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Self-Inversive Bicomplex Polynomials

Sara Ahmadi
Department of Mathematics,
Semnan University, Semnan, Iran
Email: sa_ahmadi58@yahoo.com

Mahmood Bidkham
Department of Mathematics,
Semnan University, Semnan, Iran
Email: mdbidkham@gmail.com

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Abstract. In this paper, we introduce a new class of bicomplex polynomials, namely self-inversive bicomplex polynomials, and investigate the necessary and sufficient condition for any bicomplex polynomial to be self-inversive. We also study some other properties of this class of bicomplex polynomial with restricted coefficients.

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1. Introduction and Statement of Results

Let $\mathbb{C}_2$ be the bicomplex algebra, i.e.,

$$\mathbb{C}_2 = \{x_1 + ix_2 + j(x_3 + ix_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\},$$

with $i^2 = -1, j^2 = -1$ and $ij = ji$.
We remark that one can write bicomplex number $x_1 + ix_2 + j(x_3 + ix_4)$ as $z_1 + jz_2$ where $z_1, z_2 \in \mathbb{C}_1 = \{x + iy : x, y \in \mathbb{R}, i^2 = -1\}$. Thus, $\mathbb{C}_2$ can be considered as the complexification of the usual complex numbers $\mathbb{C}_1$ and a bicomplex number can be considered as an element of $\mathbb{C}^2$. $\mathbb{C}_2$ is considerably simplified by the introduction of two bicomplex numbers e_1 and e_2 defined as $e_1 = \dfrac{1+ij}{2}$ and $e_2 = \dfrac{1-ij}{2}$. For any bicomplex number $Z = z_1 + jz_2 \in \mathbb{C}_2$, one can write:

$$Z = \alpha e_1 + \beta e_2,$$

where $\alpha = z_1 - iz_2$ and $\beta = z_1 + iz_2$ are uniquely defined complex numbers (see [14, Theorem 6.4, page 19] and [1, page 15]).
For bicomplex numbers, there are three possible conjugations. Let $Z \in \mathbb{C}_2$ and $z_1, z_2 \in \mathbb{C}_1$

such that $Z = z_1 + jz_2 \in \mathbb{C}_2$. Then we define the three conjugations as:

$$Z^{\dagger_1} = (z_1 + jz_2)^{\dagger_1} := \overline{z_1} + j\overline{z_2},$$
$$Z^{\dagger_2} = (z_1 + jz_2)^{\dagger_2} := z_1 - jz_2,$$
$$Z^{\dagger_3} = (z_1 + jz_2)^{\dagger_3} := \overline{z_1} - j\overline{z_2}.$$

In this paper, we denote $Z^{\dagger_3}$ with $\overline{Z}$, i.e.,

$$\overline{Z} = \overline{z_1} - j\overline{z_2},$$

and if we write $Z = \alpha e_1 + \beta e_2$ where $\alpha, \beta \in \mathbb{C}_1$, then $\overline{Z} = \overline{\alpha} e_1 + \overline{\beta} e_2$.
In the complex case the modulus of a complex number is intimately related with the complex conjugation. Similarly, accordingly to each of the three conjugations, three possible moduli arise:

$$|Z|_j = ZZ^{\dagger_1} = (|z_1|^2 - |z_2|^2) + 2Re(z_1\overline{z_2})j,$$
$$|Z|_i = ZZ^{\dagger_2} = z_1^2 + z_2^2,$$
$$|Z|_{ij} = ZZ^{\dagger_3} = (|z_1|^2 + |z_2|^2) - 2Im(z_1\overline{z_2})ij.$$

Also the norm of $Z = z_1 + jz_2$ define as follows

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2}.$$

We can easily show that if $Z = \alpha e_1 + \beta e_2$ where $\alpha, \beta \in \mathbb{C}_1$, then

$$\|Z\| = \sqrt{\frac{|\alpha|^2 + |\beta|^2}{2}}.$$

Suppose $Z, W \in \mathbb{C}_2$ such that $ZW = 1$, then Z and W is said to be the inverse of each other. An element which has an inverse is said to be invertible (non-singular) and an element which does not have an inverse is said to be non-invertible (singular). For a bicomplex number $Z = \alpha e_1 + \beta e_2 \in \mathbb{C}_2$, it is easy to verify that Z is invertible if and only if $\alpha, \beta \neq 0$; in this case if we denote the inverse of Z by Z^{-1}, then we have

$$Z^{-1} = \alpha^{-1} e_1 + \beta^{-1} e_2.$$

The following definition will be useful to construct a "discus" in $\mathbb{C}_2$.

