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Computational Method Based on Bernstein Polynomials for Solving a Fractional Optimal Control Problem

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Abstract. The study of optimal control problems (OCPs) are of great importance in our day life. In literature, there are many articles on the numerical solutions of OCPs on different mathematical methods. In this paper we are supposed to have the numerical solution of a fractional OCP by the help of operational matrices of Bernstein polynomials(BPs). Operational matrices of BPs are one of the most popular numerical approach for this study. This approach is quite simple in handling and very good results have been observed in literature by the virtue of this method. The simplicity and efficiency of the proposed method is checked on some illustrative examples.

AMS (MOS) Subject Classification Codes: 34A08; 49-XX; 41AXX; 65KXX
Key Words: Fractional optimal control problems, Bernstein polynomials, Operational matrices.

1. INTRODUCTION

The applications of fractional calculus can be studied in many scientific disciplines based on mathematical modeling including physics, aerodynamics, chemistry, signal and image processing, economics, electrodynamics, polymer rheology, economics, blood flow phenomena, biophysics, control theory and many others. For this, we recommend to [11, 17, 20, 21, 22]. Due to the wide range applications of fractional calculus and fractional dynamics, this area caught the interest of many researchers recently in [9, 8, 7, 6, 4].

Numerical solutions of fractional order differential equations have considered by many researchers such as in [13], H. Jafari et al. studied Abel differential equation of fractional order using Homotopy analysis method. In [5], D. Baleanu et al. consider fractional quadratic Riccati differential equations with the Riemann-Liouville derivative via the operational matrices of BPs. In [1], M. Alipour and D. Rostamy used BPs for solving Abel's integral equation. In [2], M. Alipour and D. Rostamy consider the numerical solution of nonlinear FDE via BPs. In [12], M. Inc et al. have studied the numerical solutions of one dimensional Telegraph equations of second kind by reproducing kernel Hilbert space method. In [24], V. K. Srivastava et al. studied the solution of Telegraph equation by reduced differential transform method. In [10], M. Garg et al. studied space time fractional telegraph equation by generalized differential transform method.

The OCPs have studied by many scientist in different scientific disciplines using different mathematical tools such as in [19], A. Lotfi et al. introduced Legendre orthonormal polynomial basis for the numerical solution of an OCP. In [25], E. Tohidi et al. considered and OCP by utilizing the idea of approximation by monomials and collocation technique for a uniform mesh. In [14], M. M. Khader and A. S. Hendy used Chebyshev polynomials approximation and finite difference method and produced a scheme for the numerical solution of OCPs. Our interest in this paper is to produce a scheme for the numerical solution of the fractional order OCP:
Minimize

$$J(x,u,b) = \int_0^b L(t, x(t), u(t))dt, \tag{1. 1}$$

subjected to the dynamical system

$$\mathcal{M}_1 \dot{x}(t) + \mathcal{M}_2 D^\alpha x(t) = f(t, x(t), u(t)), \quad \alpha \in (0, 1], \tag{1. 2}$$

$$x(0) = x_0, \quad x(b) = x_b, \tag{1. 3}$$

where $\mathcal{M}_1, \mathcal{M}_2$ are constants and f, L are multi-variable polynomials of $t, x(t), u(t)$. The scheme is produced by the help of operational matrices for fractional order integration of Bernstein polynomials(BPs), see for detail [16, 23]. By this numerical approach, the optimal control problem is converted to a system of algebraic equations. This scheme for the solution of optimal control is a very useful and efficient. The efficiency of the proposed technique and our numerical scheme is checked by some illustrative examples.

Organization of the Paper: In section 1, we have cited the most appropriate and related work with this scientific approach for the numerical solution of fractional OCP, as an introduction. Section 2, consisting of Preliminary results. Section 3 is for numerical scheme of the OCP (1. 1)-(1. 3). In section 4, we have checked the numerical scheme by some illustrative examples.

