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SEMT Labeling on Disjoint Union of Subdivided Stars

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Abstract. In this paper, we prove the existence of a super edge magic total (SEMT) labeling on some particular subclasses of the disjoint union of subdivided stars.

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1. Introduction

A labeling of a graph is a mapping that maps graph elements (vertices and edges) to numbers (usually non-negative integers). If domain of the mapping is union of the set of vertices and the set of edges, then such a labeling is known as a total labeling. In 1960, on the motivation of the concept of magic square in number theory, Sedlacek [22, 23] raised the problem to apply the *magic ideas* on *graphs* and introduced the notion of a *magic labeling*. Stewart (1966) [26] extended this study and proved that an $n \times n$ magic square in number theory corresponds to a super magic labeling of complete bipartite graph $K_{n,n}$. In 1970's Kotzig and Rosa [15, 16] defined the term of *magic valuation* for graphs. Later on, Ringel and Llado (1996) [20] introduced the same concept and called edge magic total (EMT) labeling. Recently, Enomoto et al. (1998) [4], defined super edge magic total (SEMT) labeling and proposed the following conjecture:

Conjecture 1.1. *Every tree admits a SEMT labeling.*

In the support of this conjecture, many authors have considered SEMT labeling for different particular classes of trees. Lee and Shah [17] verified this conjecture by a computer search for trees with at most 17 vertices. The results related to a SEMT labeling can be found for w-trees [8], extended w-trees [9], generalized extended w-trees [10], stars [18], subdivided stars [12, 13, 14, 19, 21, 29, 30], path-like trees [2], caterpillars [15, 16, 27], subdivided caterpillars [11], disjoint union of stars and books [6], wheels, fans and friendship graphs [25], paths and cycles [24] and complete bipartite graphs [1]. For detail studies of a SEMT labeling reader can see [3, 5, 6, 7].

Lu [29, 30] defined the subdivided star $T(n_1, n_2, n_3)$ and proved that it is a SEMT graph if n_1 and n_2 are odd with $n_3 = n_2 + 1$ or $n_3 = n_2 + 2$. Ngurah et al. [19] showed that $T(n_1, n_2, n_3)$ is also a SEMT graph if $n_3 = n_2 + 3$ or $n_3 = n_2 + 4$. Salman et al. [21] found a SEMT labeling of subdivided stars $T\underbrace{(n, n, n, ..., n)}_{r-times}$, where $n \in \{2, 3\}$. Recently, Javaid et al. [12, 13, 14] investigated SEMT labeling on different subclasses of subdivided stars $T(n_1, n_2, ..., n_r)$, where $n_i \geq 1$, $1 \leq i \leq r$, and $r \geq 3$. In this paper, we investigate the results related a SEMT labeling on disjoint union of subdivided stars under certain conditions.

2. Notations and Preliminaries

In this section, we define some basic notations and terminologies which are frequently used in the main results. In Definition 2.1 and Definition 2.2, the concept of an edge magic total labeling and a super edge magic total labeling is defined. Moreover, the concept of a subdivided star and its disjoint unions is defined in Definition 2.3 and Definition 2.4.

All graphs in this paper are finite, undirected and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set such that $v = |V(G)|$ and $e = |E(G)|$ are order and size of the graph G, respectively. For more basic terminologies, we refer [28].

Definition 2.1. A bijection $\lambda : V(G) \cup E(G) \to \{1, 2, ..., v + e\}$ is called an edge-magic total (EMT) labeling of a (v, e)-graph G if there exists an integral constant a such that $\lambda(x) + \lambda(xy) + \lambda(y) = a$ for all $xy \in E(G)$, where a is called a magic constant.

Definition 2.2. If λ is an EMT labeling such that the smallest labels are assigned to the vertices, then it is known as a super edge-magic total (SEMT) labeling and the graph G having such a labeling is called a SEMT graph.

Definition 2.3. The subdivided star $T(n_1, n_2, ..., n_r)$ is a tree obtained by inserting $n_i - 1$ vertices to each of the ith edge of the star $K_{1,r}$, where $1 \leq i \leq r$, $n_i \geq 1$and $r \geq 3$. The vertex-set and edge-set are defined as $V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \leq i \leq r\ ;\ 1 \leq l_i \leq n_i\}$ and $E(G) = \{c\ x_i^1 \mid 1 \leq i \leq r\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \leq i \leq r\ ;\ 1 \leq l_i \leq n_i - 1\}$, respectively. Moreover, for all $n_i = 1$, $T\underbrace{(1, 1, ..., 1)}_{r-times} \cong K_{1,r}$.

Definition 2.4. Suppose that $T_1(n_1^1, n_2^1, n_3^1, ...n_r^1)$, $T_2(n_1^2, n_2^2, n_3^2, ...n_r^2)$,..., $T_h(n_1^h, n_2^h, n_3^h, ...n_r^h)$ are h non-isomorphic subdivided stars, then $G \cong \cup_{j=1}^{h} T_j(n_1^j, n_2^j, n_3^j, ...n_r^j)$ is a disjoint union of h non-isomorphic subdivided stars with vertex-set and edge-set defined as
$V(G) = \{x_i^{(j)l_i^j} : 1 \leq i \leq r, 1 \leq l_i^j \leq n_i^j, 1 \leq j \leq h\}$
$\cup \{c_j; 1 \leq j \leq h\}$ and

$E(G) = \{x_i^{(j)l_i^j} x_i^{(j)l_i^j+1} : 1 \leq i \leq r, 1 \leq l_i^j \leq n_i^j - 1, 1 \leq j \leq h\}$
$\cup \{c_j x_i^{(j)1} : 1 \leq i \leq r, 1 \leq j \leq h\}$.

