Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 48(1)(2016) pp. 99-109

A Study of Anti-Commutativity in AG-Groupoids

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Received: 19 October, 2015 / Accepted: 14 March, 2016 / Published online: 21 March, 2016

Abstract. A magma that also satisfies the left invertive law, $(ab)\,c = (cb)\,a$ is called an AG-groupoid. Generally, an AG-groupoid is a non-associative structure lying midway between a groupoid and a commutative semigroup. We consider the notion of anti-commutativity in AG-groupoids and investigate some of their properties. A new subclass of AG-groupoids as rectangular AG-groupoid is introduced and investigated. A variety of examples and counter examples are produced using the latest computational techniques of GAP, Mace4 and Prover9.

AMS (MOS) Subject Classification Codes: 20N05, 20N02, 20N99
Key Words: Anti-commutative, AG-groupoid, unipotent, right-Bol.

1. Introduction and Preliminaries

An AG-groupoid S is the most interesting non-associative algebraic structure in which the left invertive law, $(ab)\,c = (cb)\,a$ holds. It lies midway between a groupoid and a commutative semigroup. Some new subclasses of AG-groupoids have been discovered recently in [1, 9] and interesting future work has been mentioned in these subclasses. Anti-commutative AG-groupoid is one of these newly discovered subclasses of AG-groupoids. An AG-groupoid S in which the identity $ab = ba \Rightarrow a = b$ holds for all $a, b \in S$ [9] is called anti-commutative AG-groupoid. In this

paper, we will investigate anti-commutativity in AG-groupoids. We will also find some relationships between anti-commutative AG-groupoids and other subclasses of AG-groupoids such as quasi-cancellative, right distributive and left distributive AG-groupoids. Throughout this article, S will represent an AG-groupoid otherwise stated else. S is called right quasi-cancellative if the condition, $x^2 = xy \,\&\, y^2 = yx$ implies $x = y \;\forall x, y \in S$ holds. Similarly, S is said to be a left quasi-cancellative if it satisfies the condition $x^2 = yx \,\&\, y^2 = xy$ implies $x = y$ for all $x, y \in S$. S is said to be quasi-cancellative if it is both right and left quasi-cancellative [10]. We will use the notation "$\cdot$" to avoid the frequent use of parenthesis in our calculations while proving results, e.g. $(ab \cdot c)\, d$ will be the same as $((ab)\, c)\, d$.
AG-groupoid is a well worked area of research, various articles are recently published on different concepts in the last few years. Modulo matrix AG-groupoids are recently constructed in[2, 3]. The concept of ideals and LA-rings in theory of AG-groupoids is introduced by Q. Mushtaq [6, 7]. Fuzzification of various concepts in AG-groupoids has also been done by various researchers recently. AG-groupoids have a variety of applications in flocks theory, finite mathematics, geometry and other algebras. In this article, we define a new subclass of AG-groupoids as a rectangular AG-groupoid. The existence of this AG-groupoid is proved by various non associative examples. In the following we give a table of definitions that arise in various papers like, [1, 8, 9] and is used in the rest of this article.

AG-groupoid	Defining identity	AG-groupoid	Defining identity
AG**	$a\,(bc) = b\,(ac)$	unipotent	$a^2 = b^2$
T^1	$ab = cd \Rightarrow ba = dc$	AG-3-band	$(aa)\,a = a$
T^2	$ab = cd \Rightarrow ac = bd$	right distributive	$(ab)\,c = (ac)\,(bc)$
T^3_l	$ab = ac \Rightarrow ba = ca$	left distributive	$a\,(bc) = (ab)\,(ac)$
T^3_r	$ba = ca \Rightarrow ab = ac$	paramedial	$ab \cdot cd = db \cdot ca$
T^3	both T^3_l and T^3_r	Jordan	$a\,(bb.c) = bb \cdot ac$
right-Bol	$a\,(bc \cdot b) = (ab \cdot c)\,b$	medial	$ab \cdot cd = ac \cdot bd$

Table 1. AG-groupoids with their identities

1.1. **Relation between anti-commutative- and T^1-AG-groupoids.**

Anti-commutative AG-groupoids and T^1-AG-groupoids are two different subclasses of -AG-groupoids as shown in the following examples.