Definition 1. *We say that $X \subseteq \mathbb{C}_2$ is a cartesian set determined by X_1 and X_2 if*

$$X = X_1 \times_e X_2 := \{z_1 + jz_2 \in \mathbb{C}_2 : z_1 + jz_2 = \alpha e_1 + \beta e_2, (\alpha, \beta) \in X_1 \times X_2\}.$$

A special cartesian set in $\mathbb{C}_2$, which is called a discus is defined as follows:

Definition 2. *Let $a = a_1 + jb_1 = \alpha_1 e_1 + \beta_1 e_2$ where $a_1, b_1, \alpha_1, \beta_1 \in \mathbb{C}_1$, be a fixed point in $\mathbb{C}_2$. We define the discus with center a and radii r_1 and r_2 and denote it by $D(a; r_1, r_2)$ as follows* [14, Definition 9.1, page 45]*:*

$$D(a; r_1, r_2) = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1 + jz_2 = \alpha e_1 + \beta e_2, |\alpha - \alpha_1| < r_1, |\beta - \beta_1| < r_2\}.$$

The discus $D(0; 1, 1)$ is called unit discus and when we say that a bicomplex number $Z = \alpha e_1 + \beta e_2$ lies on the unit discus it means that $|\alpha| = 1$ and $|\beta| = 1$, but if $\delta = \alpha e_1 + \beta e_2$, such that $\|\delta\| = 1$, then, it does not imply that $|\alpha| = 1$ and $|\beta| = 1$.

For an open set U of $\mathbb{C}_2$, let $f : U \subseteq \mathbb{C}_2 \to \mathbb{C}_2$ be a bicomplex function. There is a definition for the derivative of a bicomplex function which looks quite similar to its complex counterpart [6, Definition 1, page 4].

Definition 3. *The derivative of the function f at a point $Z_0 \in U$ is the limit, if it exists,*

$$f'(Z_0) := \lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0},$$

for Z in the domain of f such that $Z - Z_0$ is an invertible bicomplex number.
We shall say that the function f is bicomplex holomorphic ($\mathbb{C}_2-$holomorphic) on an open set U if and only if f is $\mathbb{C}_2-$differentiable at each point of U.
For further details on bicomplex analysis, we refer the reader to [1, 7, 8, 9, 10, 14] and references therein.

Let $\mathbb{BP}_n$ denote the class of bicomplex polynomials $P(Z) = \sum_{k=0}^{n} A_k Z^k$ of degree n with $A_k \in \mathbb{C}_2$ for all $0 \leq k \leq n$. We know that a complex polynomial P with zeros $\{z_1, ..., z_n\}$ is self-inversive if $\{z_1, ..., z_n\} = \{\frac{1}{\overline{z_1}}, ..., \frac{1}{\overline{z_n}}\}$. Some properties of complex self-inversive polynomials have been studied [12]. Here we first define the bicomplex self-inversive polynomials and then study some of its properties.

Definition 4. *Let $P \in \mathbb{BP}_n$ have at least one invertible root. $P(Z)$ is self-inversive if and only if $P(Z) = 0$ implies $P(\frac{1}{\overline{Z}}) = 0$.*

Remark 1. *All the zeros of a self-inversive bicomplex polynomial are invertible.*

Let

$$P(Z) = \sum_{k=0}^{n} A_k Z^k,$$

be a bicomplex polynomial of degree n, with $Z = z_1 + jz_2 = \alpha e_1 + \beta e_2$ and bicomplex coefficients $A_k = \gamma_k e_1 + \delta_k e_2$, for $k = 0, 1, ..., n$. Then $Z^k = \alpha^k e_1 + \beta^k e_2$ and we can rewrite $P(Z)$ as

$$P(Z) = \sum_{k=0}^{n} (\gamma_k \alpha^k) e_1 + \sum_{k=0}^{n} (\delta_k \beta^k) e_2 =: \phi(\alpha) e_1 + \psi(\beta) e_2, \tag{1. 1}$$

where $\phi(\alpha)$ and $\psi(\beta)$ are complex polynomials of degree at most n and we have the following theorem [10, Theorem 8, page 71]:

