}[-sin(2\eta\sqrt{b't})(1-erf(x_{13}) - \frac{e^{-x_{13}^2}}{2\pi x_{13}} \\
&\times \quad (1-cos(2x_{13}y_{13})) - \frac{2e^{-x_{13}^2-1/4}}{\pi(1+4x_{13}^2)} f_1(x_{13},y_{13}) - \frac{2e^{-x_{13}^2-1}}{\pi(4+4x_{13})} f_2(x_{13},y_{13})) \\
&- \quad cos(2\eta\sqrt{b't})(\frac{e^{-x_{13}^2}}{2\pi x_{13}} sin(2x_{13}y_{13}) + \frac{2e^{-x_{13}^2-1/4}}{\pi(1+4x_{13}^2)} g_1(x_{13},y_{13}) + \frac{2e^{-x_{13}^2-1}}{\pi(4+4x_{13})} \\
&\times \quad g_2(x_{13},y_{13})) + sin(2\eta\sqrt{b't})(1-erf(x_{14}) - \frac{e^{-x_{14}^2}}{2\pi x_{14}})(1-cos(2x_{14}y_{14})) \\
&- \frac{2e^{-x_{14}^2-1/4}}{\pi(1+4x_{14}^2)} f_1(x_{14},y_{14}) \frac{2e^{-x_{14}^2-1}}{\pi(4+4x_{14})} f_2(x_{14},y_{14})) - cos(2\eta\sqrt{b't})(sin(2x_{14}y_{14}) \\
&\times \quad \frac{e^{-x_{14}^2}}{2\pi x_{14}} + \frac{2e^{-x_{14}^2-1/4}}{\pi(1+4x_{14}^2)} g_1(x_{14},y_{14}) + \frac{2e^{-x_{14}^2-1}}{\pi(4+4x_{14})} g_2(x_{14},y_{14}))] - \frac{a^2a'd\,e^{-b'}}{2(b'-a)} \\
&\times \quad [-sin(2\eta\sqrt{b't})(1-erf(x_{15}) - \frac{e^{-x_{15}^2}}{2\pi x_{15}}(1-cos(2x_{15}y_{15})) - \frac{2e^{-x_{15}^2-1/4}}{\pi(1+4x_{15}^2)}
\end{aligned}
$$

$$\times\ f_1(x_{15}, y_{15}) - \frac{2e^{-x_{15}^2-1}}{\pi(4+4x_{15})} f_2(x_{15}, y_{15}))) - cos(2\eta\sqrt{b't})(\frac{e^{-x_{15}^2}}{2\pi x_{15}} sin(2x_{15}y_{15})$$
$$+\ \frac{2e^{-x_{15}^2-1/4}}{\pi(1+4x_{15}^2)} g_1(x_{15}, y_{15}) + \frac{2e^{-x_{15}^2-1}}{\pi(4+4x_{15})} g_2(x_{15}, y_{15})) + sin(2\eta\sqrt{b't})(1 - erf(x_{12})$$
$$-\ \frac{e^{-x_{16}^2}}{2\pi x_{16}}(1 - cos(2x_{16}y_{16})) - \frac{2e^{-x_{16}^2-1/4}}{\pi(1+4x_{16}^2)} f_1(x_{16}, y_{16}) - \frac{2e^{-x_{16}^2-1}}{\pi(4+4x_{16})} f_2(x_{16}, y_{16}))$$
$$-cos(2\eta\sqrt{b't})(\frac{e^{-x_{16}^2}}{2\pi x_{16}} sin(2x_{16}y_{16}) + \frac{2e^{-x_{16}^2-1/4}}{\pi(1+4x_{16}^2)} g_1(x_{16}, y_{16}) + \frac{2e^{-x_{16}^2-1}}{\pi(4+4x_{16})} g_2(x_{16}, y_{16}))], \tag{3. 35}$$