Example 1. *Let $S = \{1, 2, 3, 4\}$. Then*

(i) $(S, \cdot)$ is an anti-commutative AG-groupoid of order 4 which is not a T^1-AG-groupoid.
(ii) $(S, \star)$ is T^1-AG-groupoid of order 4 which is not an anti-commutative AG-groupoid.

$\cdot$	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

(i)

$\star$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	3	3
4	1	1	3	3

(ii)

Now, in the following theorem we investigate relationship of an anti-commutative T^1-AG-groupoid with quasi-cancellative, unipotent, AG-3-band and T^2-AG-groupoid.

Theorem 1. *Let S be an anti-commutative T^1-AG-groupoid, then the following hold;*

(i) S is quasi-cancellative
(ii) S is unipotent
(iii) S is AG-3-band
(iv) S is T^2-AG-groupoid.

Proof. Let S be an anti-commutative T^1-AG-groupoid and $x, y \in S$.

(i) Let

$$\begin{aligned} x^2 &= xy \qquad (1.1)\\ \Rightarrow xx &= xy \\ \Rightarrow xx &= yx \quad \text{by } T^1 \\ \Rightarrow x^2 &= yx \qquad (1.2)\\ \Rightarrow xy &= yx \quad \text{by (1.1) \& (1.2)} \\ \Rightarrow x &= y \quad \text{by anti-commutativity} \end{aligned}$$

Similarly let,

$$\begin{aligned} y^2 &= yx \qquad (1.3)\\ \Rightarrow yy &= yx \\ \Rightarrow yy &= xy \quad \text{by } T^1\text{property} \\ \Rightarrow y^2 &= xy \qquad (1.4)\\ \Rightarrow yx &= xy \quad \text{by 1.3 \& 1.4} \end{aligned}$$

Thus by anti-commutativity we have $x = y$. Hence S is left quasi-cancellative. Similarly, it is easy to prove that S is right quasi-cancellative and hence is quasi-cancellative.

(ii) Now we show that every anti-commutative T^1-AG-groupoid is unipotent. Let $a, b \in S$. Then by medial law, anti-commutativity and definition of T^1-AG-groupoid we have,

$$\begin{aligned} a^2b^2 &= aa \cdot bb = ab \cdot ab \\ \Rightarrow b^2a^2 &= ab \cdot ab = aa \cdot bb = a^2b^2 \\ \text{Thus } a^2 &= b^2. \end{aligned}$$

Hence $\forall a, b \in S, a^2 = b^2$, thus S is unipotent.

(iii) Let S be an anti-commutative T^1-AG-groupoid and $a \in S$. Then

$$\begin{aligned} (aa \cdot a)\, a &= aa \cdot aa \quad \text{by left invertive law} \qquad (1.5)\\ a\,(aa \cdot a) &= aa \cdot aa \quad \text{by } T^1 \qquad (1.6)\\ (aa \cdot a)\, a &= a\,(aa \cdot a) \quad \text{by (1.5) \& (1.6)} \end{aligned}$$

Thus by anti-commutativity we have $(aa)\, a = a, \forall a \in S$. Hence S is AG-3-band.