Theorem 1.1. (Analogue of the Fundamental Theorem of Algebra for bicomplex polynomials) *Consider a bicomplex polynomial $P(Z) = \sum_{k=0}^{n} A_k Z^k$. If all the coefficients A_k with the exception of the free term $A_0 = \gamma_0 e_1 + \delta_0 e_2$ are complex multiple of e_1 (respectively of e_2), but A_0 has $\delta_0 \neq 0$ (respectively $\gamma_0 \neq 0$), then $P(Z)$ has no roots. In all other cases, $P(Z)$ has at least one root.*

In recent years, the theory of bicomplex numbers and bicomplex functions has found many applications, see for instance [3, 4, 16, 17, 18]. Bicomplex numbers are a commutative ring with unity which contains the field of complex numbers and the commutative ring of hyperbolic numbers. Bicomplex (hyperbolic) numbers are unique among the complex (real) Clifford algebras in that they are commutative but not division algebras.

In this paper, we investigate the necessary and sufficient condition for a bicomplex polynomial to be self-inversive and other related problems.

Theorem 1.2. *Let $P \in \mathbb{BP}_n$, where $P(Z) = \sum_{k=0}^{n}(\gamma_k \alpha^k)e_1 + \sum_{k=0}^{n}(\delta_k \beta^k)e_2 =: \phi(\alpha)e_1 + \psi(\beta)e_2$. Then $P(Z)$ is self-inversive if and only if $\phi(\alpha)$ and $\psi(\beta)$ are self-inversive complex polynomial of degree at most n.*

Remark 2. *If $P \in \mathbb{BP}_n$ is self-inversive and $P(Z) = \sum_{k=0}^{n} A_k Z^k =: \phi(\alpha)e_1 + \psi(\beta)e_2$, since ϕ and ψ are self-inversive complex polynomials hence we have*

$$A_{n-k} = \overline{A_k} \quad \text{for } k = 0, ..., n.$$

For every complex polynomial $P(z)$, one has

$$|z^n \overline{P(\frac{1}{\overline{z}})}| = |P(z)|$$

for every complex number z with $|z| = 1$ [2, page 1]. In this respect, for bicomplex polynomials, we prove the following theorem:

Theorem 1.3. *Let $P(Z)$ be a bicomplex polynomial of degree n, then for every Z on $D(0; 1, 1)$, we have*

$$\|Z^n \overline{P(\frac{1}{\overline{\overline{Z}}})}\| = \|P(Z)\|. \tag{1. 2}$$

In the next two theorems, we study some properties of bicomplex polynomials with restricted coefficients.

Theorem 1.4. *Let $P(Z) = \sum_{k=0}^{n} A_k Z^k$ be a bicomplex polynomial of degree n $(n \geq 1)$ such that A_n is invertible. If $P(Z)$ is a self-inversive bicomplex polynomial, then, there exists $\delta \in \mathbb{C}_2$ with $\|\delta\| = 1$ such that for every invertible bicomplex number Z, we have*

$$Z^n \overline{P(\frac{1}{\overline{\overline{Z}}})} = \delta P(Z). \tag{1. 3}$$

Also if there exists a bicomplex number $\delta = \delta_1 e_1 + \delta_2 e_2$ on the unit discus such that (1.3) is true, then, $P(Z)$ is a self-inversive bicomplex polynomial.

In what follows, if $P(Z) = \sum_{k=0}^{n} A_k Z^k$, then $\overline{P}(Z)$ denotes $\sum_{k=0}^{n} \overline{A_k} Z^k$.

Theorem 1.5. *If $P(Z) = \sum_{k=0}^{n} A_k Z^k$ is a bicomplex polynomial such that A_n is invertible, then the following are equivalent:*

(i) *P is self-inversive.*

(ii) *$\overline{A_n} P(Z) = A_0 Z^n \overline{P}(\frac{1}{Z})$ for each bicomplex invertible number Z.*

(iii) *$A_0 \overline{A_k} = \overline{A_n} A_{n-k}$; $k = 0, 1, ..., n$.*

Finally, we prove the following theorem:

Theorem 1.6. *If* $P(Z) = \sum_{k=0}^{n} A_k Z^k$ *is a self-inversive bicomplex polynomial such that* A_n *is invertible, then,*

(i) $\overline{A_n}[nP(Z) - ZP'(Z)] = A_0 Z^{n-1}\overline{P'}(\frac{1}{Z})$ *for each* $Z \in \mathbb{C}_2$.