where

$$\begin{aligned}
x_1 &= \eta\sqrt{Pr},\ \ y_1 = -\sqrt{b'},\ \ x_2 = \eta\sqrt{Pr},\ \ y_2 = \sqrt{b'},\ \ x_3 = \eta\sqrt{Sc},\ \ y_3 = -\sqrt{b'},\\
x_4 &= \eta\sqrt{Sc},\ \ y_4 = \sqrt{b'},\ \ x_5 = \eta + \sqrt{\frac{wt}{2}},\ \ y_5 = \sqrt{\frac{wt}{2}},\ \ x_6 = \eta - \sqrt{\frac{wt}{2}},\\
y_6 &= -\sqrt{\frac{wt}{2}},\ \ x_7 = \eta + \sqrt{\frac{wt}{2}},\ \ y_7 = -\sqrt{\frac{wt}{2}},\ \ x_8 = \eta - \sqrt{\frac{wt}{2}},\ \ y_8 = \sqrt{\frac{wt}{2}},\\
x_9 &= \eta,\ \ y_9 = -\sqrt{b't},\ \ x_{10} = \eta,\ \ y_{10} = \sqrt{b't},\ \ x_{11} = \eta,\ \ y_{11} = -\sqrt{b't},\\
x_{12} &= \eta,\ \ y_{12} = \sqrt{b't},\ \ x_{13} = \eta\sqrt{Pr},\ \ y_{13} = -\sqrt{b't},\ \ x_{14} = \eta\sqrt{Pr},\ \ y_{14} = \sqrt{b't},\\
x_{15} &= \eta\sqrt{Sc},\ \ y_{15} = -\sqrt{(a+k)t},\ \ x_{16} = \eta\sqrt{Sc},\ \ y_{16} = \sqrt{(a+k)t}.
\end{aligned}$$

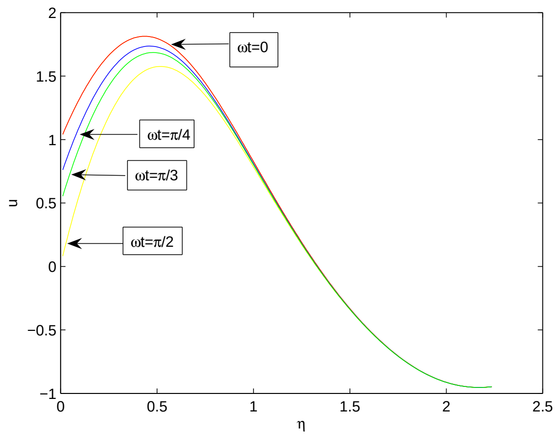

FIGURE 1. Profile of the expression for velocity given by Eq. (3.28) versus similarity parameter for $Pr = 7$, $K = 2$, $Gr = 2$, $Gc = 2$, $Sc = 0.6$, $t = 0.2$, $a = 1$, $b = 2$ and for different values of phase angle.

4. DISCUSSION OF RESULTS

For physical review, we have plotted values of velocity and concentration for real parts against the values of η. We observed the effect on velocity and concentration profiles

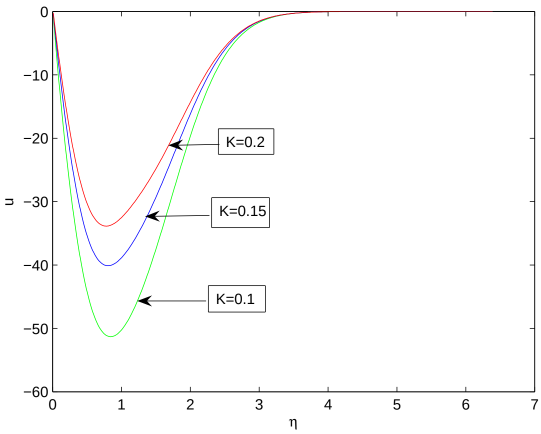

FIGURE 2. Profile of the expression for velocity given by Eq. (3.28) versus similarity parameter for $Pr = 7,\ Sc = 0.6,\ t = 0.3,\ \omega t = \pi/3,\ Gr = 5,\ Gc = 10,\ a = 1,\ b = 2$ and for different values of K.

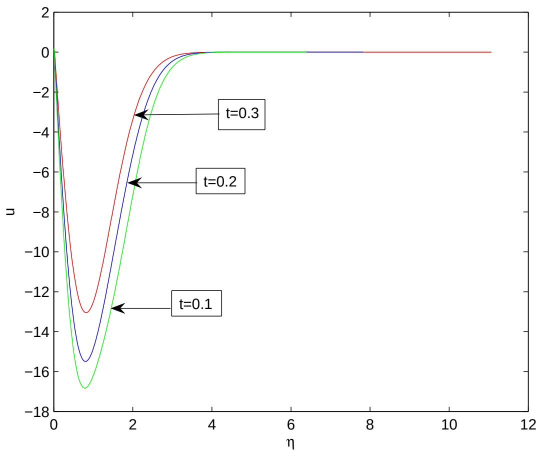

FIGURE 3. Profile of the expression for velocity given by Eq. (3.28) versus similarity parameter for $Pr = 7,\ K = 0.2,\ \omega t = \pi/3,\ Gr = 5,\ Gc = 5,\ Sr = 0.6,\ a = 1,\ b = 2$ and for different values of time.

varying the values of involved parameters like reaction of chemical parameter, Schmidt number, time and phase angle. The nature of flow and transport is also observed by varying K, Sc, t and ωt.