2. Preliminary Results

DEFINITION 1. [6] *The Caputo fractional derivative for a real value α is defined by*

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\frac{d^n}{d\tau^n} f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau, & n-1 < \alpha < n, \quad n \in I\!N, \\ \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2. 4}$$

DEFINITION 2. [6] *For an arbitrary order α, the Riemann-Liouville integral is denied by*

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0. \tag{2. 5}$$

We give some properties of fractional derivative and integral from the available resources in [20, 21, 22]:

$$(i) \qquad D_t^{\alpha} C = 0, \qquad (C\ is\ a\ constant),$$

$$(ii) \qquad D_t^{\alpha} t^{\beta} = \begin{cases} 0 & \beta \in I\!N_0,\ \beta < \lceil \alpha \rceil \\ \\ \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, & \beta \in I\!N_0,\ \beta \geq \lceil \alpha \rceil\ or\ \beta \notin I\!N_0,\ \beta > \lfloor \alpha \rfloor, \end{cases} \tag{2. 6}$$

$$(iii) \qquad I_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \qquad n-1 < \alpha \leq n. \tag{2. 7}$$

DEFINITION 3. [7] *The Bernstein polynomilas of mth degree on $[0, b]$ is defined as*

$$B_{i,m}(t) = \binom{m}{i} \frac{t^i (b-t)^{m-i}}{b^m},\ 0 \leq i \leq m. \tag{2. 8}$$

For the detail description of the following results, we recommend the readers to [18, 3, 16, 23, 15].

LEMMA 2.1. *For A an upper triangular matrix of size $(m+1)\times(m+1)$ with entries of coefficients of BPs and $\mathcal{T} = [1, t, t^2, \ldots, t^m]$, The array $\Phi(t) = [\beta_{0,m}(t), \beta_{1,m}(t), \ldots, \beta_{m,m}(t)]^T$, can be expressed by $\Phi(t) = A\mathcal{T}$.*

Let Q be a square matrix of size $(m+1) \times (m+1)$, defined by

$$Q = \int_0^b \Phi(t)\Phi^T(t) dt. \tag{2. 9}$$

LEMMA 2.2. *Let $L^2[0, b]$ be a Hilbert space with the inner product $\langle f, g \rangle = \int_0^b f(t)g(t)dt$ and $y \in L^2[0, b]$. The we can fined a unique vector $c = [c_0, c_1, \ldots, c_m]^T$ such that*

$$y(t) \approx \sum_{i=0}^{m} c_i \beta_{i,m}(t) = c^T \Phi(t),$$

where $c^T = \langle y, \Phi \rangle Q^{-1}$, such that

$$\langle y, \Phi \rangle = \int_0^b y(t)\Phi(t)^T dt = [\langle y, \beta_{0,m} \rangle, \langle y, \beta_{1,m} \rangle, \ldots, \langle y, \beta_{m,m} \rangle],$$

and $Q = (Q_{i,j})_{i,j=1}^{m+1}$ *is as follows*

$$Q_{i+1,j+1} = \int_0^b \beta_{i,m}(t)\beta_{j,m}(t)dt = \frac{b\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}, \quad i,j = 0,1,\ldots,m.$$

LEMMA 2.3. *Suppose that* $C_{(m_1)\times 1}$ *is an arbitrary vector, The operational matrix of product* $\hat{C}_{(m+1)\times(m+1)}$ *using BPs can be given*

$$c^T\Phi(t)\Phi(t)^T \approx \Phi(t)^T\hat{C}. \tag{2. 10}$$

Corrollary 2.6a *Suppose* $y(t) \approx c^T\Phi(t)$, $x(t) \approx d^T\Phi(t)$ *and* $\hat{C}_{(m+1)\times(m+1)}$ *be the operational matrix of product using BPs for vector* c. *Then the approximation of product* $x(t)y(t)$ *can be expressed by*

$$x(t)y(t) \approx \Phi(t)\hat{C}d. \tag{2. 11}$$

Corrollary 2.6b *Suppose* $y(t) \approx c^T\Phi(t)$ *and* $\hat{C}_{(m+1)\times(m+1)}$ *be the operational matrix of product using BPs for vector* c, *then the approximation of function* $y^k(t)$, $(k \in N)$ *is given by*

$$y^k(t) \approx \Phi(t)\hat{C}_k. \tag{2. 12}$$

where $\hat{C}_k \approx \hat{C}^{k-1}c$.