(ii) $\|\frac{nP(Z)}{ZP'(Z)} - 1\| = 1$ *for each* Z *on* $D(0;1,1)$.

2. Lemmas

To prove these theorems, we require the following lemmas.

Lemma 2.1. *Let* X_1 *and* X_2 *be open sets in* $\mathbb{C}_1$. *If* $f_{e_1} : X_1 \longrightarrow \mathbb{C}_1$ *and* $f_{e_2} : X_2 \longrightarrow \mathbb{C}_1$ *are holomorphic functions of* $\mathbb{C}_1$ *on* X_1 *and* X_2 *respectively, then the function* $f : X_1 \times_e X_2 \longrightarrow \mathbb{C}_2$ *defined as*

$$f(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2, \quad \forall z_1 + jz_2 \in X_1 \times_e X_2,$$

is $\mathbb{C}_2-$*holomorphic on the open set* $X_1 \times_e X_2$ *and*

$$f'(z_1 + jz_2) = f'_{e_1}(z_1 - iz_2)e_1 + f'_{e_2}(z_1 + iz_2)e_2, \quad \forall z_1 + jz_2 \in X_1 \times_e X_2.$$

This lemma for derivative of a polynomial is proved by Charak et al. [5, Theorem 2.6, page 60] (see also [6, Theorem 2, page 6] and [15, page 136]).

Remark 3. *Let* $P(Z) = \sum_{k=0}^{n} A_k Z^k = \phi(\alpha)e_1 + \psi(\beta)e_2$ *be a bicomplex polynomial. In the above lemma, if we take* $X_1 = X_2 = \mathbb{C}_2$, *then* $P(Z)$ *is* $\mathbb{C}_2-$*holomorphic on* $\mathbb{C}_2$ *and*

$$P'(Z) = P'(z_1 + jz_2) = \phi'(z_1 - iz_2)e_1 + \psi'(z_1 + iz_2)e_2 =: \phi'(\alpha)e_1 + \psi'(\beta)e_2.$$

The following properties of self-inversive complex polynomial have been noted by O'hara et al.[12, Lemma 1, page 1].

Lemma 2.2. *If* $P(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$ *is a complex polynomial, then, the following are equivalent:*

(i) P *is self-inversive.*

(ii) $\overline{a_n}P(z) = a_0 z^n \overline{P}(\frac{1}{z})$ *for each complex number* z.

(iii) $a_0\overline{a_k} = \overline{a_n}a_{n-k}; \quad k = 0, 1, ..., n.$

The next lemma proved by O'hara et al. [12, Lemma 2, page 1].

Lemma 2.3. *If* $P(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$ *is a self-inversive complex polynomial and* $a_n \neq 0$, *then,*

(i) $\overline{a_n}[nP(z) - zP'(z)] = a_0 z^{n-1}\overline{P'}(\frac{1}{z})$ *for each* $z \in \mathbb{C}_1$.

(ii) $|\frac{nP(z)}{zP'(z)} - 1| = 1$ *for each* z *on* $|z| = 1$.

Regarding the number of zeros of a bicomplex polynomial, we have the following result[10, Corollary 9, page 71]:

Lemma 2.4. *Assume that a bicomplex polynomial $P(Z)$ of degree $n \geqslant 1$ has at least one root. Then,*

(1) *If at least one of the coefficients A_k, for $k = 0, ..., n$, is invertible, then $P(Z)$ has at most n^2 distinct roots.*
(2) *If all coefficients are complex multiples of e_1 (respectively e_2) then $P(Z)$ has infinitely many roots.*

Note that zeros of bicomplex polynomials were originally investigated in [13]. In this respect, we prove the following lemma:

Lemma 2.5. *If $P \in \mathbb{BP}_n$ is a self-inversive polynomial, then, $P(Z)$ has at most n^2 zeros.*

Proof. Suppose that $P(Z)$ has infinitely many roots, then $\psi \equiv 0$ (respectively $\phi \equiv 0$) and $\phi(\alpha)$ is a complex polynomial of degree n (similarly, $\psi(\beta)$ is a complex polynomial of degree n). If $\phi(a_1) = 0$, then $P(a_1e_1) = \phi(a_1)e_1 = 0$, but a_1e_1 is singular and this contradicts that $P(Z)$ is self-inversive (similarly, if $\phi \equiv 0$, we get a contradiction). Since $P(Z)$ is self-inversive, it has at least one root, so by Lemma 2.4, $P(z)$ has at most n^2 roots.