(iv) Let S be an anti-commutative T^1-AG-groupoid and $a,b,c,d \in S$. Assume that $ab = cd$ we prove that $ac = bd$. Using the assumption, definition of T^1-AG-groupoid, medial law and anti-commutativity since,

$$\begin{aligned}
ac \cdot bd &= ab \cdot cd \\
&= cd \cdot ab = ca \cdot db \quad \text{by assumption \& medial law} \qquad (1.7)\\
\Rightarrow bd \cdot ac &= db \cdot ca \quad \text{by } T^1 \qquad (1.8)\\
\Rightarrow ba \cdot dc &= dc \cdot ba \quad \text{by medial law}\\
\Rightarrow ba &= dc \quad \text{by anti-commutativity} \qquad (1.9)\\
\Rightarrow bd \cdot ac &= dc \cdot ba \quad \text{by (1.1) \& medial law}\\
\Rightarrow ac \cdot bd &= ba \cdot dc \quad \text{by } T^1\\
\Rightarrow ac \cdot bd &= dc \cdot ba \quad \text{by (1.8)}\\
\Rightarrow ac \cdot bd &= db \cdot ca \quad \text{by medial law} \qquad (1.10)\\
bd \cdot ac &= ac \cdot bd \quad \text{by (1.7) \& (1.9)}
\end{aligned}$$

Equvalently, by anti-commutativity $ac = bd$. Hence S is T^2-AG-groupoid.

Therefore the theorem is proved.

□

2. **Relation of distributive AG-groupoids with anti-commutative AG-groupoids**

Here we establish some relationships of left and right distributive (LD and RD) AG-groupoids with anti-commutative AG-groupoids. In general, there is no direct relation between LD- and RD-AG-groupoids and the anti-commutative AG-groupoids, but the combination of anti-commutativity with any one of these properties leads to another subclass of AG-groupoid as given in the following theorem.

Theorem 2. *For anti-commutative AG-groupoid S, the following are equivalent:*

(i) S is a left distributive AG-groupoid
(ii) S is a right distributive AG-groupoid
(iii) S is a distributive AG-groupoid.

Proof. (1) $(i) \Rightarrow (ii)$. Let S be an anti-commutative left distributive AG-groupoid and $a,b,c \in S$. Now, by medial and left invertive laws, left distributive property and anti-commutativity we have,

$$\begin{aligned}
(ab \cdot c)(ac \cdot bc) &= (ab \cdot c)(ab \cdot cc) = (ab \cdot ab)(c \cdot cc)\\
&= (ab \cdot ab)(cc \cdot cc) = ((cc \cdot cc)\, ab)(ab)\\
&= ((ab \cdot cc)\, cc)(ab) = (((ab \cdot c)(ab \cdot c))\, cc)(ab)\\
&= ((ab \cdot c)\, c\,(ab \cdot c)\, c)(ab) = ((ab \cdot c)(cc))(ab)\\
&= (ab \cdot cc)(ab \cdot c) = (ab \cdot cc)(ab \cdot c) = (ac \cdot bc)(ab \cdot c)
\end{aligned}$$

Thus by anti-commutativity we have $(ab \cdot c) = (ac \cdot bc)$.

(2) $(ii) \Rightarrow (i)$. Let S be an anti-commutative right distributive AG-groupoid and $a,b,c \in S$. Using medial law, right distributive property, left invertive law and

anti-commutativity we get,

$$\begin{aligned}
(ab \cdot ac)(a \cdot bc) &= (aa \cdot bc)(a \cdot bc) = (aa \cdot a)(bc \cdot bc) = (aa \cdot aa)(bc \cdot bc) \\
&= ((bc \cdot bc)\, aa)\, aa = ((aa \cdot bc)\, bc)\, aa \\
&= ((aa \cdot bc)\, a)(bc \cdot a) = ((aa \cdot bc)(bc))\, a \\
&= (a \cdot bc)(aa \cdot bc) = (a \cdot bc)(ab \cdot ac) \\
&= (ab \cdot ac)(a \cdot bc) = (a \cdot bc)(ab \cdot ac) \\
\text{Thus } a \cdot bc &= ab \cdot ac.
\end{aligned}$$

(3) $(ii) \Rightarrow (iii)$ By (ii) and (2).
(4) $(iii) \Rightarrow (i)$. Obvious.
Hence the theorem is proved.

