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Unitary Irreducible Representation (UIR) Matrix Elements of Finite Rotations of SO$(2,1)$ Decomposed According to the Subgroup T_1

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Abstract. Using a technique of Kalnins, unitary irreducible representation (UIR) of principle series of SO$(2,1)$, decomposed according to the group T_1, are realized in the space of homogeneous functions on the cone

$$\xi_0^2 - \xi_1^2 - \xi_2^2 = 0$$

as the carrier space. It is then shown that the matrix element of an arbitrary finite rotation of SO$(2,1)$ are determined by those of two specific types of finite rotations, each depending on a single parameter; matrix elements of these two specific types of finite rotations are then explicitly computed. Finally, a number of new relations between special functions appearing in these matrix elements, are obtained by using the usual standard techniques of deriving such relations with the help of group representation theory.

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1. Introduction

The problem of determination of UIR matrix elements of finite rotations of compact and non-compact rotation groups, has a pretty long history. it was originated by Wigner [28] in 1930s, when he obtained his famous d-functions, which are simply the matrix elements of finite rotations of the ordinary rotation group SO(3). The next step was taken by Bargmann [1] in 1947, when he obtained the matrix elements of finite rotations of the Lorentz groups SO$(2,1)$ and SO$(3,1)$. Ever since the sixties, there has been a large number of papers (see references given in Syed [23, 24, 25]) , not only concerning these simpler groups, but also the general ones SO(n), SO$(n,1)$ and SO$(n,2)$.

An important aspect of these studies concerns the selection of basis for the carrier space of the representation. Generally, the choice is such that its elements are eigenvectors of not only the Casimir operators of the group itself but also of the Casimir operators of some maximal subgroups. This situation is expressed by saying that the representation has been DECOMPOSED according to the particular subgroup. The matter becomes specially important in the case of non-compact groups as here we have compact as well as non-compact maximal subgroups. The problem is obviously simpler for decomposition according to compact subgroups, so that most of the papers, specially the earlier ones, dealt with this case only. For decomposition according to non-compact subgroups, some progress was made in seventies and eighties by Syed [23, 24, 25], Mukunda [17, 18, 19, 20], Boyer [4, 5], Wolf [29], Kalnins [11], Kuznetsov [13], Basu [3], Lindblad [14], MacFadyen [15] for SO$(2,1)$, for decomposition according to its maximal non-compact subgroups and for the most degenerate representations (in which all Casimir operators except one, are zero) of SO$(n,1)$ and SO$(n,2)$, for the decomposition

$$SO\,(n,1) \supset SO\,(n-1,1)\,,$$

$$SO\,(n,2) \supset SO\,(n,1)\,,$$

and for the matrix elements of SO$(3,1)$ for the decomposition

$$SO\,(3,1) \supset SO\,(2,1)\,.$$

In many of the above works, the carrier space is chosen as the set of functions on hyperboloids, on which, only the most degenerate representations can be realized, and these also as multiplier or induced representations (developed by Mackey in 1950s) rather than ordinary i.e. non-induced ones. However, a group Russian authors such as Kalnins [11], Kuznetsev et. el. [13] etc. have realized representations of SO$(2,1)$ and SO$(3,1)$ on spaces of functions on cones which have the following advantages:

(1) even the non-degenerate representations can be realized on these spaces.
(2) the representations are ordinary ones rather than being induced ones.
(3) representations decomposed according to various maximal subgroups, are simply obtained by choosing appropriate parameterization of points on cone.

As the situation for the problem of determination of UIR matrix elements of finite rotations of SO$(2,1)$ and SO$(3,1)$ in ALL subgroups decompositions, appear more hopeful with these realizations, we have started working with them. The present paper is the first step in this direction; in it, we consider the simpler group SO$(2,1)$. As already mentioned, Bargmann [1] was the first to obtain the d-functions of this group; he did this when the representation was decomposed according to the compact subgroup SO(2). The same result was obtained by Barut and Fronsdal [2] also, by using a somewhat different method. Later on, Mukunda [17], [18], [19], [20]and Macfadyen [15] obtained these functions for the decomposition according to the subgroup SO$(1,1)$. Now, in addition to SO(2)and SO$(1,1)$, SO$(2,1)$ has one more maximal subgroup T_1 which is generated by $M_0 - N_1$, where M_0 is the generator of (ordinary) rotations while N_1 and N_2 are generators of pure Lorentz transformations in the direction of x_2 and x_1 respectively. Although Vilenkin [27] and Itzykson [10] have realized some representations of SO$(2,1)$ decomposed according to the subgroup T_1, they have not considered the UIR matrix elements of finite rotations. In fact, they have only determined the action of the representation operator $V_l\left(g\right)$, g an element of SO$(2,1)$, of the representation D^l of principle series, on the function $F\left(\lambda\right)$, where the collection $F\left(\lambda\right)$ constitutes the carrier space with the property that its elements are eigenfunctions of elements of T_1:

$$V_l\left(t\left(\alpha\right)\right)F\left(\lambda\right)=e^{i\lambda\alpha}F\left(\lambda\right),$$

(see equation(3), p. 369, of [27] and equation (24), p.1114, of [10]) . This action is given as an integral operator

$$V_l\left(g\right)F\left(\lambda\right)=\int_0^{\infty}K_g^l\left(\lambda,\mu\right)F\left(\mu\right)d\mu,$$

with the kernel $K_g^l\left(\lambda,\mu\right)$, and what they determine are explicit expressions for this kernel for various choices of $g\in SO\left(2,1\right)$. Basu and Wolf [3] do obtain the matrix elements in all subgroup reductions but they use the technique of canonical transform realization of $SL\left(2,R\right)$ rather than the Lie transformation group realization as we do. Lindblad and Nagel [14] also obtain matrix elements of finite rotations, but they use a method very different from ours; thus, for example, in their computations, eigenvectors of the compact generator plays a fundamental role in the sense that eigenvectors of all non-compact generators are expressed as linear combinations of those of compact generator while we consider the eigenvectors of the relevant non-compact generator quite independently, without any reference to those of compact generator. In addition, they find the UIR matrix elements only of the elements

$$r_0\left(\theta\right),l_2\left(\zeta\right)$$

of SO$(2,1)$, which means that for arbitrary element g of SO$(2,1)$ which is parameterized as

$$g=r_0\left(\theta\right)l_2\left(\zeta\right)r_0\left(\acute{\theta}\right)$$

they will have to insert two complete sets of states which obviously makes the calculations much more complicated than ours in which, essentially, no insertions are needed to be made. We, in the present paper, compute the UIR matrix elements of arbitrary element of $SO\left(2,1\right)$, for representations of Principle Series of continuous class and of integral type, and obtain explicit expressions for them in closed form. Actually, these representations are realized on a space of functions on a 3-dimensional cone, with coordinates on it being chosen in such a way that the representations are decomposed according to the non-compact maximal subgroup T_1.The action of the representative operators, as usual, consist of a change in the argument of the functions. The UIR matrix elements of finite rotations of $SO\left(2,1\right)$ are then cast in the form of an integral, which is easily evaluated in closed form (in terms of a new modified Bessel function $L_s\left(z\right)$) for two specific elements of $SO\left(2,1\right)$, each of which depends on a single parameter. As it is shown that the matrix element of an arbitrary element of $SO\left(2,1\right)$, is trivially obtained from those of the above 2 specific elements, complete solution of the problem under discussion, is automatically obtained. Proceeding further, we use the standard technique of group representation theory to obtain a number of new relations between the special functions appearing in these matrix elements.

Quite a long time ago, the author published a paper [23] in which he derived the most degenerate representation matrix elements of finite rotations of $SO\left(n-2,2\right)$; however, the technique used there, was completely different from the one that we use in the present paper. Thus the carrier space there was the space of functions on a two-sheeted hyperboloid rather than on a cone, while the representation there was decomposed according to the subgroup $SO\,(n-2,1)$ which corresponds to the decomposition $SO\,(2,1) \supset SO\,(1,1)$ for the group being considered now, rather than the decomposition $SO\,(2,1) \supset T_1$ being considered here. Finally, the representation chosen there, was an induced representation obtained as an extension of a similar representation given for $SO\,(n-1,1)$ by Boyer and Ardalan [5], while here, we use ordinary (i.e. non-induced) representation given by Kalnins[11].

Although most of the work on UIR matrix elements of SO(2, 1)was done in sixties and seventies, there is an important reason for reviving the subject after a gap of more than thirty years. During the last ten years or so, it has been found that UIR of Lorentz group SO(3, 1) and its compact and non-compact subgroups, play a crucial role in a certain approach to Quantum Gravity [22], [26], [21]. it was found by Conrady [6] and Conrady and Hnybida [7] that what were needed in the theory were UIR of SL$(2, C)$ decomposed according to the non-compact maximal subgroup SL$(2, R) \approx$ SU(1, 1), which were themselves decomposed according to the non-compact subgroup SO(1, 1). Keeping this in mind, Conrady and Hnybida [7] obtain UIR matrix elements of the generators of SL$(2, C)$ when the representation is decomposed according to SU(1, 1) (which is homomorphic to SO(2, 1)) , which is itself decomposed according to both its non-compact maximal group SO(1, 1) and its compact maximal subgroup SO(2). The importance of decomposition of UIR of SO(2, 1) according to its non-compact maximal subgroups thus becomes obvious. Hence our efforts to obtain UIR matrix elements of finite rotations of SO(2, 1) when the representation is decomposed according to its non-compact subgroup T_1, which has not been obtained in a simple form up to now, using the Lie transformation group realization of SO(2, 1). From the point of view of Mathematics also, this derivation is important as firstly, there is no reason to leave this case aside when decomposition according to all other maximal subgroups have been satisfactorily treated, and secondly, it leads to derivation of new properties of special functions appearing in the matrix elements.