3. Proofs of the Theorems

Proof of Theorem 1.2. First, we suppose that $P(Z)$ be self-inversive. Then by Theorem 1.1, $\phi(\alpha)$ and $\psi(\beta)$ are complex polynomials with at least one root. Let $\alpha_1, \beta_1 \in \mathbb{C}_1$ and $\phi(\alpha_1) = \psi(\beta_1) = 0$, then we have $P(\alpha_1e_1 + \beta_1e_2) = 0$.
Since $P(Z)$ is self-inversive

$$P(\frac{1}{\overline{\alpha_1e_1 + \beta_1e_2}}) = P(\frac{1}{\overline{\alpha_1}}e_1 + \frac{1}{\overline{\beta_1}}e_2) = 0,$$

hence, $\phi(\frac{1}{\overline{\alpha_1}}) = \psi(\frac{1}{\overline{\beta_1}}) = 0$.
This implies that ϕ and ψ are self-inversive complex polynomials.
Conversly, if ϕ and ψ are self-inversive complex polynomials, then $P(Z)$ has at least one invertible root. Let $Z_1 = \alpha_1e_1 + \beta_1e_2 \neq 0$ such that $P(Z_1) = 0$, then, $\phi(\alpha_1) = \psi(\beta_1) = 0$, therefore

$$\phi(\frac{1}{\overline{\alpha_1}}) = \psi(\frac{1}{\overline{\beta_1}}) = 0$$

or

$$P(\frac{1}{\overline{\alpha_1}}e_1 + \frac{1}{\overline{\beta_1}}e_2) = 0.$$

This implies that $P(\frac{1}{\overline{Z_1}}) = 0$ and completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let $P(Z) = \phi(\alpha)e_1 + \psi(\beta)e_2$ where $\phi(\alpha)$ and $\psi(\beta)$ are complex polynomials of degree at most n. For every invertible bicomplex number $Z = \alpha e_1 + \beta e_2$, we have

$$\begin{aligned} Z^n\overline{P(\frac{1}{\overline{Z}})} &= (\alpha^n e_1 + \beta^n e_2)(\overline{\phi(\frac{1}{\overline{\alpha}})}e_1 + \overline{\psi(\frac{1}{\overline{\beta}})}e_2) \\ &= (\alpha^n\overline{\phi(\frac{1}{\overline{\alpha}})})e_1 + (\beta^n\overline{\psi(\frac{1}{\overline{\beta}})})e_2. \end{aligned}$$

If Z lies on the unit discus, then, we have $|\alpha| = |\beta| = 1$ and

$$|\alpha^n \overline{\phi(\frac{1}{\overline{\alpha}})}| = |\phi(\alpha)| \quad , \quad |\beta^n \overline{\psi(\frac{1}{\overline{\beta}})}| = |\psi(\beta)|,$$

hence

$$\begin{aligned}\|Z^n \overline{P(\frac{1}{\overline{Z}})}\| &= \sqrt{\frac{|\alpha^n \overline{\phi(\frac{1}{\overline{\alpha}})}|^2 + |\beta^n \overline{\psi(\frac{1}{\overline{\beta}})}|^2}{2}} \\ &= \sqrt{\frac{|\phi(\alpha)|^2 + |\psi(\beta)|^2}{2}} \\ &= \|P(Z)\|.\end{aligned}$$