□

Although, anti-commutative AG-groupoid and left distributive AG-groupoids are not T^3-AG-groupoids, however here we establish a relation among the three classes. This relationship is given in the following theorem.

Theorem 3. *Let S be an anti-commutative left distributive AG-groupoid. Then any of the following hold:*

(i) S is T^3-AG-groupoid
(ii) S is quasi-cancellative AG-groupoid
(iii) S is AG-3-band.

Proof. (i) Let S be an anti-commutative left distributive AG-groupoid and $a, b, c \in S$. We have to prove that S is right-T^3-AG-groupoid for this let $ab = ac$. Now using the medial and left invertive laws and the assumption we have,

$$\begin{aligned}
ba \cdot ca &= (bc \cdot aa) = (bc \cdot a)(bc \cdot a) = (bc \cdot bc)(aa) \\
&= (aa \cdot bc)(bc) = ((aa \cdot b)(aa \cdot c))(bc) \\
&= ((aa \cdot aa)\, bc)(bc) = ((aa \cdot aa)\, bc)(bc) \\
&= ((a \cdot aa)\, bc)(bc) = ((ab)(aa \cdot c))(bc) \\
&= ((ac)(aa \cdot c))(bc) = ((ac)(aa \cdot c))(bc) \\
&= ((a \cdot aa)(cc))(bc) = ((aa \cdot aa)(cc))(bc) \\
&= ((cc \cdot aa)(aa))(bc) = ((ca \cdot ca)(aa))(bc) \\
&= ((ca \cdot a)(ca \cdot a))\, bc = (ca \cdot aa)(bc) \\
&= (bc \cdot aa)(ca) = (ba \cdot ca)(ca) = (ba \cdot ca)(ca) \\
&= (ca \cdot ca)(ba) = (ca \cdot b)(ca \cdot a) = ca \cdot ba
\end{aligned}$$

Thus by anti-commutativity $ba = ca$.

(ii) Now, we have to prove that S is left-T^3-AG-groupoid, for this let $ba = ca$. Using the assumption, medial law and left invertive law we get,

$$\begin{aligned}
ab \cdot ac &= aa \cdot bc = (aa \cdot b)(aa \cdot c) = (aa \cdot aa)\, bc \\
&= (bc \cdot aa)\, aa = (ba \cdot ca)\, aa = ((ba \cdot c)(ba \cdot a))\, aa \\
&= ((ba \cdot c)(ca \cdot a))\, aa = ((ca \cdot b)(ca \cdot a))\, aa \\
&= (ca \cdot ba)\, aa = (ca \cdot a)(ba \cdot a) \\
&= (aa \cdot c)(aa \cdot b) = aa \cdot cb = ac \cdot ab
\end{aligned}$$

Thus $ab = ac$. Hence S is T^3-AG-groupoid.

(iii) Let S be an anti-commutative left distributive AG-groupoid and $x, y \in S$. Let $x^2 = xy$ and using the assumption, left invertive law and medial law we get,

$$\begin{aligned}
xy \cdot yx &= xx \cdot yx = (xx \cdot y)(xx \cdot x) = (xy \cdot y)(xx \cdot x) \\
&= (xy \cdot y)(xx \cdot x) = (yy \cdot x)(xx \cdot x) \\
&= (yy \cdot xx)\, xx = (yx \cdot yx)\, xx \\
&= (yx \cdot x)(yx \cdot x) = yx \cdot xx = yx \cdot xy
\end{aligned}$$

Thus $xy = yx$ by anti-commutativityand equivalently $x = y$.
Similarly, let $y^2 = yx$ and using the assumption, left invertive and medial laws and anti-commutativity we get,

$$\begin{aligned}
xy \cdot yx &= xy \cdot yy = (xy \cdot y)(xy \cdot y) = (xy \cdot y)(yy \cdot x) \\
&= (xy \cdot y)(yx \cdot x) = (xy \cdot yx)\, yx \\
&= (yx \cdot yx)\, xy = (yx \cdot x)(yx \cdot y) = yx \cdot xy \\
\text{i.e. } xy &= yx \quad \text{by anti-commutativity}
\end{aligned}$$