LEMMA 2.4. *For* $\Phi(t) = [\beta_{0,m}(t), \beta_{1,m}(t), \ldots, \beta_{m,m}(t)]^T$, *the operational matrix of fractional order integral* I^α *for* α *an arbitrary real valued number is given by*

$$I_t^\alpha\Phi(t) \approx I^\alpha\Phi(t), \tag{2. 13}$$

where $I^\alpha = ADE^T$, *is a square matrix of* $(m+1)\times(m+1)$.

3. SCHEME FOR THE APPROXIMATE SOLUTION OF FRACTIONAL OPTIMAL CONTROL PROBLEM

The approximate solution of fractional order optimal control problems have considered by many scientist as discussed in introduction. Their approach to their approximations were on different mathematical methods. Here we are producing a numerical scheme for the solutions of fractional order optimal control problem (1. 1), (1. 2) by the help of operational matrices of Bernstein polynomials. The approach is too simple and powerful. For this purpose we take start from the approximation of $\dot{x}(t)$ using lemma (2.2) as under

$$\dot{x}(t) \approx \Lambda_1^T\Phi(t), \tag{3. 14}$$

where $\Lambda_1^T = [\mu_0, \mu_1, \ldots, \mu_m]$. Integrating (3. 14) and using initial condition in (1. 2) we have

$$x(t) \approx \Lambda_1^T I^1\Phi(t) + x_0 = \Lambda_1^T I^1\Phi(t) + X_0^T\Phi(t) = \mathcal{H}_1^T\Phi(t), \tag{3. 15}$$

were $X_0^T = [x_0, x_0, \ldots, x_0]$ and $\mathcal{H}_1^T = \Lambda_1^T I^1 + X_0^T$. Similarly, by the use of (2. 7) and (3. 14) we can have the approximation

$$D^\alpha x(t) = I^{1-\alpha}\dot{x}(t) \approx I^{1-\alpha}\Lambda_1^T\Phi(t) = \mathcal{H}_2^T\Phi(t) \tag{3. 16}$$

where $\mathcal{H}_2^T = \Lambda_1^T I^{1-\alpha}$, and using Lemma (2.2), we have

$$u(t) \approx \Lambda_2^T\Phi(t) \tag{3. 17}$$

where $\Lambda_2^T = [\nu_0, \nu_1, \ldots, \nu_m]$. Thus

$$J(x,u,T) \approx \int_0^b L(t, \mathcal{H}_1^T\Phi(t), \Lambda_2^T\Phi(t))dt \tag{3. 18}$$

from lemma 2.3 and Corollaries $2.6a, 2.6b$ we can approximate $L(t, \mathcal{H}_1^T\Phi(t), \Lambda_2^T\Phi(t))$ by

$$L(t, \mathcal{H}_1^T\Phi(t), \Lambda_2^T\Phi(t)) \approx \mathcal{L}(\mathcal{H}_1, \Lambda_2)\Phi(t) \qquad (3.\ 19)$$

where $\mathcal{L} : \mathbb{R}^{(m+1)\times 1} \times \mathbb{R}^{(m+1)\times 1} \to \mathbb{R}^{1\times(m+1)}$, thus

$$J^*(\mathcal{H}_1, \Lambda_2) = \int_0^b \mathcal{L}(\mathcal{H}_1, \Lambda_2)\Phi(t)dt \qquad (3.\ 20)$$

by the help of (3. 14)–(3. 17) the dynamical system (1. 2) becomes

$$\mathcal{M}_1\Lambda_1^T\Phi(t) + \mathcal{M}_2\mathcal{H}_2^T\Phi(t) = f(t, \mathcal{H}_1^T\Phi(t), \Lambda_2^T\Phi(t)) \qquad (3.\ 21)$$

by the help of Lemma 2.3 and Corollaries $2.6a, 2.6b$, we can get the approximation

$$f(t, \mathcal{H}_1^T\Phi(t), \Lambda_2^T\Phi(t)) \approx \mathcal{F}(\mathcal{H}_1, \Lambda_2)\Phi(t) \qquad (3.\ 22)$$