Proof of Theorem 1.4. Let $P(Z) = \sum_{k=0}^{n} A_k Z^k = \phi(\alpha)e_1 + \psi(\beta)e_2$. First, we suppose that $P(Z)$ is a self-inversive bicomplex polynomial of degree n. By Theorem 1.2, $\phi(\alpha)$ and $\psi(\beta)$ are self-inversive complex polynomials and since A_n is invertible, hence $\phi(\alpha)$ and $\psi(\beta)$ are polynomials of degree n. Therefore, there exist $\delta_1, \delta_2 \in \mathbb{C}_1$, such that $|\delta_1| = |\delta_2| = 1$ and

$$\alpha^n \overline{\phi(\frac{1}{\overline{\alpha}})} = \delta_1 \phi(\alpha) \quad , \quad \beta^n \overline{\psi(\frac{1}{\overline{\beta}})} = \delta_2 \psi(\beta), \tag{3. 4}$$

for every $\alpha, \beta \in \mathbb{C}_1 - \{0\}$.
Let $Z = \alpha_1 e_1 + \beta_1 e_2$ be an invertible bicomplex number, then $\alpha_1, \beta_1 \neq 0$ and

$$\begin{aligned}Z^n \overline{P(\frac{1}{\overline{Z}})} &= (\alpha_1^n e_1 + \beta_1^n e_2)\overline{P(\frac{1}{\overline{\alpha_1}} e_1 + \frac{1}{\overline{\beta_1 e_2}})} \\ &= (\alpha_1^n e_1 + \beta_1^n e_2)\overline{(\phi(\frac{1}{\overline{\alpha_1}})e_1 + \psi(\frac{1}{\overline{\beta_1}})e_2)} \\ &= \alpha_1^n \overline{\phi(\frac{1}{\overline{\alpha_1}})} e_1 + \beta_1^n \overline{\psi(\frac{1}{\overline{\beta_1}})} e_2 \\ &= \delta_1 \phi(\alpha_1)e_1 + \delta_2 \psi(\beta_1)e_2 \qquad \text{(by (3. 4))} \\ &= (\delta_1 e_1 + \delta_2 e_2)P(\alpha_1 e_1 + \beta_1 e_2) \\ &= \delta P(Z), \end{aligned} \tag{3. 5}$$

where $\delta = \delta_1 e_1 + \delta_2 e_2$ and $\|\delta\| = \|\delta_1 e_1 + \delta_2 e_2\| = \sqrt{\frac{|\delta_1|^2 + |\delta_2|^2}{2}} = 1$.
Now suppose that there exists $\delta = \delta_1 e_1 + \delta_2 e_2$ on the unit discus such that for every invertible bicomplex $Z = \alpha_1 e_1 + \beta_1 e_2$, we have

$$Z^n \overline{P(\frac{1}{\overline{Z}})} = \delta P(Z). \tag{3. 6}$$

By (3.5), we have

$$Z^n \overline{P(\frac{1}{\overline{Z}})} = \alpha_1^n \overline{\phi(\frac{1}{\overline{\alpha_1}})} e_1 + \beta_1^n \overline{\psi(\frac{1}{\overline{\beta_1}})} e_2,$$

and also

$$\delta P(Z) = (\delta_1 e_1 + \delta_2 e_2)(\phi(\alpha_1)e_1 + \psi(\beta_1)e_2) = \delta_1\phi(\alpha_1)e_1 + \delta_2\psi(\beta_1)e_2.$$

Hence, by applying (3.6), we get

$$\alpha_1^n\overline{\phi(\frac{1}{\overline{\alpha_1}})} = \delta_1\phi(\alpha_1) \quad , \quad \beta_1^n\overline{\psi(\frac{1}{\overline{\beta_1}})} = \delta_2\psi(\beta_1).$$

Since δ lies on the unit discus, $|\delta_1| = |\delta_2| = 1$ and it follows that $\phi(\alpha)$ and $\psi(\beta)$ are self-inversive complex polynomial. Therefore by Theorem 1.2, $P(Z)$ is a self-inversive bicomplex polynomial and this completes the proof of Theorem 1.4.