Thus by anti-commutativity we conclude that $x = y$.
Next we prove that S is right quasi-cancellative. To do this let $x^2 = yx$ and using the assumption, left distributive property, medial and left invertive laws and anti commutativity we get,

$$\begin{aligned}
(xy)(yx) &= xy \cdot xx = (xy \cdot x)(xy \cdot x) = (xy \cdot xy)\, xx \\
&= ((xx)(xy))xy = (xx \cdot xy)\, xy = ((xx \cdot x)(xx \cdot y))\, xy \\
&= (xy\,(xx \cdot y))(xx \cdot x) = (xy\,(yx \cdot x))(xx \cdot x) \\
&= ((x \cdot yx)(yx))(xx \cdot x) = ((xx \cdot x)\, x)(yx \cdot yx) \\
&= ((xx)(xx))((yx)(yx)) = (xx \cdot yx)(yx \cdot yx) \\
&= ((yx \cdot yx)\, yx)\, xx = ((yx \cdot y)(yx \cdot x))\, xx \\
&= (yx \cdot yx)\, xx = (yy \cdot xx)\, xx = (xx \cdot xx)\, yy \\
&= (xx \cdot yx)\, yy = (xy \cdot xx)\, yy = ((yy)(xx))xy \\
&= (yx \cdot yx)\, xy = (yx \cdot x)(yx \cdot y) = (yx)(xy)
\end{aligned}$$

Thus $xy = yx$ by anti-commutativityand equivalently $x = y$. Similarly, let $y^2 = xy$ and using the assumption, medial and left invertive laws and anti-commutativity we get,

$$\begin{aligned}
xy \cdot yx &= (xy \cdot y)(xy \cdot x) = (xy \cdot xy)\, yx = (yy \cdot xy)\, yx \\
&= ((yy \cdot x)(yy \cdot y))\, yx = ((xy \cdot y)(yy \cdot y))\, yx \\
&= ((xy \cdot y)(xy \cdot y))\, yx = ((yy \cdot y)(xy \cdot y))\, yx \\
&= ((yy \cdot xy)\, yy)\, yx = ((yx \cdot yy)\, yy)\, yx \\
&= ((yy \cdot yy)\, yx)\, yx = ((yy \cdot y)(yy \cdot x))\, yx \\
&= (yy \cdot yx)\, yx = (yx \cdot yx)\, yy = (yx \cdot y)(yx \cdot y) = yx \cdot yy \\
xy \cdot yx &= yx \cdot xy \quad \text{by assumption of } y^2 = xy
\end{aligned}$$

Thus $xy = yx$ by anti-commutativityand equivalently $x = y$. Hence S is quasi-cancellative.

(iv) Let S be an anti-commutative left distributive AG-groupoid and $a \in S$ then using the left invertive law, left distributivity, the medial law and anti-commutativity we get,

$$\begin{aligned} (aa \cdot a)\, a &= aa \cdot aa = (aa \cdot a)\,(aa \cdot a) = (aa \cdot aa)\, aa \\ &= (a \cdot aa)\, aa = a\,(aa \cdot a) \\ \text{i.e. } (aa \cdot a)\, a &= a\,(aa \cdot a) \end{aligned}$$

Thus by anti-commutativity $(aa \cdot a) = a$. Hence S is AG-3-band.

□

The following corollaries are now obvious using Theorem 2.