where $\mathcal{F} : \mathbb{R}^{(m+1)\times 1} \times \mathbb{R}^{(m+1)\times 1} \to \mathbb{R}^{1\times(m+1)}$. By the help of (3. 21), (3. 22) we have the following equation

$$\mathcal{M}_1\Lambda_1^T\Phi(t) + \mathcal{M}_2\mathcal{H}_2^T\Phi(t) - \mathcal{F}(\mathcal{H}_1, \Lambda_2)\Phi(t) = 0 \qquad (3.\ 23)$$

and thus the dynamical system (1. 2) gets the form

$$\mathcal{M}_1\Lambda_1^T + \mathcal{M}_2\mathcal{H}_2^T - \mathcal{F}(\mathcal{H}_1^T, \Lambda_2^T) = 0. \qquad (3.\ 24)$$

defining, the Lagrangian function as:

$$\mathcal{L}^*[\mathcal{H}_1, \Lambda_2, \lambda] = J^*[\mathcal{H}_1, \Lambda_2] + (\mathcal{M}_1\Lambda_1^T + \mathcal{M}_2\mathcal{H}_2^T - \mathcal{F}(\mathcal{H}_1, \Lambda_2))\lambda + \hat{\lambda}(\mathcal{H}_1^T\Phi(b) - x_b), \qquad (3.\ 25)$$

where $\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_m]^T$, $\hat{\lambda}$ are the unknown Lagrange multipliers. For the minimization of the optimal control problem (1. 1), (1. 2) we have the following necessary conditions

$$\frac{\partial}{\partial\Lambda_2}\mathcal{L}^*[\mathcal{H}_1, \Lambda_2, \lambda] = 0, \quad \frac{\partial}{\partial\mathcal{H}_1}\mathcal{L}^*[\mathcal{H}_1, \Lambda_2, \lambda] = 0, \quad \frac{\partial}{\partial\lambda}\mathcal{L}^*[\mathcal{H}_1, \Lambda_2, \lambda] = 0, \quad \mathcal{H}_1^T\Phi(b) - x_b = 0 \qquad (3.\ 26)$$

from here, we can get $\Lambda_2, \mathcal{H}_1$ and ultimately we can approximate $x(t), u(t)$ from (3. 15), (3. 17) respectively.

4. Numerical Examples

Below we use the presented approach in order to get the numerical solutions of two OCPs for different values of α and the approximating terms m.

EXAMPLE 1. *We consider the OCP*

$$\textit{Minimize } J = \int_0^1 (tu(t) - (\alpha+2)x(t))^2 dt, \qquad (4.\ 27)$$

subjected to the dynamical system

$$x'(t) + D^\alpha x(t) = u(t) + t^2, \qquad (4.\ 28)$$

with initial and boundary conditions

$$x(0) = 0, \quad x(1) = \frac{2}{\Gamma(\alpha+3)}, \qquad (4.\ 29)$$

the exact solution is given by $x(t) = \frac{2t^\alpha+2}{\Gamma(\alpha+3)}, \quad u(t) = \frac{2t^\alpha+1}{\Gamma(\alpha+2)}$

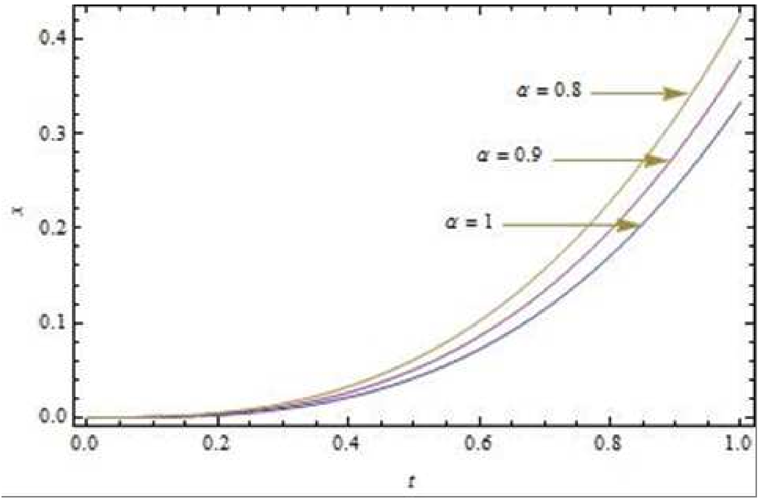

FIGURE 1. Plot of $x(t)$ for $m = 7$ and $\alpha = .8, .9, 1$ in example 1.