Proof of Theorem 1.5. Let $P(Z) = \sum_{k=0}^{n} A_k Z^k = P(\alpha e_1 + \beta e_2) = \phi(\alpha)e_1 + \psi(\beta)e_2$. Since A_n is invertible, $\phi(\alpha)$ and $\psi(\beta)$ are complex polynomials of degree n.
First, we suppose that $P(Z)$ is a self-inversive bicomplex polynomial. By Theorem 1.2, $\phi(\alpha)$ and $\psi(\beta)$ are also self-inversive complex polynomials, so, by Lemma 2.2, we have

$$\overline{\gamma_n}\phi(\alpha) = \gamma_0\alpha^n\overline{\phi}(\frac{1}{\alpha}) \quad \text{and} \quad \overline{\delta_n}\psi(\beta) = \delta_0\beta^n\overline{\psi}(\frac{1}{\beta}), \tag{3. 7}$$

where $A_k = \gamma_k e_1 + \delta_k e_2$ for $0 \leq k \leq n$. Therefore

$$\begin{aligned}\overline{A_n}P(Z) &= (\overline{\gamma_n}e_1 + \overline{\delta_n}e_2)(\phi(\alpha)e_1 + \psi(\beta)e_2)\\ &= (\overline{\gamma_n}\phi(\alpha))e_1 + (\overline{\delta_n}\psi(\beta))e_2\\ &= (\gamma_0\alpha^n\overline{\phi}(\frac{1}{\alpha}))e_1 + (\delta_0\beta^n\overline{\psi}(\frac{1}{\beta}))e_2 \qquad \text{(by (3.7))}\\ &= (\gamma_0 e_1 + \delta_0 e_2)(\alpha e_1 + \beta e_2)^n(\overline{\phi}(\frac{1}{\alpha})e_1 + \overline{\psi}(\frac{1}{\beta})e_2)\\ &= A_0 Z^n\overline{P}(\frac{1}{Z}).\end{aligned}$$

Next we suppose $\overline{A_n}P(Z) = A_0Z^n\overline{P}(\frac{1}{Z})$ for every bicomplex invertible number Z. It follows that

$$(\overline{\gamma_n}e_1 + \overline{\delta_n}e_2)(\phi(\alpha)e_1 + \psi(\beta)e_2) = (\gamma_0 e_1 + \delta_0 e_2)(\alpha^n e_1 + \beta^n e_2)(\overline{\phi}(\frac{1}{\alpha})e_1 + \overline{\psi}(\frac{1}{\beta})e_2),$$

or

$$\overline{\gamma_n}\phi(\alpha) = \gamma_0\alpha^n\overline{\phi}(\frac{1}{\alpha}) \quad \text{and} \quad \overline{\delta_n}\psi(\beta) = \delta_0\beta^n\overline{\psi}(\frac{1}{\beta}).$$

Now using Lemma 2.2, we have

$$\gamma_0\overline{\gamma_k} = \overline{\gamma_n}\gamma_{n-k} \quad \text{and} \quad \delta_0\overline{\delta_k} = \overline{\delta_n}\delta_{n-k} \quad ; \quad k = 0, 1, ..., n, \tag{3. 8}$$

therefore by using equality (3.8), we have

$$\begin{aligned}A_0\overline{A_k} &= (\gamma_0 e_1 + \delta_0 e_2)(\overline{\gamma_k}e_1 + \overline{\delta_k}e_2)\\ &= (\gamma_0\overline{\gamma_k})e_1 + (\delta_0\overline{\delta_k})e_2\\ &= (\overline{\gamma_n}\gamma_{n-k})e_1 + (\overline{\delta_n}\delta_{n-k})e_2\\ &= (\overline{\gamma_n}e_1 + \overline{\delta_n}e_2)(\gamma_{n-k}e_1 + \delta_{n-k}e_2)\\ &= \overline{A_n}A_{n-k}.\end{aligned}$$

Finally to complete the proof of Theorem 1.5, we suppose that $A_0\overline{A_k} = \overline{A_n}A_{n-k}$; $(k = 0, 1, ..., n)$. It follows that

$$(\gamma_0 e_1 + \delta_0 e_2)(\overline{\gamma_k}e_1 + \overline{\delta_k}e_2) = (\overline{\gamma_n}e_1 + \overline{\delta_n}e_2)(\gamma_{n-k}e_1 + \delta_{n-k}e_2),$$

i.e.,

$$\gamma_0\overline{\gamma_k} = \overline{\gamma_n}\gamma_{n-k} \quad \text{and} \quad \delta_0\overline{\delta_k} = \overline{\delta_n}\delta_{n-k}, \qquad ; \qquad k = 0, 1, ..., n.$$

Hence $\phi(\alpha)$ and $\psi(\beta)$ are self-inversive polynomials and by Theorem 1.2, $P(Z)$ is self-inversive.