The arrangement of the material in the paper is as follows. In Section 2 below, we use a technique of Kalnins [11] to set up a realization of the relevant representation, and then obtain an expression for the matrix element of an arbitrary element SO(2, 1), in an integral form. We then use, in Section 3, a parameterization of SL$(2, R)$ similar to the one given by Vilenkin [27], to obtain a parameterization of SO(2, 1) which shows that in order to find the matrix element of an arbitrary element of SO(2, 1), we need to find it only for two specific elements, each of which depends on a single parameter. In Section 4 and Section 5, we actually compute these two matrix elements, and obtain results in closed form. These results are then used in Section 6, to obtain some new relations involving a new kind of modified Bessel function, the Whittaker function and the confluent hypergeometric function.

2. A Realization of the Representation $D^{\frac{-1}{2}+i\rho}$ of SO(2, 1)

Following Kalnins [11], we denote functions on the 2-dimensional cone

$$[\xi,\xi] \equiv \xi_0^2 - \xi_1^2 - \xi_2^2 = 0 \tag{2. 1}$$

by $|\xi\rangle$, and the action of the operator

$$U\,(g)\,, g \in SO\,(2,1)\,, \text{the Lorentz group in 3 dimension,}$$

on them by

(2. 2) $$U\,(g)\,|\xi\rangle = |g^{-1}\xi\rangle.$$

We decompose these functions into homogeneous components of degree σ, by

(2. 3) $$|\xi,\sigma\rangle = \int_0^\infty |t\xi\rangle t^{-\sigma-1}dt,$$

which satisfy the homogeneity condition Kalnins [11]

(2. 4) $$|a\xi,\sigma\rangle = a^\sigma|\xi,\sigma\rangle, \quad \text{a real.}$$

$U\,(g)$ acting on these homogeneous functions $|\xi,\sigma\rangle$ for

(2. 5) $$\sigma = \frac{-1}{2} + i\rho, \; -\infty < \rho < \infty,$$

gives the UIR $D^{\frac{-1}{2}+i\rho}$ of principle series of SO$(2,1)$. For convenience, we denote

$$|\xi, \frac{-1}{2} + i\rho\rangle \qquad by \qquad |\xi,\rho\rangle.$$

Again following Kalnins [11], we parameterize ξ on the cone by

(2. 6) $$\xi \equiv \xi_r = \omega\left(r^2+1, r^2-1, 2r\right), 0 < \omega < \infty, -\infty < r < \infty.$$

It is easy to check that the effect of $t\,(s) \in T_1$ on ξ_r is simply given by

(2. 7) $$t\,(s)\,\xi_r = \xi_{r-s};$$

here, of course, we have used the fact that

$$t\,(s) = e^{Ts}, T = M_0 - N_1,$$

so as to get

(2. 8) $$t\,(s) = \begin{bmatrix} 1+s^2/2 & -s^2 & -s \\ s^2/2 & 1-s^2/2 & -s \\ -s & s & 1 \end{bmatrix}$$

The homogeneity condition (2. 4) gives

(2. 9) $$|\xi,\rho\rangle = \omega^{\left(-\frac{1}{2}+i\rho\right)}|r,\rho\rangle;$$

expanding $|r,\rho\rangle$ by means of a Fourier integral

$$|r,\rho\rangle = \int_{-\infty}^{\infty} dp|p,\rho\rangle e^{ipr},$$

we will have

(2. 10) $$|\xi,\rho\rangle = \int_{-\infty}^{\infty} dp|p,\rho\rangle\omega^{\left(-\frac{1}{2}+i\rho\right)}e^{ipr}.$$

It is easy to verify that

$$U\,(T)\,|p,\rho\rangle = ip|p,\rho\rangle$$

i.e. $|p,\rho\rangle$ are the eigenvectors of the generator T, corresponding to the eigenvalue ip. For (2. 7) and (2. 2) imply that

$$U\,(t\,(s))\,|\xi_r,\rho\rangle = |t^{-1}\,(s)\,\xi_r,\rho\rangle = |\xi_{r+s},\rho\rangle,$$

so that (2. 10) gives

$$U\,(t\,(s))\int_{-\infty}^{\infty} dp|p,\rho\rangle\omega^{-\frac{1}{2}+i\rho}e^{ipr} = \int_{-\infty}^{\infty} dp|p,\rho\rangle\omega^{-\frac{1}{2}+i\rho}e^{ip(r+s)}$$

$$\Rightarrow \int_{-\infty}^{\infty} dpU\left(t\left(s\right)\right)|p,\rho\rangle\omega^{-\frac{1}{2}+i\rho}e^{ipr} = \int_{-\infty}^{\infty} dpe^{ips}|p,\rho\rangle\omega^{\frac{-1}{2}+i\rho}e^{ipr}$$

$$\Rightarrow U\left(e^{Ts}\right)|p,\rho\rangle \equiv U\left(t\left(s\right)\right)|p,\rho\rangle = e^{ips}|p,\rho\rangle \tag{2. 11}$$

$$\Rightarrow U\left(T\right)|p,\rho\rangle = ip|p,\rho\rangle$$

as asserted.

Now we have the following

THEOREM:- If g is an arbitrary element of $SO\left(2,1\right)$ and

$$U\left(g\right)|\xi,\rho\rangle = |\xi^{/},\rho\rangle$$

where

$$\xi = \omega\left(r^2+1, r^2-1, 2r\right), \ \acute{\xi} = \acute{\omega}\left(\acute{r}^2+1, \acute{r}^2-1, 2\acute{r}\right), \tag{2. 12}$$

then

$$\langle p,\rho|U\left(g\right)|\acute{p},\rho\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr\left(\frac{\acute{\omega}}{\omega}\right)^{-\frac{1}{2}+i\rho} e^{i\left(p\acute{r}-\acute{p}r\right)} \tag{2. 13}$$

gives the matrix element of g in an integral form.

Proof:- Note first of all that according to equation (2) ,

$$\acute{\xi} = g^{-1}\xi \tag{2. 14}$$

so that equations (12, 14) determine $\acute{\omega}, \acute{r}$ in terms of ω, r. Next as

$$|\acute{\xi},\rho\rangle = \int_{-\infty}^{\infty} d\acute{p}|\acute{p},\rho\rangle\left(\acute{\omega}\right)^{-\frac{1}{2}+i\rho} e^{i\acute{p}\acute{r}}$$

according to equation (10) , we get

$$\begin{aligned} &\int_{-\infty}^{\infty} dp\ U\left(g\right)|p,\rho\rangle\omega^{-\frac{1}{2}+i\rho} \\ &= U\left(g\right)|\xi,\rho\rangle = |\acute{\xi},\rho\rangle \\ &= \int_{-\infty}^{\infty} d\acute{p}|\acute{p},\rho\rangle\acute{\omega}^{-\frac{1}{2}+i\rho}e^{i\acute{p}\acute{r}}; \end{aligned}$$

multiplying this equation by $e^{ip''r}$, integrating with respect to r, and using

$$\langle p,\rho|p',\rho\rangle = \delta\left(p-p'\right),$$

we get

$$2\pi\langle p,\rho|U\left(g\right)|p'',\rho\rangle = \int_{-\infty}^{\infty} dr\left(\frac{\acute{\omega}}{\omega}\right)^{-\frac{1}{2}+i\rho} e^{i\left(pr'-p''r\right)}$$

$$\Rightarrow \langle p,\rho|U\left(g\right)|p',\rho\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr\left(\frac{\acute{\omega}}{\omega}\right)^{-\frac{1}{2}+i\rho} e^{i\left(pr'-p'r\right)},$$

as required.

3. An Appropriate Parameterization of SO(2, 1)

To obtain a parameterization of SO(2, 1) suitable for our purpose, we prove the following

THEOREM:- An arbitrary element $g \in SL\,(2,R)$, the group of 2×2 real matrices with determinant $=1$, is of one of the following two forms:

$$(i)\, g = s_1\delta,\ \ (ii)\, g = s_1\delta\tau s_2, \tag{3. 15}$$

where

$$s_1 = \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix}, s_2 = \begin{bmatrix} 1 & s_2 \\ 0 & 1 \end{bmatrix}, \delta = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3. 16}$$

Proof:- Note first of all that this is simply a slight modification of the parameterization given by Vilenkin [27], and we verify it in exactly the same way as he did. So let us first suppose that

$$g = \begin{bmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{bmatrix};$$

then

$$g = \begin{bmatrix} 1 & \alpha\beta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix},$$

which is of the form $s_1\delta$. Next let

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \gamma \neq 0;$$

then

$$s_1^{-1} g s_2^{-1} = \begin{bmatrix} \alpha - s_1\gamma & -\alpha s_2 + \beta + \gamma s_1 s_2 - \delta s_1 \\ \gamma & -\gamma s_2 + \delta \end{bmatrix},$$

so that choosing $s_1 = \frac{\alpha}{\gamma}$, $s_2 = \frac{\delta}{\gamma}$ and using $\alpha\delta - \beta\gamma = 1$, we get

$$s_1^{-1} g s_2^{-1} = \begin{bmatrix} 0 & -\frac{1}{\gamma} \\ \gamma & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\gamma} & 0 \\ 0 & -\gamma \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \delta\tau,$$

so that

$$g = s_1\delta\tau s_2$$

as required. This proves the Theorem.