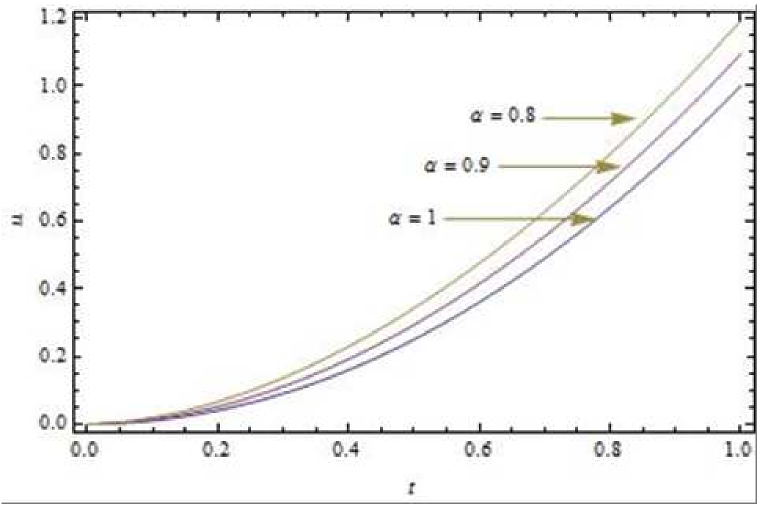

FIGURE 2. Plot of $u(t)$ for $m = 7$ and $\alpha = .8, .9, 1$ in example 1.

For $m = 7$, the approximate solutions of the states functions and the control functions are plotted in the figures 1 and 2. These figures show that as $\alpha \to 1$ the approximate solutions get close to the exact solution. Also, the error in the norm $L^2(\|f(t)\|_{L^2[a,b]} = \sqrt{\int_a^b |f(t)|^2 dt})$ for approximate solutions obtained are given in tables 1, 2 for $x(t)$, $u(t)$ respectively for different values of approximating terms m.

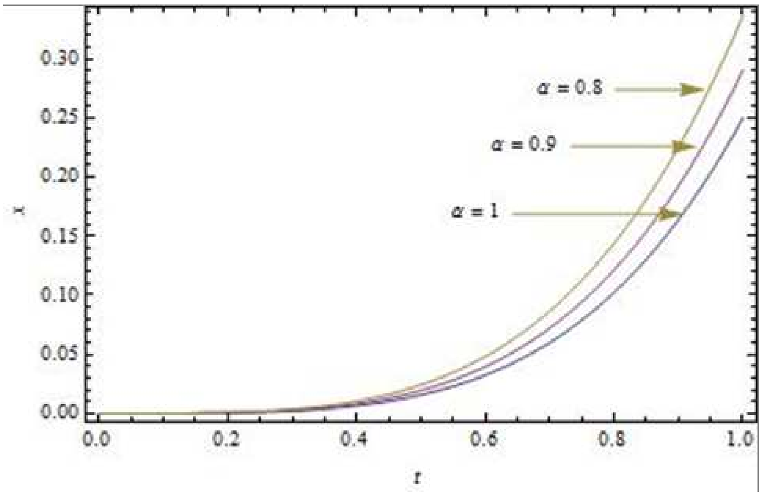

FIGURE 3. Plot of $x(t)$ for $m = 8$ and $\alpha = .8, .9, 1$ in example 2.