Corollary 1. *Anti-commutative right distributive AG-groupoid is T^3-AG-groupoid.*

Corollary 2. *Anti-commutative right distributive AG-groupoid is quasi-cancellative.*

Corollary 3. *Anti-commutative right distributive AG-groupoid is AG-3-band.*

3. Relation among anti-commutative, Jordan and paramedial AG-groupoids

In this section, we give the relationship among anti-commutative, Jordan and paramedial AG-groupoids is discussed. In fact there is no direct relation among the anti-commutative AG-groupoids, Jordan AG-groupoids and paramedial AG-groupoids. However, on combining these properties in an AG-groupoid give rise to another subclass of AG-groupoids. Here we give an important result among these three subclasses that will leads us to an important result directly in the form of a corollary.

Consider the following examples which show that anti-commutative AG-groupoid is not paramedial. Similarly, Jordan AG-groupoid is also not a paramedial AG-groupoid.

Example 2. *Let $S = \{1, 2, 3, 4\}$. Then*

(i) $(S, \cdot)$ is an anti-commutative AG-groupoid that is not a paramedial AG-groupoid.
(ii) $(S, \star)$ is Jordan AG-groupoid that is not a paramedial AG-groupoid.
(iii) $(S, \circ)$ is Jordan AG-groupoid which is not AG-3-band.

$\cdot$	1	2	3	4
1	2	3	1	4
2	4	1	3	2
3	3	2	4	1
4	1	4	2	3

(i)

$\star$	1	2	3	4
1	1	1	1	1
2	1	1	1	3
3	1	4	1	1
4	1	1	2	1

(ii)

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
. 3	3	4	2	1
4	4	3	1	2

(iii)

Theorem 4. *For an anti-commutative Jordan AG-groupoid S, the following hold:*

(i) S is paramedial AG-groupoid
(ii) S is T^1-AG-groupoid
(iii) S is AG-3-band.

Proof. Let S be an anti-commutative Jordan AG-groupoid and $a, b, c, d \in S$.

(i) Now using the medial and left invertive laws, Jordan identity and anti-commutativity we get,

$$\begin{aligned}
(ab \cdot cd)(db \cdot ca) &= (ab \cdot db)(cd \cdot ca) = (ad \cdot bb)(cc \cdot da) \\
&= ((bb \cdot d)a)(cc \cdot da) = ((db \cdot b)a)(cc \cdot da) \\
&= ((db \cdot b)a)(d(cc \cdot a)) = (ab \cdot db)(d(cc \cdot a)) \\
&= (ab \cdot d)(db(cc \cdot a)) = (ab \cdot d)(db(cc \cdot a)) \\
&= (db \cdot a)(cc(db \cdot a)) = (db \cdot a)(cc(ab \cdot d)) \\
&= ((cc(ab \cdot d))a)db = ((ab(cc \cdot d))a)db \\
&= ((a(cc \cdot d))ab)db = ((cc \cdot ad)ab)db \\
&= ((ca \cdot cd)ab)db = ((ab \cdot cd)ca)db = (db \cdot ca)(ab \cdot cd)
\end{aligned}$$

Thus by anti-commutativity, $(ab \cdot cd) = (db \cdot ca)$.

(ii) Let S be an anti-commutative Jordan AG-groupoid and $a, b, c, d \in S$. Let $ab = cd$ and consider,

$$ba \cdot dc = ca \cdot db = cd \cdot ab = ab \cdot cd = ab \cdot cd = dc \cdot ba$$

Thus by anti-commutativity it follows that $ba = dc$.

(iii) Let S be an anti-commutative Jordan AG-groupoid and $a \in S$ then,

$$(aa \cdot a)a = aa \cdot aa = a(aa \cdot a)$$

Thus $aa \cdot a = a$ by anti-commutativity.

Hence the theorem is proved. □

3.1. **Relation of anti-commutative AG-groupoid with rectangular and paramedial AG-groupoids.** In this subsection we define a new subclass of AG-groupoids which will be called rectangular AG-groupoid and will find its relation with anti-commutative and paramedial AG-groupoids.