Proof of Theorem 1.6. Let $P(Z) = \sum\limits_{k=0}^{n} A_k Z^k = P(\alpha e_1 + \beta e_2) = \phi(\alpha)e_1 + \psi(\beta)e_2$ be a self-inversive bicomplex polynomial of degree n. Since A_n is invertible, $\phi(\alpha)$ and $\psi(\beta)$ are complex polynomials of degree n. Also by Theorem 1.2, $\phi(\alpha)$ and $\psi(\beta)$ are self-inversive complex polynomials, therefore, by Lemma 2.3, we have for each $Z = \alpha e_1 + \beta e_2$,

$$\overline{\gamma_n}[n\phi(\alpha) - \alpha\phi'(\alpha)] = \gamma_0\alpha^{n-1}\overline{\phi'}(\frac{1}{\alpha}) \qquad \text{for each } \alpha \in \mathbb{C}_1, \tag{3. 9}$$

and

$$\overline{\delta_n}[n\psi(\beta) - \beta\psi'(\beta)] = \delta_0\beta^{n-1}\overline{\psi'}(\frac{1}{\beta}) \qquad \text{for each } \beta \in \mathbb{C}_1, \tag{3. 10}$$

also

$$|\frac{n\phi(\alpha)}{\alpha\phi'(\alpha)} - 1| = 1 \quad \text{for each } \alpha \text{ with } |\alpha| = 1, \tag{3. 11}$$

and

$$|\frac{n\psi(\beta)}{\beta\psi'(\beta)} - 1| = 1 \quad \text{for each } \beta \text{ with } |\beta| = 1. \tag{3. 12}$$

Hence

$$\begin{aligned}
\overline{A_n}[nP(Z) - ZP'(Z)] &= \overline{(\gamma_n e_1 + \delta_n e_2)}[n(\phi(\alpha)e_1 + \psi(\beta)e_2) \\
&\quad - (\alpha e_1 + \beta e_2)(\phi'(\alpha)e_1 + \psi'(\beta)e_2)] \\
&= \overline{\gamma_n}[n\phi(\alpha) - \alpha\phi'(\alpha)]e_1 + \overline{\delta_n}[n\psi(\beta) - \beta\psi'(\beta)]e_2 \\
&= \gamma_0\alpha^{n-1}\overline{\phi'}(\frac{1}{\alpha})e_1 + \delta_0\beta^{n-1}\overline{\psi'}(\frac{1}{\beta})e_2 \qquad \text{(by (3.9) and (3.10))} \\
&= (\gamma_0 e_1 + \delta_0 e_2)(\alpha e_1 + \beta e_2)^{n-1}(\phi'(\frac{1}{\alpha})e_1 + \psi'(\frac{1}{\beta})e_2) \\
&= A_0 Z^{n-1}P'(\frac{1}{\alpha}e_1 + \frac{1}{\beta}e_2) \\
&= A_0 Z^{n-1}P'(\frac{1}{\alpha e_1 + \beta e_2}) \\
&= A_0 Z^{n-1}P'(\frac{1}{Z}).
\end{aligned}$$

Also, for each $Z = \alpha e_1 + \beta e_2$ on $D(0;1,1)$, we have

$$\begin{aligned}\|\frac{nP(Z)}{ZP'(Z)} - 1\| &= \|\frac{n(\phi(\alpha)e_1 + \psi(\beta)e_2)}{(\alpha e_1 + \beta e_2)(\phi'(\alpha)e_1 + \psi'(\beta)e_2)} - 1\| \\ &= \|\frac{n\phi(\alpha)}{\alpha\phi'(\alpha)}e_1 + \frac{n\psi(\beta)}{\beta\psi'(\beta)}e_2 - e_1 - e_2\| \\ &= \sqrt{\frac{|\frac{n\phi(\alpha)}{\alpha\phi'(\alpha)} - 1|^2 + |\frac{n\psi(\beta)}{\beta\psi'(\beta)} - 1|^2}{2}} \\ &= 1. \qquad \text{(by (3.11) and (3.12))}\end{aligned}$$

This completes the proof of Theorem 1.6. □

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