Table 1. $\|x(t) - x_m(t)\|$ for different values of α and m in example 1.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
7	$2.89933 * 10^{-17}$	$7.5999 * 10^{-7}$	$1.89168 * 10^{-6}$
9	$2.41324 * 10^{-17}$	$1.65651 * 10^{-7}$	$4.27512 * 10^{-7}$
12	$1.03578 * 10^{-17}$	$2.89815 * 10^{-8}$	$7.7282 * 10^{-8}$

Table 2. $\|u(t) - u_m(t)\|$ for different values of α and m in example 1.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
7	$1.93584 * 10^{-16}$	0.0000120377	0.0000314255
9	$3.53488 * 10^{-16}$	$4.09026 * 10^{-6}$	0.0000112193
12	$8.26356 * 10^{-17}$	$1.16833 * 10^{-6}$	$3.38702 * 10^{-6}$

EXAMPLE 2. *We consider the OCP*

$$\textit{Minimize } J = \int_0^1 (u(t) - x(t))^2 dt, \tag{4. 30}$$

subjected to the dynamical system

$$x'(t) + D^\alpha x(t) = u(t) - x(t) + \frac{6t^\alpha + 2}{\Gamma(\alpha + 3)}, \tag{4. 31}$$

with initial and boundary conditions

$$x(0) = 0, \quad x(1) = \frac{6}{\Gamma(\alpha + 4)}, \tag{4. 32}$$

the exact solution is given by $x(t) = u(t) = \frac{6t^\alpha + 3}{\Gamma(\alpha+4)}$

In figures 3 and 4, we can observe plots of the approximate solutions of the states function and the control function for $m = 8$ and $\alpha = 0.8, 0.9, 1.0$. Moreover, in tables 3 and 4, we can see the errors in the norm L^2 for the obtained solutions.

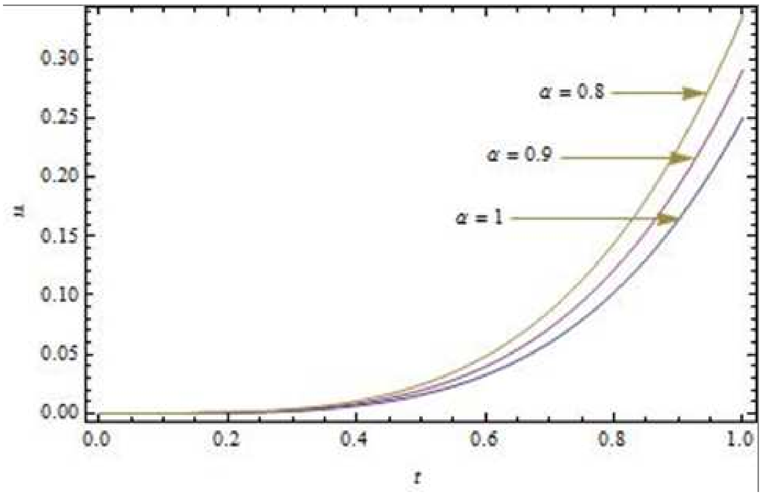

FIGURE 4. Plot of $u(t)$ for $m = 8$ and $\alpha = .8, .9, 1$ in example 2.

Table 3. $\|x(t) - x_m(t)\|$ for different values of α and m in example 2.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
4	$1.11714 * 10^{-18}$	0.0000288778	0.0000575561
8	$7.53703 * 10^{-19}$	$7.17619 * 10^{-8}$	$1.80442 * 10^{-7}$
12	$1.06883 * 10^{-20}$	$3.14699 * 10^{-9}$	$8.29973 * 10^{-9}$

Table 4. $\|u(t) - u_m(t)\|$ for different values of α and m in example 2.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
4	$5.87904 * 10^{-18}$	0.000112289	0.000221961
8	$9.3817 * 10^{-19}$	$2.92602 * 10^{-7}$	$7.30618 * 10^{-7}$
12	$7.79996 * 10^{-20}$	$1.31311 * 10^{-8}$	$3.42575 * 10^{-8}$

5. CONCLUSION

In this paper, we have estimated the functions, product of functions, power of function and fractional order integral of function by Bernstein polynomials. Then, using these approximations, we have reduced the original fractional optimal control problem to an optimization problem. So, by the Lagrange multipliers method for the obtained optimization problem, a system of algebraic equations was resulted which is simple in handling. For the efficiency of the scheme, we considered two illustrative examples for different number of approximating terms and it was observed that the results obtained in both examples are too close to the exact solutions and the results become too better as we exceeded the value of m. By these examples, it is confirmed that the scheme produced is a simple and efficient for the numerical solutions of OCPs and further better results can be obtained by increasing the number of approximating terms.

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