To proceed further, we note that if $SU\,(1,1)$ is the pseudo-unitary group in 2 dimension i.e. the group of 2×2 complex matrices of the form

$$\begin{bmatrix} \alpha & \beta \\ \beta^{\times} & \alpha^{\times} \end{bmatrix}$$

with unit determinant, then as indicated by Knapp [12],

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1} SL\,(2,R) \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = SU\,(1,1)$$

gives an isomorphism of $\mathrm{SL}(2,R)$ onto $\mathrm{SU}(1,1)$; as

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} SL\left(2,R\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = SL\left(2,R\right)$$

is obviously an isomorphism of $\mathrm{SL}(2,R)$ onto itself, it follows that

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} SL\left(2,R\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = SU\left(1,1\right)$$

is also an isomorphism of $\mathrm{SL}(2,R)$ onto $\mathrm{SU}(1,1)$. It is easy to check that under this isomorphism

$$\begin{bmatrix} e^{\frac{\zeta}{2}} & 0 \\ 0 & e^{-\frac{\zeta}{2}} \end{bmatrix} \leftrightarrow \begin{bmatrix} ch\frac{\zeta}{2} & sh\frac{\zeta}{2} \\ sh\frac{\zeta}{2} & ch\frac{\zeta}{2} \end{bmatrix},$$

$$\tau \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \leftrightarrow i \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1-\frac{1}{2}is & \frac{1}{2}is \\ -\frac{1}{2}is & 1+\frac{1}{2}is \end{bmatrix}.$$

Finally, we apply to the above elements of $\mathrm{SU}(1,1)$, the two-to-one onto homomorphism

$$SU\left(1,1\right) \to SO\left(2,1\right)$$

given by Bargmann [1]; we find that

$$\pm \begin{bmatrix} ch\frac{\zeta}{2} & sh\frac{\zeta}{2} \\ sh\frac{\zeta}{2} & ch\frac{\zeta}{2} \end{bmatrix} \leftrightarrow \begin{bmatrix} ch\frac{\zeta}{2} & sh\frac{\zeta}{2} & 0 \\ sh\frac{\zeta}{2} & ch\frac{\zeta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\pm i \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\pm \begin{bmatrix} 1-\frac{1}{2}is & \frac{1}{2}is \\ -\frac{1}{2}is & 1+\frac{1}{2}is \end{bmatrix} \leftrightarrow \begin{bmatrix} 1+\frac{1}{2}s^2 & -\frac{1}{2}s^2 & -s \\ \frac{1}{2}s^2 & 1-\frac{1}{2}s^2 & -s \\ -s & s & 1 \end{bmatrix}.$$

Combining this homomorphism to the earlier isomorphism, we see that there exists a two-to-one onto homomorphism

$$SL\left(2,R\right) \to SO\left(2,1\right)$$

which gives the following associations:

$$\pm \begin{bmatrix} e^{\frac{\zeta}{2}} & 0 \\ 0 & e^{-\frac{\zeta}{2}} \end{bmatrix} \leftrightarrow \begin{bmatrix} ch\frac{\zeta}{2} & sh\frac{\zeta}{2} & 0 \\ sh\frac{\zeta}{2} & ch\frac{\zeta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv l_2\left(\zeta\right), \tag{3. 17}$$

$$\pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \equiv \tau_0, say, \tag{3. 18}$$

(3. 19)
$$\pm\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1+\frac{1}{2}s^2 & -\frac{1}{2}s^2 & -s \\ \frac{1}{2}s^2 & 1-\frac{1}{2}s^2 & -s \\ -s & s & 1 \end{bmatrix} \equiv t\left(s\right).$$

Applying this homomorphism to equation (15) [which says that an arbitrary element g of $\mathrm{SL}(2, R)$ can be expressed either as $g = s_1\delta$ or as $g = s_1\delta\tau s_2$], we see that an arbitrary element g of $\mathrm{SO}(2, 1)$ can be parameterized either as

(3. 20)
$$g = t\left(s_1\right) l_2\left(\zeta\right) \quad \text{or as } g = t\left(s_1\right) l_2\left(\zeta\right) \tau_0 t\left(s_2\right);$$

this is the required parameterization that we need.

Now using (11) and its immediate consequence

$$\langle p, \rho| U\left(t\left(-s\right)\right)| = e^{-ips}\langle p, \rho|,$$

we see that for any element A of $\mathrm{SO}(2, 1)$,

$$\begin{aligned} &\langle p, \rho| U\left(t\left(s\right) At\left(s'\right)\right)|p', \rho\rangle \\ &= \langle p, \rho| U\left(t\left(s'\right)\right) U\left(A\right) U\left(t\left(s\right)\right)|p', \rho\rangle \\ &= e^{i\left(ps'+p's\right)}\langle p, \rho| U\left(A\right)|p', \rho\rangle \end{aligned}$$

which implies that the matrix elements of $t(s)At(s')$ are determined by those of A. Hence it follows that in order to obtain the matrix element of an arbitrary rotation of $\mathrm{SO}(2, 1)$, it is sufficient to obtain them for $l_2\left(\zeta\right)$ and $l_2\left(\zeta\right)\tau_0$; this we now do.

4. Matrix Elements of $l_2\left(\zeta\right)$

We obtain the matrix elements of $l_2\left(\zeta\right)$ in the present section and that of $l_2\left(\zeta\right)\tau_0$ in the next section.

Now when $g = l_2\left(\zeta\right)$, we have

$$\begin{aligned} \xi' &= g^{-1}\xi \\ &= l_2^{-1}\left(\zeta\right)\xi \\ &= \begin{bmatrix} ch\frac{\zeta}{2} & -sh\frac{\zeta}{2} & 0 \\ -sh\frac{\zeta}{2} & ch\frac{\zeta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \omega\left(r^2+1\right) \\ \omega\left(r^2-1\right) \\ 2\omega r \end{bmatrix} \\ &= \begin{bmatrix} \omega\left(r^2+1\right) ch\zeta - \omega\left(r^2-1\right) sh\zeta \\ -\omega\left(r^2+1\right) sh\zeta + \omega\left(r^2-1\right) ch\zeta \\ 2\omega r \end{bmatrix} \\ &= \begin{bmatrix} \omega e^{\zeta}\left(r^2 e^{-2\zeta}+1\right) \\ \omega e^{\zeta}\left(r^2 e^{-2\zeta}-1\right) \\ 2\omega e^{\zeta}.re^{-\zeta} \end{bmatrix} \end{aligned}$$

after a bit of simplification. As this must be equal to

$$\begin{bmatrix} \omega'\left(\left(r'\right)^2+1\right) \\ \omega'\left(\left(r'\right)^2-1\right) \\ 2\omega' r' \end{bmatrix}$$

we get

(4. 21) $$\omega' = \omega e^{\zeta}, r' = re^{-\zeta};$$

equation (13) therefore gives the required matrix element as

$$\langle p, \rho| U\left(l_2\left(\zeta\right)\right) |p', \rho\rangle$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dr \left(e^{\zeta}\right)^{-\frac{1}{2}+i\rho} exp\left(i\left(pe^{-\zeta}r - p'r\right)\right)$$

$$= \left(e^{\zeta}\right)^{-\frac{1}{2}+i\rho} \delta\left(pe^{-\zeta} - p'\right)$$

(4. 22) $$\Rightarrow \langle p, \rho| U\left(l_2\left(\zeta\right)\right) |p', \rho\rangle = \left(e^{\zeta}\right)^{\frac{1}{2}+i\rho} \delta\left(p - p'e^{\zeta}\right).$$

Note the extreme simplicity of this expression; however, this is to be expected as it is easy to verify that

$$U\left(l_2\left(\zeta\right)\right) |p', \rho\rangle = \left(e^{\zeta}\right)^{\frac{1}{2}+i\rho} |p'e^{\zeta}, \rho\rangle.$$

5. Matrix Elements of $l_2\left(\zeta\right)\tau_0$

We start with the following

DEFINITION:- In analogy with the well known modified Bessel function K_s which is given by

$$K_s\left(z\right) = \frac{\pi}{2} \frac{1}{\sin\ \left(\pi s\right)} [I_{-s}\left(z\right) - I_s\left(z\right)],$$

we define another modified Bessel function L_s by

$$L_s\left(z\right) = \frac{\pi}{2} \frac{1}{\sin\left(\pi s\right)} [J_{-s}\left(z\right) - J_s\left(z\right)].$$

We next prove the following

THEOREM:- The matrix elements of $l_2\left(\zeta\right)\tau_0$ are given by

(5. 23)

$$\langle p, \rho| U\left(l_2\left(\zeta\right)\tau_0\right) |p', \rho\rangle = \begin{cases} 0 \text{ if } \operatorname{Sign} p = -\operatorname{Sign} p', \\ \frac{2}{\pi} ch\left(\pi\rho\right) e^{\frac{\zeta}{2}} \left(pp'\right)^{i\rho} L_{2i\rho}\left(2\sqrt{pp'}e^{\zeta}\right) \text{ if } \operatorname{Sign} p = \operatorname{Sign} p'. \end{cases}$$

Proof:- When

$$g = l_2\left(\zeta\right)\tau_0$$

$$= \begin{bmatrix} ch\zeta & sh\zeta & 0 \\ sh\zeta & ch\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} ch\zeta & -sh\zeta & 0 \\ sh\zeta & -ch\zeta & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$g^{-1} = \begin{bmatrix} ch\zeta & -sh\zeta & 0 \\ sh\zeta & -ch\zeta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

so that

$$\xi' = g^{-1}\xi$$

$$= \begin{bmatrix} ch\zeta & -sh\zeta & 0 \\ sh\zeta & -ch\zeta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \omega\left(r^2+1\right) \\ \omega\left(r^2-1\right) \\ 2\omega r \end{bmatrix}$$

$$\begin{bmatrix} \omega e^{\zeta}\left(r^2 e^{-2\zeta}+1\right) \\ -\omega e^{\zeta}\left(r^2 e^{-2\zeta}-1\right) \\ -2\omega r \end{bmatrix}$$

which gives

$$\omega' = \omega r^2 e^{-\zeta}, \quad r' = -e^{\zeta}/r. \tag{5. 24}$$

Equation (13) therefore gives

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|p',\rho\rangle$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dr\left(r^2 e^{-\zeta}\right)^{-\frac{1}{2}+i\rho} exp\left(i\left(-pe^{\zeta}/r - p'r\right)\right)$$

$$= \frac{1}{2\pi}\left(e^{-\zeta}\right)^{-\frac{1}{2}+i\rho}\int_{-\infty}^{\infty} dr\left(r^2\right)^{-\frac{1}{2}+i\rho} exp\left(-i\left(pe^{\zeta}/r + p'r\right)\right).$$