Definition 1. *An AG-groupoid S is called a rectangular AG-groupoid in which the identity $ab \cdot ac = db \cdot dc$ holds for all $a, b, c, d \in S$.*

The identity given in the above definition of rectangular AG-groupoid is present in the book "Latin squares and their applications" on page no.60[4].

Example 3. *Rectangular AG-groupoid of order 4.*

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	4	3	1	2
4	3	4	2	1

Theorem 5. *For an anti-commutative paramedial AG-groupoid S, the following hold.*

(i) S is rectangular AG-groupoid
(ii) S is unipotent
(iii) S is AG^{**}-groupoid
(iv) S is T^1 -AG-groupoid.

Proof. (i) Let S be anti-commutative paramedial AG-groupoid and $a, b, c, d \in S$. Then by the medial and paramedial laws and anti-commutativity we get,

$$\begin{aligned}(ab \cdot ad)(cb \cdot cd) &= (aa \cdot bd)(cc \cdot bd) = (aa \cdot cc)(bd \cdot bd) \\ &= (ca \cdot ca)(bd \cdot bd) = (cc \cdot aa)(bd \cdot bd) \\ &= (cc \cdot bd)(aa \cdot bd) = (cb \cdot cd)(ab \cdot ad)\end{aligned}$$

Thus by anti-commutativity $ab \cdot ad = cb \cdot cd$. Hence S is rectangular.

(ii) Let S be an anti-commutative paramedial AG-groupoid and $a, b \in S$ then,

$$\begin{aligned} a^2 b^2 &= aa \cdot bb = ba \cdot ba = bb \cdot aa = b^2 a^2 \\ \text{i.e. } a^2 &= b^2 \quad \text{by anti-commutativity}\end{aligned}$$

Thus by anti-commutativity $a^2 = b^2$. Hence S is unipotent.

(iii) Let S be an anti-commutative paramedial AG-groupoid and $a, b, c \in S$. Then by the paramedial law, medial law, left invertive law and anti-commutativity we have,

$$\begin{aligned}(a \cdot bc)(b \cdot ac) &= (ac \cdot bc)\, ba = (ac \cdot b)(bc \cdot a) = (bc \cdot a)(ac \cdot b) \\ &= (bc \cdot ac)\, ab = (b \cdot ac)(a \cdot bc)\end{aligned}$$

Thus by anti-commutativity it follows that $a \cdot bc = b \cdot ac$. Hence S is AG**-groupoid.

(iv) Let S be an anti-commutative Jordan AG-groupoid and $a, b, c, d \in S$. Assume $ab = cd$, now using the paramedial property, medial law, assumption and anti-commutativity we get,

$$(ba)(dc) = (ca)(db) = (cd)(ab) = (ab)(cd) = (db)(ca) = (dc)(ba).$$

This implies that $(ba)(dc) = (dc)(ba)$ and thus by anti-commutativity $ba = dc$. Hence S is T^1 -AG-groupoid.

□

Using the above theorem with Theorem 4 and 5 one can easily prove the following:

Corollary 4. *Every anti-commutative paramedial AG-groupoid is quasi-cancellative.*

Corollary 5. *Every anti-commutative Jordan AG-groupoid is quasi-cancellative.*

The following examples show that anti-commutative or paramedial AG-groupoids are not necessarily AG**-groupoids.

Example 4. *Let $S = \{1, 2, 3, 4\}$ and $T = \{a, b, c, d\}$. Then*

(i) $(S, \cdot)$ is an anti-commutative AG-groupoid which is not AG**-groupoid.
(ii) $(T, \cdot)$ is a paramedial AG-groupoid which is not AG**-groupoid.