In Fig. 1, velocity profile given by Eq. (3.28) is depicted against phase angle ωt, we analyzed that the velocity will increase when the decrease occur in phase angle. In Fig. 2, opposite effect is observed in case of chemical reaction parameter against velocity that is the velocity will increase with increase in K keeping the time constant. The effect of velocity for different values of time is shown in Fig. 3, it is observed that velocity increases with increase in time with respect to the other parameters involved like Pr, Gr,

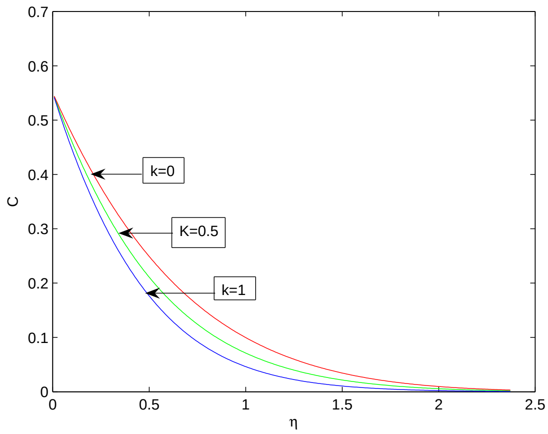

FIGURE 4. Profile of the expression for concentration given by Eq. (3.23) versus similarity parameter for $Sc = 0.6,\ t = 0.4,\ a = 1,\ b = 2$ and for different values of K.

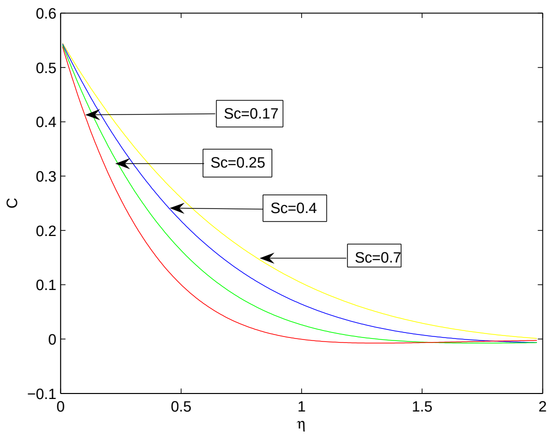

FIGURE 5. Profile of the expression for concentration given by Eq. (3.23) versus similarity parameter for $K = 2,\ t = 0.4,\ a = 1,\ b = 2$ and for different values of Schmidt number.

Gc, Sc and ωt. The effects of concentration for reaction of chemical parameter (K) and Schmidt number (Sc) are shown in Figs. (4) and (5). All of these figures showing that the concentration is decreasing when increase occur in K with respect to other two constant parameters (Schmidt number and time) while it is an increasing function for Schmidt number for constant values of K and t.

5. Conclusion

In this paper, we considered unsteady flow of incompressible and viscous fluid over an infinite vertical plate. Initially, the plate and fluid is in rest position with same temperature T_∞ and concentration C'_∞. We take x'-axis is along the plate and y'-axis be at the horizontal direction perpendicular to the x'-axis. After that when time $t' = 0^+$, the plate begins moving about its initial position with velocity $u_0 cos\omega' t'$, where $\omega' t'$ is phase angle and u_0 is the velocity of plate. The temperature of the plate raised to $T_\infty + T_\omega(1 - a' exp(-b't'))$ that changes exponentially, where T_ω is constant temperature. Assuming that the effect of viscous dissipation is trivial. We obtained the governing equations by considering the boundary layer and Boussinesq approximation.

Firstly the obtained governing equations are linearized and then the exact solution corresponding to unsteady flow of a viscous in-compressible fluid passing through an infinite oscillating wall vertically with different temperature and mass diffusion are obtained by taking into consideration the homogeneous chemical reaction of first order. The temperature and concentration level of the plate increased linearly with respect to time. The technique of Laplace transform is used to solve the linearized dimensionless equations. The plate will oscillate in its own plane harmonically. The profiles of velocity and concentration are analyzed for different physical parameters like phase angle, chemical reaction parameter, mass Grashöf number, thermal Grashöf number, Schmidt number and time. The numerical values corresponding to the different parameters are designed for physical vision. In the light of above discussion and results, we are able to give the following remarks:

- Velocity of the fluid increases with decrease in phase angle, while it is increasing with increase in chemical reaction parameter and time.
- Concentration of the fluid increases with decrease in chemical reaction parameter while it is increasing with increase in Schmidt number.

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