It is proved in the Appendix A that the above integral vanishes unless p and p' have the same sign; we therefore assume, from now onwards, that they have the same sign. Then proceeding further, we get

(5. 25)

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|p',\rho\rangle = \frac{1}{2\pi}\left(e^{-\zeta}\right)^{-\frac{1}{2}+i\rho}\left[\int_0^{\infty} dr\left(r^2\right)^{-\frac{1}{2}+i\rho} e^{\left(-i\left(pe^{\zeta}/r+p'r\right)\right)}\right.$$

$$\left. + \int_{-\infty}^{0} dr\left(r^2\right)^{-\frac{1}{2}+i\rho} e^{\left(-i\left(pe^{\zeta}/r+p'r\right)\right)}\right]$$

$$= \frac{1}{2\pi}\left(e^{-\zeta}\right)^{-\frac{1}{2}+i\rho}\cdot 2\int_0^{\infty} dr\left(r^2\right)^{-\frac{1}{2}+i\rho}\cos p'\left(r+\frac{pe^{\zeta}}{p'r}\right)$$

$$= \frac{1}{\pi}\left(e^{-\zeta}\right)^{-\frac{1}{2}+i\rho}\int_0^{\infty} drr^{-1+2i\rho}\cos p'\left(r+\frac{pe^{\zeta}}{p'}\frac{1}{r}\right). \tag{5. 26}$$

Now, according to the Formula 35, p.321 of Erdelyi et el [8], we have

$$\int_0^{\infty} dxx^{s-1}\cos[a\left(x+b^2/x\right)] = \left(-\pi\right)b^s[J_s\left(2ab\right)\sin\left(\frac{1}{2}\pi s\right) + Y_s\left(2ab\right)\cos\left(\frac{1}{2}\pi s\right)]$$

$$\left(-1 < Res < 1\right),\ \left(a>0\right),\ \left(b>0\right),$$

$$= 2b^s\cos\left(\frac{1}{2}\pi s\right)\cdot\frac{\pi}{2\sin\left(\pi s\right)}[J_{-s}\left(2ab\right) - J_s\left(2ab\right)],$$

using the expression for Y_s in terms of J_s and J_{-s}. Hence putting (for $p>0, p'>0$)

$$x = r,\ s = 2i\rho,\ a = p',\ b^2 = pe^{\zeta}/p',$$

we will get

$$\int_0^\infty drr^{-1+2i\rho}\cos[p'\left(r+\frac{pe^\zeta}{p'}\frac{1}{r}\right)]$$

$$=2\left(\frac{pe^\zeta}{p'}\right)^{i\rho}ch\,(\pi\rho)\,L_{2i\rho}\left(2\sqrt{pp'}e^{\frac{\zeta}{2}}\right).$$

Equation (25) therefore gives

$$\langle p,\rho|U\,(l_2\,(\zeta)\,\tau_0)\,|p',\rho\rangle$$

$$=\frac{2}{\pi}\left(e^{-\zeta}\right)^{-\frac{1}{2}+i\rho}\left(\frac{pe^\zeta}{p'}\right)^{i\rho}ch\,(\pi\rho)\,L_{2i\rho}\left(2\sqrt{pp'}e^{\frac{\zeta}{2}}\right)$$

$$=\frac{2}{\pi}ch\,(\pi\rho)\,e^{\frac{\zeta}{2}}\,(p/p')^{i\rho}\,L_{2i\rho}\left(2\sqrt{pp'}e^{\frac{\zeta}{2}}\right),\quad p>0,p'>0.$$

By replacing p,p' by $-p,-p'$, we see that we get the same result when $p<0,p'<0$; this completes the proof of the Theorem.

Thus, we have obtained the value of this matrix element also in closed form, as asserted in the Introduction. A trivial consequence of equation (23) is

$$\langle p,\rho|U\,(\tau_0)\,|p',\rho\rangle=\frac{2}{\pi}ch\,(\pi\rho)\,(p/p')^{i\rho}\,L_{2i\rho}\left(2\sqrt{pp'}\right). \tag{5. 27}$$

6. Properties of the L-function

It is a well known fact (Vilenkin [27]) that expressions for UIR matrix elements of various Lie groups, in terms of special functions, lead to a number of properties of these functions obtained by using group theoretical arguments, some of which may be new ones although some others may be already known ones. We therefore obtain some properties of the L-functions introduced above, by using the fact that the UIR of matrix elements of $\mathrm{SO}(2,1)$ discussed in this paper, have been expressed in terms of these functions. For the purpose, we need the following simple relations between certain elements of $\mathrm{SO}(2,1)$, which are easily verified by going over to the corresponding elements of $\mathrm{SL}(2,R)$.

$$\tau_0 l_2\,(\zeta)\,\tau_0=l_2\,(-\zeta)\,, \tag{6. 28 a}$$

$$l_2\,(\zeta)\,t\,(s)=t\left(se^\zeta\right)l_2\,(\zeta)\,, \tag{6. 28 b}$$

$$\tau_0 t\,(s)\,\tau_0=\hat{t}\,(-s)\,, \tag{6. 28 c}$$

$$\hat{t}\,(-s)=t\,(-s)\,l_2\left(\ln\frac{1}{|s|}\right)\tau_0 t\left(-\frac{1}{s}\right),s\neq 0. \tag{6. 28 d}$$

where $\hat{t}\,(s)$ is the element of $\mathrm{SO}(2,1)$ corresponding to the element

$$\hat{S}\,(s)=\begin{bmatrix}1&0\\s&1\end{bmatrix}$$

of $\mathrm{SL}(2,R)$. Let us now obtain the properties of L-functions one by one.
I: Using the relation

$$l_2\,(\zeta)\,\tau_0 l_2\,(\zeta')\,\tau_0=l_2\,(\zeta-\zeta')$$

which follows immediately from equation (27a) , we get

$$\left(e^{\zeta-\zeta'}\right)^{\frac{1}{2}+i\rho}\delta\left(p-p'e^{\zeta-\zeta'}\right)$$

$$= \int_0^\infty d\hat{p}\langle p,\rho|U\left(l_2\left(\zeta'\right)\tau_0\right)|\hat{p},\rho\rangle\langle\hat{p},\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|p',\rho\rangle, \quad \text{when} \quad p>0, p'>0,$$

$$= 4e^{\left(\zeta+\zeta'\right)/2}\left(p/p'\right)^{i\rho}ch^2\left(\pi\rho\right)\int_0^\infty d\hat{p}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta'}{2}}\right)L_{2i\rho}\left(2\sqrt{\hat{p}p'}e^{\frac{\zeta}{2}}\right)$$

$$\Rightarrow \int_0^\infty d\hat{p}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta'}{2}}\right)L_{2i\rho}\left(2\sqrt{\hat{p}p'}e^{\frac{\zeta}{2}}\right)$$

$$= \frac{1}{4}\left(\frac{p'e^{\zeta}}{pe^{\zeta'}}\right)^{i\rho}\operatorname{sech}^2\left(\pi\rho\right)\delta\left(pe^{\zeta'}-p'e^{\zeta}\right).$$

Hence, putting

$$2\sqrt{pe^{\zeta'}}=a, \qquad 2\sqrt{p'e^{\zeta}}=b, \qquad \hat{p}=x,$$

(so that as $a>0$ and $b>0$) we get

$$\int_0^\infty dxL_{2i\rho}\left(a\sqrt{x}\right)L_{2i\rho}\left(b\sqrt{x}\right)=\frac{1}{4}\operatorname{sech}^2\left(\pi\rho\right)\left(\frac{a}{b}\right)^{2i\rho}\delta\left(a^2-b^2\right),$$

which, using $\delta\left(a+b\right)=0$, gives

$$\text{(6. 29)} \qquad \int_0^\infty dxL_{2i\rho}\left(a\sqrt{x}\right)L_{2i\rho}\left(b\sqrt{x}\right)=\frac{1}{8}\left(\frac{a}{b}\right)^{2i\rho}\operatorname{sech}^2\left(\pi\rho\right)\delta\left(a-b\right).$$

II: For $s\neq 0$, we have

$\langle p,\ \rho|U\left(l_2\left(\zeta\right)\tau_0 t\left(s\right)l_2\left(\zeta'\right)\tau_0\right)|p',\rho\ \rangle$

$$= \int_{-\infty}^\infty d\hat{p}\left\langle p,\rho|U\left(l_2\left(\zeta'\right)\tau_0\right)U\left(t\left(s\right)\right)|\hat{p},\rho\right\rangle\left\langle\hat{p}.\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|p',\rho\right\rangle$$

$$= \int_{-\infty}^\infty d\hat{p}e^{is\hat{p}}\left\langle p,\rho|U\left(l_2\left(\zeta'\right)\tau_0\right)|\hat{p},\rho\right\rangle\left\langle\hat{p},\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|p',\rho\right\rangle$$

$$= \int_{-\infty}^\infty d\hat{p}e^{is\hat{p}}\cdot 2e^{\frac{\zeta'}{2}}\left(\frac{p}{\hat{p}}\right)^{i\rho}ch\pi\rho L_{2i\rho}\left(2\sqrt{pp'}e^{\frac{\zeta'}{2}}\right)\cdot$$

$$2e^{\frac{\zeta}{2}}\left(\frac{\hat{p}}{p'}\right)^{i\rho}ch\pi\rho L_2 i\rho\left(2\sqrt{\hat{p}p'}e^{\frac{\zeta}{2}}\right) \text{ if } p,p'>0,$$

$$= 4e^{\frac{\left(\zeta+\zeta'\right)}{2}}\left(\frac{p}{p'}\right)^{i\rho}ch^2\pi\rho\int_0^\infty d\hat{p}e^{is\hat{p}}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta'}{2}}\right)L_{2i\rho}\left(2\sqrt{\hat{p}p'}e^{\frac{\zeta}{2}}\right).$$