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

(i)

$\cdot$	a	b	c	d
a	a	b	b	b
b	b	a	a	a
c	b	a	a	a
d	c	a	a	a

(ii)

Corollary 6. *Every anti-commutative Jordan AG-groupoid is AG**-groupoid.*

Corollary 7. *Every anti-commutative Jordan AG-groupoid is unipotent.*

3.2. **Relation between anti-commutative and AG**-groupoid.** This subsection contains some relations between anti-commutative AG-groupoids and AG**-groupoids with various other subclasses of AG-groupoids. Although, there is no relation between anti-commutative AG-groupoid and AG**-groupoid but these subclasses give various other AG-groupoids as in the following theorem:

Example 5. *we give the following as counter examples.*

(i) Anti-commutative AG-groupoid which is not unipotent.
*(ii) AG**-groupoids which is not unipotent.*

·	1	2	3	4
1	2	3	1	4
2	4	1	3	2
3	3	2	4	1
4	1	4	2	3

(i)

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

(ii)

Theorem 6. *For an anti-commutative AG**-groupoid S, the following hold:*

(i) S is T^3-AG-groupoid
(ii) S is right-Bol-AG-groupoid
(iii) S is unipotent.

Proof. Let S be an anti-commutative AG**-groupoid and $a, b, c, d \in S$.

(i) To prove that S is T^3-AG-groupoid. Assume that $ab = ac$ for T^3_r. Using medial law, AG** property, left invertive law the paramedial law, the anti-commutativity and the assumption we get,

$$\begin{aligned} ba \cdot ca &= bc \cdot aa = a\,(bc \cdot a) = a\,(ac \cdot b) \\ &= ac \cdot ab = aa \cdot cb = ac \cdot ab = ab \cdot ac \\ &= aa \cdot bc = ca \cdot ba. \end{aligned}$$

This gives further by the anti-commutativity of S we get, $ba \cdot ca = ca \cdot ba$ and thus $ba = ca$. Hence S is T^3_r AG-groupoid.
Let us assume that $ba = ca$ for T^3_r. Now using the medial law, AG** property and left invertive law we get,

$$ab \cdot ac = aa \cdot bc = b\,(aa \cdot c) = b\,(ca \cdot a)\,.$$

Again using the AG** property, assumption, medial law, paramedial law and anti-commutativity we get,

$$\begin{aligned} ab \cdot ac &= b\,(ca \cdot a) = ca \cdot ba = ba \cdot ca \\ &= bc \cdot aa = ac \cdot ab. \end{aligned}$$

Thus $ab \cdot ac = ac \cdot ab$ and hence by anti-commutativity we conclude that $ab = ac$.

(ii) Assume that S is an anti-commutative AG**-groupoid and $a, b, c \in S$. Now, by AG** property, left invertive law and anti-commutativity we get,

$$\begin{aligned}(a(bc \cdot b))((ab \cdot c)b) &= (bc \cdot ab)((ab \cdot c)b) = (bc \cdot ab)(bc \cdot ab) \\ &= ((ab \cdot c)b)(bc \cdot ab) = ((ab \cdot c)b)(a(bc \cdot b))\end{aligned}$$

Thus by anti-commutativity we conclude that $a(bc \cdot b) = (ab \cdot c)b$.

(iii) Let a, b be elements of an anti-commutative AG-groupoid S. Now, using the definition of AG**, left invertive law and anti-commutativity we get,

$$\begin{aligned}a^2 \cdot b^2 &= (aa)(bb) = b(aa \cdot b) = b(ba \cdot a) \\ &= (ba)(ba) = (bb)(aa) = b^2 \cdot a^2\end{aligned}$$

Thus by anti-commutativity it follows that $a^2 = b^2 \forall a, b \in S$. Hence S is unipotent.

□

4. Conclusion

In this article, various relations of anti-commutative AG-groupoids with other subclasses of AG-groupoids are investigated. A new subclass of AG-groupoids named as rectangular AG-groupoid is introduced and investigated. Various examples and counter examples are provided with latest computational techniques of GAP, Mace4 and Prover9. The researchers are motivated to find other relations of these subclasses with other known subclasses and to investigate other properties of these AG-groupoids.

5. Acknowledgments

The authors are very thankful to the editor and referees for their helpful suggestions which improved the presentation of the paper.

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