On the other hand, for $s\neq 0$, we also have

$$\begin{aligned} l_2\left(\zeta\right)\tau_0 t\left(s\right)l_2\left(\zeta'\right)\tau_0 &= l_2\left(\zeta\right)\tau_0 t\left(s\right)\tau_0 l_2\left(-\zeta'\right)=l_2\left(\zeta\right)\hat{t}\left(-s\right)l_2\left(-\zeta'\right) \\ &= l_2\left(\zeta\right)t\left(-\frac{1}{s}\right)l_2\left(\ln\frac{1}{|s|}\right)\tau_0 t\left(-\frac{1}{s}\right)l_2\left(-\zeta'\right) \\ &= t\left(-\frac{e^{\zeta}}{s}\right)l_2\left(\zeta+\zeta'+\ln\left(\frac{1}{|s|}\right)\right)\tau_0 t\left(-\frac{e^{-\zeta'}}{s}\right), \end{aligned}$$

so that

$$
\begin{aligned}
&\langle p,\rho|U\left(l_2\left(\zeta\right)\tau_0 t\left(s\right)l_2\left(\zeta'\right)\tau_0\right)|p',\rho\rangle\\
&=\left\langle p,\rho|U\left(t\left(-\frac{e^{-\zeta'}}{s}\right)\right)U\left(l_2\left(\zeta+\zeta'+\ln\left(\frac{1}{|s|}\right)\right)\tau_0\right)U\left(t\left(-\frac{e^{-\zeta'}}{s}\right)\right)|p',\rho\right\rangle\\
&=e^{-i\ \ pe^{\zeta'}+P'e^{\zeta}\ \ /s}\cdot 2e^{\left(\zeta+\zeta'+\ln\left(\frac{1}{|s|}\right)\right)/2}\cdot ch\pi\rho\left(p/p'\right)^{i\rho}L_{2i\rho}\left(2\sqrt{p'p}e^{\left(\zeta+\zeta'+\ln\left(\frac{1}{|s|}\right)\right)/2}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&4e^{\left(\zeta+\zeta'\right)/2}\left(p/p'\right)^{i\rho}ch^2\pi\rho\int_0^\infty d\hat{p}e^{is\hat{p}}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta'}{2}}\right)L_{2i\rho}\left(2\sqrt{\hat{p}p'}e^{\frac{\zeta}{2}}\right)\\
&=2e^{-i\ \ pe^{\zeta'}+p'e^{\zeta}\ \ /s}e^{\left(\zeta+\zeta'+\ln\left(\frac{1}{|s|}\right)\right)/2}ch\pi\rho\left(p/p'\right)^{i\rho}L_{2i\rho}\left(2\sqrt{p'p}e^{\left(\zeta+\zeta'+\ln\left(\frac{1}{|s|}\right)\right)/2}\right)\\
&\Rightarrow\int_0^\infty d\hat{p}e^{is\hat{p}}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta'}{2}}\right)L_{2i\rho}\left(2\sqrt{\hat{p}p'}e^{\frac{\zeta}{2}}\right)\\
&=\frac{\operatorname{sech}\pi\rho}{2\sqrt{|s|}}e^{-i\ \ pe^{\zeta'}+p'e^{\zeta}\ \ /s}L_{2i\rho}\left(\frac{2}{\sqrt{|s|}}\sqrt{pp'}e^{\left(\zeta+\zeta'\right)/2}\right)
\end{aligned}
$$

(6. 30)

$$
\Rightarrow\int_0^\infty dxe^{isx}L_{2i\rho}\left(a\sqrt{x}\right)L_{2i\rho}\left(b\sqrt{x}\right)=\frac{\operatorname{sech}\pi\rho}{\sqrt{|s|}}e^{-i\left(a^2+b^2\right)/(4s)}L_{2i\rho}\left(\frac{ab}{2\sqrt{|s|}}\right),s\neq 0,
$$

where, as before

$$
a=2\sqrt{p}e^{\frac{\zeta'}{2}},\qquad b=2\sqrt{p'}e^{\frac{\zeta}{2}}.
$$

To get additional relations involving L-functions, we consider the mixed basis matrix elements of SO$(2,1)$ obtained by Kalnins [11]. He calls the coordinate systems corresponding to the three subgroup reductions

$$
SO\left(2,1\right)\supset SO\left(2\right),\qquad SO\left(2,1\right)\supset SO\left(1,1\right),\qquad SO\left(2,1\right)\supset T_1,
$$

as Spherical system **S**,
Hyperbolic System **H**,
Horospherical System **HO**,
respectively, and takes

$$
|\rho;M\rangle,\ \ |\rho;\pm;\tau\rangle,\ \ |\rho;S\rangle,
$$

as basis vectors in these systems (our basis vectors $|p,\rho\rangle$ are clearly to be identified with $|\rho;S\rangle$ of Kalnins) . He then says in his equation (2.5) that if

$$
h_1\left(a\right)=e^{N_1a}\quad\left(\equiv\text{our}\quad e^{N_2a}\right),
$$

then

$$
\langle\rho;M|h_1\left(a\right)|\rho;S\rangle=\frac{\left(-1\right)^M S^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\left(2S\right)^{\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2e^{-a}S\right),\quad S>0,
$$

where $W_{\mu\nu}\left(Z\right)$ is the Whittaker function as defined in Magnus et al [16]. However, we show in the Appendix B that there is an error in the expression on the RHS, and that the correct formula is

$$\langle\rho;M|h_1(a)|\rho;S\rangle=\frac{(-1)^M S^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}(S/2)^{\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2e^{-a}S\right),$$

which, in our notation, becomes

$$\text{(6. 31)}\qquad \langle M;\rho|U(l_2(\zeta))|p,S\rangle=\frac{(-1)^{M+1}p^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}(p/2)^{\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2pe^{\zeta}\right);$$

note that Kalnins e^{-a} becomes e^{ζ} in our notation as Kalnins take

$$U(g)|\xi\rangle=|\xi g\rangle,$$

while we take

$$U(g)|\xi\rangle=|g^{-1}\xi\rangle.$$

This formula has the following trivial consequences, used quite frequently, latter:

$$\text{(6. 32 a)}\qquad \langle p,\rho|U(l_2(\zeta))|M,\rho\rangle=\frac{(-1)^{M+1}p^{-1}}{\Gamma\left(\frac{1}{2}+i\rho-M\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}+i\rho}W_{-M,i\rho}\left(2pe^{-\zeta}\right),$$

$$\text{(6. 32 b)}\qquad \langle M,\rho|p,\rho\rangle=\frac{(-1)^{M+1}p^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}-i\rho}W_{-M,i\rho}(2p);$$

$$\text{(6. 32 c)}\qquad \langle p,\rho|M,\rho\rangle=\frac{(-1)^{M+1}p^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}+i\rho}W_{-M,i\rho}(2p);$$

$$\text{(6. 32 d)}\qquad \langle M,\rho|U(l_2(\zeta)\tau_0)|p,\rho\rangle=\frac{(-p)^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2pe^{\zeta}\right),$$

$$\text{(6. 32 e)}\qquad \langle p,\rho|U(l_2(\zeta)\tau_0)|M,\rho\rangle=\frac{-p^{-1}}{\Gamma\left(\frac{1}{2}+i\rho-M\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}+i\rho}W_{-M,i\rho}\left(2pe^{\zeta}\right);$$

note that here, we have used the property of Whittaker functions

$$W_{-M,-i\rho}(z)=W_{-M,i\rho}(z)$$

and the fact that

$$\langle M,\rho|U(\tau_0)\equiv\langle M,\rho|U(r_0(\pi))=e^{iM\pi}\langle M,\rho|.$$

III: We have

$$\left(e^{\zeta}\right)^{\frac{1}{2}+i\rho}\delta\left(p-p'e^{\zeta}\right)=\langle p,\rho|U(l_2(\zeta))|p',\rho\rangle$$
$$=\sum_M\langle p,\rho|M,\rho\rangle\langle M,\rho|U(l_2(\zeta))|p',\rho\rangle$$
$$=\sum_M\frac{(-1)^{M+1}p^{-1}}{\Gamma\left(\frac{1}{2}+i\rho-M\right)}\frac{(-1)^{M+1}p'^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}+i\rho}\left(\frac{p'}{2}\right)^{\frac{1}{2}-i\rho}W_{-M,i\rho}(2p)W_{-M,i\rho}\left(2p'e^{\zeta}\right)$$
$$\Rightarrow\sum_M\frac{\left(4pp'e^{\zeta}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-M+i\rho\right)\Gamma\left(\frac{1}{2}-M-i\rho\right)}\left(\frac{p}{p'e^{\zeta}}\right)^{i\rho}W_{-M,i\rho}(2p)W_{-M,i\rho}\left(2p'e^{\zeta}\right)$$
$$=\delta\left(p-p'e^{\zeta}\right).$$

Setting now

$$2p=a,\quad 2p'e^{\zeta}=b\Rightarrow 4pp'e^{\zeta}=ab,\quad \frac{p}{p'e^{\zeta}}=\frac{a}{b},$$

we get

$$\sum_M \frac{(ab)^{-\frac{1}{2}}\left(\frac{a}{b}\right)^{i\rho}}{\Gamma\left(\frac{1}{2}-M+i\rho\right)\Gamma\left(\frac{1}{2}-M-i\rho\right)} W_{-M,i\rho}(a)\, W_{-M,i\rho}(b) = 2\delta(a-b). \tag{6. 33}$$

IV: We have

$$\begin{aligned}
&2ch\pi\rho e^{\frac{\zeta}{2}}\left(p/p'\right)^{i\rho} L_{2i\rho}\left(2\sqrt{pp'}e^{\frac{\zeta}{2}}\right) = \langle p,\rho|U\left(l_2(\zeta)\,\tau_0\right)|p',\rho\rangle \\
&= \sum_M e^{iM\pi}\langle p,\rho|M,\rho\rangle\langle M,\rho|U\left(l_2(\zeta)\right)|p',\rho\rangle \qquad \text{as} \quad \tau_0 = r_0(\pi), \\
&= \sum_M (-1)^M \frac{(-1)^{M+1}p^{-1}}{\Gamma\left(\frac{1}{2}-M+i\rho\right)} \frac{(-1)^{M+1}p'^{-1}}{\Gamma\left(\frac{1}{2}-M-i\rho\right)} \times \\
&\qquad \left(\frac{p}{2}\right)^{\frac{1}{2}+i\rho}\left(\frac{p'}{2}\right)^{\frac{1}{2}-i\rho} W_{-M,i\rho}(2p)\, W_{-M,i\rho}\left(2p'e^{\zeta}\right)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \sum_M \frac{(-1)^M\left(4pp'e^{\zeta}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-M+i\rho\right)\Gamma\left(\frac{1}{2}-M-i\rho\right)} W_{-M,i\rho}(2p)\, W_{-M,i\rho}\left(2p'e^{\zeta}\right) \\
&= 2ch\pi\rho L_{2i\rho}\left(2\sqrt{pp'}e^{\frac{\zeta}{2}}\right) \\
&\Rightarrow \sum_M \frac{(-1)^M(ab)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-M+i\rho\right)\Gamma\left(\frac{1}{2}-M-i\rho\right)} W_{-M,i\rho}(a)\, W_{-M,i\rho}(b) \\
&= 2ch\pi\rho L_{2i\rho}\left(2\sqrt{ab}\right).
\end{aligned} \tag{6. 34}$$

where

$$a = 2p, b = 2p'e^{\zeta}.$$

V: We have

$$\begin{aligned}
&\frac{-p^{-1}}{\Gamma\left(\frac{1}{2}-M+i\rho\right)}\left(\frac{p}{2}\right)^{\frac{1}{2}-i\rho} W_{-M,i\rho}\left(2pe^{\zeta}\right) = \langle M,\rho|U\left(l_2(\zeta)\,\tau_0\right)|p,\rho\rangle \\
&= \int_0^\infty d\hat{p}\,\langle M,\rho|\hat{p},\rho\rangle\langle\hat{p},\rho|U\left(l_2(\zeta)\,\tau_0\right)|p,\rho\rangle, \qquad \text{when} \quad p>0, \\
&= \int_0^\infty d\hat{p}\frac{(-1)^{M+1}\hat{p}^{-1}}{\Gamma\left(\frac{1}{2}-M-i\rho\right)}\left(\frac{\hat{p}}{2}\right)^{\frac{1}{2}-i\rho} W_{-M,i\rho}(2\hat{p}) \cdot 2ch\pi\rho e^{\frac{\zeta}{2}}\left(\frac{\hat{p}}{p}\right)^{i\rho} L_{2i\rho}\left(2\sqrt{\hat{p}p}e^{\frac{\zeta}{2}}\right) \\
&\Rightarrow \int_0^\infty d\hat{p}\hat{p}^{-\frac{1}{2}} W_{-M,i\rho}(2\hat{p})\, L_{2i\rho}\left(2\sqrt{\hat{p}p}e^{\frac{\zeta}{2}}\right) \\
&= \left(\frac{1}{2}\right)(-1)^{M+1}\operatorname{sech}\pi\rho\left(pe^{\zeta}\right)^{-\frac{1}{2}} W_{-M,i\rho}\left(2pe^{\zeta}\right)
\end{aligned}$$

(6. 35)

$$\Rightarrow \int_0^\infty dx W_{-M,i\rho}\left(x^2\right) L_{2i\rho}(xa) = \frac{1}{2a}(-1)^{M+1}\operatorname{sech}\pi\rho W_{-M,i\rho}\left(a^2\right)$$

where

$$x = \sqrt{2\hat{p}}, a = \sqrt{2pe^{\zeta}}.$$

Equations (32) to (34) are three new relations involving the L-functions and Whittaker functions. Finally, we obtain a couple of relations involving L-functions, Whittaker functions and confluent hypergeometric functions by using equations (2.26) on page 656 and equation (2.27) on page 657 of Kalnins [11]. Actually, there are some errors in these equations and the corrected equations, in our notation, are

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\right)|\tau,+,\rho\rangle = \frac{1}{2\pi}[\left(-\frac{1}{2ip}\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho+i\tau\right)W_{-i\tau-i\rho}\left(-2ipe^{-\zeta}\right)$$

$$+\left(\frac{1}{2ip}\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho-i\tau\right)W_{i\tau-i\rho}\left(2ipe^{-\zeta}\right)]. \tag{6. 36 a}$$

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\right)|\tau,-,\rho\rangle = \frac{1}{2\pi}\left(\frac{e^{\zeta}}{4}\right)^{-\frac{1}{2}+i\rho}e^{-ipe^{-\zeta}}$$

$$B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;2ipe^{-\zeta}\right). \tag{6. 36 b}$$

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|\tau,+,\rho\rangle = \frac{1}{2\pi}\left(\frac{e^{-\zeta}}{4}\right)^{-\frac{1}{2}+i\rho}e^{ipe^{\zeta}}$$

$$B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;-2ipe^{\zeta}\right). \tag{6. 36 c}$$

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|\tau,-,\rho\rangle = \frac{1}{2\pi}\left[\left(\frac{1}{2ip}\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho+i\tau\right)W_{-i\tau-i\rho}\left(2ipe^{\zeta}\right)\right.$$

$$\left.+\left(-\frac{1}{2ip}\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho-t\tau\right)W_{i\tau-i\rho}\left(-2ipe^{\zeta}\right)\right]. \tag{6. 36 d}$$

Note that

(1) Equation (35 a) is the error removed form of Kalnins [11] equation (2.27) on page 657.
(2) Equation (35 b) is Kalnins [11] equation (2.26) on page 656, with the middle part deleted.
(3) Equation (35 c) is the error removed form of Kalnins [11] equation (2.26), page 656, with its first part deleted.
(4) Equation (35 d) has not been mentioned by Kalnins.

We give the proof of equation (35 a) in Appendix C; the rest of equations (35 b, 35c, 35d) are proved in a similar manner. We now proceed with the derivation of the couple of relations mentioned earlier.

VI: Using $(34c)$, we get

$$\frac{1}{2\pi}\left(\frac{e^{-\zeta}}{4}\right)^{-\frac{1}{2}+i\rho} B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)e^{ipe^{\zeta}}F_{11}\left(\frac{1}{2i\rho}-i\tau,1-2i\rho;-2ipe^{\zeta}\right)$$
$$=\langle p,\rho|U\left(l_2\left(\zeta\right)\right)|\tau,-,\rho\rangle$$
$$=\int_0^\infty d\hat{p}\langle p,\rho|U\left(l_2\left(\zeta\right)\tau_0\right)|\hat{p},\rho\rangle\langle\hat{p},\rho|\tau,+,\rho\rangle \quad \text{for} \quad p>0,$$
$$=\int_0^\infty d\hat{p}\ 2e^{\frac{\zeta}{2}}\left(\frac{p}{\hat{p}}\right)^{i\rho}\ ch\pi\rho\ L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta}{2}}\right)\cdot$$
$$\frac{1}{2\pi}\{(-i\hat{p}/2)^{-\frac{1}{2}+i\rho}\ \Gamma\left(\frac{1}{2}-i\rho+i\tau\right)\ W_{-i\tau-i\rho}\left(-2i\hat{p}\right)$$
$$+\left(i\hat{p}/2\right)^{-\frac{1}{2}+i\rho}\ \Gamma\left(\frac{1}{2}-i\rho-i\tau\right)\ W_{i\tau-i\rho}\left(2i\hat{p}\right)\}$$

using (6. 36 a) with $\zeta=0$,

$$\Rightarrow\int_0^\infty d\hat{p}\ \hat{p}^{-\frac{1}{2}}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{\frac{\zeta}{2}}\right)\{(-2i)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho+i\tau\right)W_{-i\tau-i\rho}\left(-2i\hat{p}\right)$$
$$+\left(2i\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho-i\tau\right)W_{i\tau-i\rho}\left(2i\hat{p}\right)\}$$
$$=\frac{1}{2}\operatorname{sech}\pi\rho\left(pe^{\zeta}\right)^{-i\rho}e^{ipe^{\zeta}}\cdot$$
$$B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;-2ipe^{\zeta}\right).$$

Putting

$$\hat{p}=x,\ \ \textit{and}\ \ pe^{\zeta}=a,$$

we get

$$\int_0^\infty dx\ \ x^{-\frac{1}{2}}L_{2i\rho}\left(2\sqrt{ax}\right)\{(-2i)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho+i\tau\right)W_{-i\tau-i\rho}\left(-2ix\right)$$

$$+\left(2i\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho-i\tau\right)\ W_{i\tau-i\rho}\left(2ix\right)\}$$

(6. 37)
$$=\frac{1}{2}\operatorname{sech}\pi\rho\ a^{-i\rho}e^{ia}\ B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)\ F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;-2ia\right).$$

VII: Using (6. 36 a) , we get

$$\frac{1}{2\pi}\{(-1/2ip\ )^{-\frac{1}{2}+i\rho}\ \Gamma\left(\frac{1}{2}-i\rho+i\tau\right)\ W_{-i\tau-i\rho}\left(-2ipe^{-\zeta}\right)$$

$$+\left(ip/2\right)^{-\frac{1}{2}+i\rho}\Gamma\left(\frac{1}{2}-i\rho-i\tau\right)\ W_{i\tau-i\rho}\left(2ipe^{-\zeta}\right)\}$$

$$=\langle p,\rho|U\left(l_2\left(\zeta\right)\right)|\tau,+,\rho\rangle$$

$$=\int_0^\infty d\hat{p}\,\langle p,\rho|U\left(l_2\left(-\zeta\right)\tau_0\right)|\hat{p},\rho\rangle\,\langle\hat{p},\rho|U\left(\tau_0\right)|\tau,+,\rho\rangle\quad for\quad p>0,$$

$$=\int_0^\infty d\hat{p}2e^{-\zeta/2}\left(\frac{p}{\hat{p}}\right)^{i\rho}\ ch\pi\rho\ \ L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{-\zeta/2}\right).\frac{1}{2\pi}\left(1/4\right)^{-\frac{1}{2i\rho}}$$

$$\cdot\ B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)e^{i\hat{p}}F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;-2i\hat{p}\right)\tag{6. 38}$$

using (6. 36 c) with $\zeta=0$,

$$\Rightarrow\int_0^\infty d\hat{p}e^{i\hat{p}}\hat{p}^{-i\rho}L_{2i\rho}\left(2\sqrt{p\hat{p}}e^{-\zeta/2}\right)F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;-2i\hat{p}\right)$$

$$=\frac{1}{2}\frac{e^{\frac{\zeta}{2}}\ \operatorname{sech}\pi\rho\ \ p^{-\frac{1}{2}}}{B\left(\frac{1}{2}-i\rho_1\tau,\frac{1}{2}-i\rho+i\tau\right)}(-2i)^{-\frac{1}{2}+i\rho}\,\Gamma\left(\frac{1}{2}-i\rho+i\tau\right)W_{-i\tau-i\rho}\left(-2ipe^{-\zeta}\right)$$

$$+\left(2i\right)^{-\frac{1}{2}+i\rho}\Gamma\left(1/-i\rho-i\tau\right)W_{i\tau-i\rho}\left(2ipe^{-\zeta}\right),$$

so that putting

$$\hat{p}=x,\ \ pe^{-\zeta}=a,$$

we get

$$\int_0^\infty dx\ \ x^{-i\rho}e^{ix}L_{2i\rho}\left(2\sqrt{ax}\right)F_{11}\left(\frac{1}{2}-i\rho-i\tau,1-2i\rho;2ix\right)$$

$$=\frac{\operatorname{sech}\pi\rho}{2\sqrt{a}B\left(\frac{1}{2}-i\rho-i\tau,\frac{1}{2}-i\rho+i\tau\right)}\{(-2i)^{-\frac{1}{2}+i\rho}\,\Gamma\left(\frac{1}{2}-i\rho+i\tau\right)$$

$$\cdot\ W_{-i\tau-i\rho}\left(-2ia\right)+\left(2i\right)^{-\frac{1}{2}+i\rho}\ \Gamma\left(\frac{1}{2}-i\rho-i\tau\right)\ W_{i\tau-i\rho}\left(2ia\right)\}.\tag{6. 39}$$

(6. 37), & (6. 39) are the two relations between the L-functions, Whittaker functions and the confluent hypergeometric functions, mentioned earlier. However, using the "orthogonality" of the L-functions given by Equation (6. 29), each one of these relations can be shown to be derivable from the other one. Thus, for example, if we multiply (32) by $L_{2i\rho}\left(2\sqrt{ab}\right)$ and integrate with respect to a, we are lead to equation Equation (33) ; conversely, equation (32) can also be obtained from equation (33) in a similar manner.

7. Conclusion

By using a particular parameterization of points on a 3-dimensional cone, as suggested by Kalnins[11], we are able to obtain the UIR matrix elements of arbitrary finite rotations of SO(2, 1) in a representation of principle series and of integral type, which is decomposed according to the non-compact subgroup T_1. We note the extreme simplicity of our computations as compared to those for decomposition according to the subgroup SO$(1,1)$ which were carried out (only for two one parameter subgroups of SO$(2,1)$) by Mukunda[19] much earlier. The matrix elements have been expressed in closed form in terms of a new type of Bessel function $L_\nu(z)$ which may be considered as a comparison function of the well known modified Bessel function $K_\nu(z)$. We have also obtained many new relations between these new Bessel functions $L_\nu(z)$, Whittaker functions and the confluent hypergeometric functions, by using the standard techniques of group representation theory. In the end, we would like to mention that some errors in some of the equations of Kalnins [11] have been pointed out and the corresponding error-free equations have been given.

Appendix A

Let I be the integral

$$I = \int_{-\infty}^{\infty} dx \left(x^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(p'x+\frac{p}{x}\right)};$$

we show in this Appendix that it vanishes if p and p' has opposite signs, as stated in the text. If C is the contour as shown in the following diagram (Figure 1)

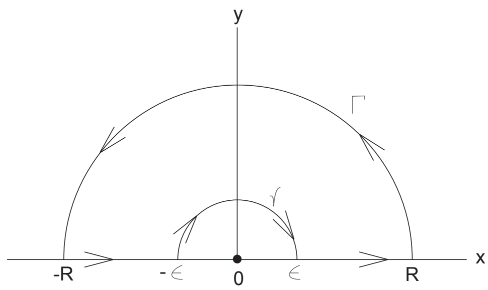

Figure 1. The Contour C

consisting of the real axis from ϵ to R (where ϵ is a very small and R is a very large positive real number) , the upper semicircle $\Gamma : |z| = R$, the real axis from $-R$ to $-\epsilon$ and the upper semicircle $\gamma : |z| = \epsilon$, then

$$\left(z^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(\acute{p}z+\frac{p}{z}\right)}$$

is regular throughout C, and so we get

$$\int_C dz \left(z^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(\acute{p}z+\frac{p}{z}\right)} = 0$$

$$\text{(A-1)} \qquad \Rightarrow \left(\int_\Gamma dz + \int_{-R}^{-\epsilon} dz + \int_\gamma dz + \int_\epsilon^R dz\right)\left(\left(z^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(\acute{p}z+\frac{p}{z}\right)}\right) = 0.$$

We now show that as $\epsilon \to 0$ and $R \to \infty$, the integrals over γ and Γ vanish whenever $p > 0$ and $\acute{p} < 0$. For, on Γ, we have

$$z = Re^{i\theta} \Rightarrow \frac{dz}{z} = i d\theta,$$

$$z = R\left(\cos\theta + i\sin\theta\right), \frac{1}{z} = \frac{1}{R}\left(\cos\theta - i\sin\theta\right),$$

so that

$$\int_{\Gamma} dz \left(z^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(p'z+\frac{p}{z}\right)}$$

$$= \int_0^{\pi} id\theta.Re^{i\theta}\left(R^2e^{2i\theta}\right)^{-\frac{1}{2}+i\rho} e^{-i\left\{p'R(\cos\theta+i\sin\theta)+\frac{p}{R}(\cos\theta-i\sin\theta)\right\}}$$

$$= i\int_0^{\pi} d\theta e^{2i\rho(LnR+i\theta)-ip'R(\cos\theta+i\sin\theta)-i\frac{p}{R}(\cos\theta-i\sin\theta)}$$

$$= i\int_0^{\pi} d\theta e^{i\left(2\rho LnR-\left(p'R+\frac{p}{R}\right)\cos\theta\right)} e^{-2\rho\theta+\left(p'R-\frac{p}{R}\right)\sin\theta}$$

$$\to 0 \text{ as } R\to\infty \quad \text{whenever } p'<0, \ as \ \sin\theta>0.$$

Similarly, on γ, which is described clockwise, we will have

$$z = \epsilon e^{i\theta}$$

so that

$$\int_{\gamma} dz \left(z^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(p'z+\frac{p}{z}\right)}$$

$$= -i\int_0^{\pi} d\theta e^{i\left(2\rho Ln\epsilon-\left(p'\epsilon+\frac{p}{\epsilon}\right)\cos\theta\right)} \cdot e^{-2\rho\theta+\left(p'\epsilon+\frac{p}{\epsilon}\right)\sin\theta}$$

$$\to 0 \text{ as } \epsilon\to 0, \text{whenever } p>0.$$

Hence, taking the limit as $\epsilon\to 0$ and $R\to\infty$, equation (A-1) will give

$$I = \int_{-\infty}^{\infty} dx \left(x^2\right)^{-\frac{1}{2}+i\rho} e^{-i\left(p'x+\frac{p}{x}\right)} = 0$$

Whenever $p>0$ and $p'<0$. That $I=0$ whenever $p<0$ and $p'>0$, can be proved ion exactly the same manner by choosing the contour C' which is the mirror image of C in the real axis. Hence, it follows that

$$I = 0$$

whenever p and p' have opposite signs, as asserted at the beginning.

Appendix B

In this Appendix, we show that there is an error in equation $(2.5)\,,p.655$, of Kalnins[11] paper by obtaining its error-free form. The integral on the RHS of equation $(2.4)\,,p.655$, of this reference

$$\frac{1}{2\pi}\int_{\infty}^{\infty} dr \left(\frac{\bar{\omega_s}}{\omega_E}\right)^{-\frac{1}{2}+i\rho} e^{iM\phi}e^{-iSr}$$

with

$$\frac{\bar{\omega_s}}{\omega_E} = e^a\left(r^2+e^{-2a}\right), \quad e^{i\phi} = \frac{r+ie^{-a}}{r-ie^{-a}},$$

becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dr\,(e^a)^{-\frac{1}{2}+i\rho}\left(r^2+e^{-2a}\right)^{-\frac{1}{2}+i\rho}\left(\frac{r+ie^{-a}}{r-ie^{-a}}\right)^M e^{-iSr}$$

$$=\frac{1}{2\pi}(e^a)^{-\frac{1}{2}+i\rho}\int_{-\infty}^{\infty} dr\left(e^{-a}+ir\right)^{-\frac{1}{2}+i\rho}\left(e^{-a}-ir\right)^{-\frac{1}{2}+i\rho}\left(\frac{i\left(e^{-a}-ir\right)}{-i\left(e^{-a}+ir\right)}\right)^M e^{-iSr}$$

$$=\frac{1}{2\pi}(-1)^M(e^a)^{-\frac{1}{2}+i\rho}\int_{-\infty}^{\infty} dr\left(e^{-a}+ir\right)^{-\frac{1}{2}+i\rho-M}\left(e^{-a}-ir\right)^{-\frac{1}{2}+i\rho+M}e^{-iSr}.$$

Now in Erdelyi et el [9], Formula (12) on $p.119$, says

$$\int_{-\infty}^{\infty} dx\,(\alpha+ix\,)^{-2\mu}(\beta-ix)^{-2\nu}e^{-ixy}$$
$$=-2\pi(\alpha+\beta)^{-\nu-\mu}[\Gamma(2\nu)]^{-1}y^{\nu+\mu+1}W_{\nu-\mu,\frac{1}{2}-\nu-\mu}\left((\alpha+\beta)\,y\right),y>0.$$

Hence, taking

$$x=r,y=S,\alpha=\beta=e^{-a},-2\mu=-\frac{1}{2}+i\rho-M,-2\nu=-\frac{1}{2}+i\rho+M,$$

so that

$$\nu+\mu-1=-\frac{1}{2}-i\rho,\quad \frac{1}{2}-\nu-\mu=i\rho,$$

we will get

$$\frac{1}{2\pi}(-1)^M(e^a)^{-\frac{1}{2}+i\rho}\int_{-\infty}^{\infty} dr\left(e^{-a}+ir\right)^{-\frac{1}{2}+i\rho-M}\left(e^{-a}-ir\right)^{-\frac{1}{2}+i\rho+M}e^{-iSr}$$

$$=\frac{1}{2\pi}(-1)^M(e^a)^{-\frac{1}{2}+i\rho}\cdot\frac{(-2\pi)\left(2e^{-a}\right)^{-\frac{1}{2}+i\rho}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\cdot e^0S^{-\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2e^{-a}S\right),\quad S>0,$$

$$=\frac{(-1)^M(2)^{-\frac{1}{2}+i\rho}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}S^{-1}S^{\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2e^{-a}S\right)$$

$$=\frac{(-1)^{M+1}\cdot S^{-1}}{\Gamma\left(\frac{1}{2}-i\rho-M\right)}\left(\frac{S}{2}\right)^{\frac{1}{2}-i\rho}W_{-M,i\rho}\left(2e^{-a}S\right).$$

This is the error-free expression for the RHS of equation (2.5) of Kalnins[11].

Appendix C

In this Appendix, we prove the equation (6. 36 a) of the text, which gives the value of the matrix element

$$\langle p,\rho|U\left(l_2\left(\zeta\right)\right)|\tau,+,\rho\rangle\,.$$

As $|\tau,+,\rho\rangle$ is a vector of Kalnins H-system, we parameterize ξ according to Kalnin's [11] Equation (1.10) as

$$\xi=\omega\left(ch\theta,1,sh\theta\right).$$

Then

$$\acute{\xi} = l_2^{-1}(\zeta)\,\xi = \omega \begin{bmatrix} ch\zeta ch\theta - sh\zeta \\ -sh\zeta ch\theta + ch\zeta \\ sh\theta \end{bmatrix} = \acute{\omega} \begin{bmatrix} (\acute{r}^2+1) \\ (\acute{r}^2-1) \\ 2r \end{bmatrix}$$

as $\acute{\xi}$ has to be parameterized according to Kalnins HO-system. This gives

$$\frac{\acute{\omega}}{\omega} = e^{\zeta} sh^2\theta/2, \quad \acute{r} = e^{-\zeta}\coth\theta/2,$$

so that Kalnins[11] equation(2.25) (in our notation) gives

$$\begin{aligned}\langle p,\rho| U(l_2(\zeta))|\tau,+,\rho\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\left(e^{\zeta}sh^2\theta/2\right)^{-\frac{1}{2}+i\rho} e^{ipe^{-\zeta}}\coth\theta/2\ \ e^{-ir\theta} \\ &= \frac{\left(e^{\zeta}\right)^{-\frac{1}{2}+i\rho}}{2\pi}(I_1+I_2)\end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{\infty} d\theta\left(sh^2\theta/2\right)^{-\frac{1}{2}+i\rho} e^{ipe^{-\zeta}}\coth\theta/2\left(e^{\theta}\right)^{-ir}, \\ I_2 &= \int_{-\infty}^{0} d\theta\left(sh^2\theta/2\right)^{-\frac{1}{2}+i\rho} e^{ipe^{-\zeta}}\coth\theta/2\left(e^{\theta}\right)^{-ir}. \end{aligned}$$

To evaluate I_1, we put

$$\coth\theta/2 = s \Rightarrow d\theta = -2sh^2\theta/2ds, sh^2\theta/2 = \frac{1}{(s-1)(s+1)}, e^{\theta} = \frac{s+1}{s-1},$$

$$as\ \theta\to 0^+,\ s\to\infty, \quad as\ \theta\to\infty,\ s\to 1,$$

so that

$$I_1 = 2\int_1^{\infty} ds\left(sh^2\theta/2\right)^{\frac{1}{2}+i\rho} e^{ipe^{-\zeta}s}\left(e^{\theta}\right)^{-ir}$$

$$= 2\int_1^{\infty} ds\,(s-1)^{-\frac{1}{2}-i\rho}(s+1)^{-\frac{1}{2}-i\rho} e^{ipe^{-\zeta}s}(s-1)^{ir}(s+1)^{-ir}$$

$$= 2\int_1^{\infty} ds\,(s-1)^{-\frac{1}{2}-i\rho+i\tau}(s+1)^{-\frac{1}{2}-i\rho-i\tau} e^{ipe^{-\zeta}s}$$

$$= 2\int_0^{\infty} 2dt\,(2t)^{-\frac{1}{2}-i\rho+i\tau}(2(t+1))^{-\frac{1}{2}-i\rho-i\tau} e^{ipe^{-\zeta(2t+1)}}, (s-1=2t)$$

$$= 2^{1-2i\rho}e^{ipe^{-\zeta}}\int_0^{\infty} dtt^{-\frac{1}{2}-i\rho+i\tau}(t+1)^{-\frac{1}{2}-i\rho-i\tau} e^{2ipe^{-\zeta}t}.$$

Now, according to the formula (18) on p.274 of Erdelye et el [9]

$$\Gamma\left(\frac{1}{2}-\kappa+\mu\right)W_{\kappa,\mu}(x) = e^{-\frac{1}{2}\ \ x}x^{\mu+\frac{1}{2}}\int_0^{\infty} dte^{-tx}t^{-\frac{1}{2}-\kappa+\mu}(t+1)^{-\frac{1}{2}+\kappa+\mu}$$

$$\Rightarrow \int_0^{\infty} dte^{-tx}t^{-\frac{1}{2}-\kappa+\mu}(t+1)^{-\frac{1}{2}+\kappa+\mu} = e^{\frac{1}{2x}}x^{-\mu-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\kappa+\mu\right)W_{\kappa,\mu}(x),$$

so that taking

$$x = -2ipe^{-\zeta}, \quad \kappa = -i\tau, \quad \mu = -i\rho,$$

we get

$$I_1 = \left(\frac{1}{4}\right)^{-\frac{1}{2}+i\rho} e^{ipe^{-\zeta}} e^{-ipe^{-\zeta}} \left(-2ipe^{-\zeta}\right)^{-\frac{1}{2}+i\rho} \Gamma\left(\frac{1}{2} - i\rho + i\tau\right) W_{-i\tau,-i\rho}\left(-2ipe^{-\zeta}\right)$$

$$= \left(-\frac{1}{2ip} e^{-\zeta}\right)^{-\frac{1}{2}+i\rho} \Gamma\left(\frac{1}{2} - i\rho + i\tau\right) W_{-i\tau,-i\rho}\left(-2ipe^{-\zeta}\right).$$

Next

$$I_2 = \int_{-\infty}^{0} d\theta \left(sh^2\theta\right)^{-\frac{1}{2}+i\rho} e^{ipe^{-\zeta}} \coth\theta/2e^{-i\tau\theta}$$

$$= \int_{0}^{\infty} d\theta \left(sh^2\theta\right)^{-\frac{1}{2}+i\rho} e^{-ipe^{-\zeta}} \coth\theta/2e^{i\tau\theta}$$

$$= I_1, \text{ with } p \text{ and } \tau \text{ replaced by } -p \text{ and } -\tau \text{ respectively,}$$

$$= \left(\frac{1}{2} ipe^{-\zeta}\right)^{-\frac{1}{2}+i\rho} \Gamma\left(\frac{1}{2} - i\rho - i\tau\right) W_{i\tau,-i\rho}\left(2ipe^{-\zeta}\right),$$

so that we finally get

$$\langle p, \rho | U\left(l_2\left(\zeta\right)\right) \tau, +, \rho\rangle$$
$$= \frac{1}{2\pi}\{\left(-\frac{1}{2}ip\right)^{-\frac{1}{2}+i\rho} \Gamma\left(\frac{1}{2} - i\rho + i\tau\right) W_{-i\tau,-i\rho}\left(-2ipe^{-\zeta}\right)$$
$$+ \left(\frac{1}{2}ip\right)^{-\frac{1}{2}+i\rho} \Gamma\left(\frac{1}{2} - i\rho - i\tau\right) W_{i\tau,-i\rho}\left(2ipe^{-\zeta}\right)\}$$

as stated in equation (6. 36 a) of the text.

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