The following lemma provides a necessary and sufficient condition for a graph to be a SEMT graph.
Lemma 2.1 [5] A graph G with v vertices and e edges is a super edge-magic total if and only if there exists a bijective function $\lambda : V(G) \to \{1, 2, \cdots, v\}$ such that the set $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of e consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G with magic constant $c = v + e + s$, where $s = min(S)$ and

$$S = \{\lambda(x) + \lambda(y) | xy \in E(G)\} = \{c - (v + 1), c - (v + 2), \cdots, c - (v + e)\}.$$

□

3. Main Results

In this section, we present the main results on SEMT labeling for the disjoint union of subdivided stars under certain conditions. The results of SEMT labeling on the union of two disjoint copies of subdivided stars are presented in Theorem 3.1 and Theorem 3.2. Moreover, Theorem 3.3 and Theorem 3.4 study the existence of SEMT labeling for the union of three disjoint copies of subdivided stars.

Theorem 3.1. For any odd $n \geq 3$, and $r \geq 5$, $G \cong T_1(n_1^1, n_2^1, n_3^1, ..., n_r^1) \cup T_2(n_1^2, n_2^2, n_3^2, ..., n_r^2, n_{r+1}^2)$ admits a SEMT labeling if

$$\lambda(n_i^j) = \begin{cases} n, & \text{for} \quad 1 \leq i \leq 4, \\ 2^{i-4}(n-1)+1, & \text{for} \quad 5 \leq i \leq r-1+j, \end{cases}$$

where, $j = 1, 2$.

Proof: Consider $v = \sum_{j=1}^{2} [\sum_{i=1}^{r-1+j} n_i^j] + 2$, and $e = \sum_{j=1}^{2} [\sum_{i=1}^{r-1+j} n_i^j]$. Define the vertex-labeling $\lambda : V(G) \to \{1, 2, 3, ...v\}$ as given below:

$$\lambda(c_j) = \begin{cases} n, & \text{for} \quad j = 1, \\ (8n+3) + \sum_{m=5}^{r} \{2^{m-3}(n-1)+2\}, & \text{for} \quad j = 2. \end{cases}$$

When l_i^j is even, $1 \leq l_i^j \leq n_i^j$:

For $j=1$,

$$\lambda(u)=\begin{cases} \frac{l_1^1}{2}, & \text{for} \quad u=x_1^{(1)l_1^1}, \\ n-\frac{l_2^1}{2}, & \text{for} \quad u=x_2^{(1)l_2^1}, \\ n+\frac{l_3^1}{2}, & \text{for} \quad u=x_3^{(1)l_3^1}, \\ 2n-\frac{l_4^1}{2}, & \text{for} \quad u=x_4^{(1)l_4^1}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^1})=2n+\sum_{m=5}^{i}2^{m-5}(n-1)-\frac{l_i^1}{2}$, where $5\le i\le r$.

For $j=2$, and $\alpha_1=(7n+3)+\sum_{m=5}^{r}\{2^{m-3}(n-1)+2\}$,

$$\lambda(u)=\begin{cases} \alpha_1+\frac{l_1^2}{2}, & \text{for} \quad u=x_1^{(2)l_1^2}, \\ \alpha_1+n-\frac{l_2^2}{2}, & \text{for} \quad u=x_2^{(2)l_2^2}, \\ \alpha_1+n+\frac{l_3^2}{2}, & \text{for} \quad u=x_3^{(2)l_3^2}, \\ \alpha_1+2n-\frac{l_4^2}{2}, & \text{for} \quad u=x_4^{(2)l_4^2}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^2})=(\alpha_1+2n)+\sum_{m=5}^{i}2^{m-5}(n-1)-\frac{l_i^2}{2}$, where $5\le i\le r$.
For $j=2$ and $i=r+1$,

$$\lambda(x_i^{(j)l_i^j})=(4n+1)+\sum_{m=5}^{r}\{2^{m-4}(n-1)+1\}+\frac{l_i^j}{2}.$$

When l_i^j is odd, $1\le l_i^j\le n_i^j$:

For $j=1$ and $\beta_1=(5n+1)+\sum_{m=5}^{r}\{2^{m-5}(3n-3)+1\}$,

$$\lambda(u)=\begin{cases} \beta_1+\frac{l_1^1+1}{2}, & \text{for} \quad u=x_1^{(1)l_1^1}, \\ (\beta_1+n+2)-\frac{l_2^1+1}{2}, & \text{for} \quad u=x_2^{(1)l_2^1}, \\ (\beta_1+n+1)+\frac{l_3^1+1}{2}, & \text{for} \quad u=x_3^{(1)l_3^1}, \\ (\beta_1+2n+3)-\frac{l_4^1+1}{2}, & \text{for} \quad u=x_4^{(1)l_4^1}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^1})=(\beta_1+2n+3)+\sum_{m=5}^{i}\{2^{m-5}(n-1)+1\}-\frac{l_i^1+1}{2}$, where $5\le i\le r$.

For $j=2$ and $\alpha_2=(2n-1)+\sum_{m=5}^{r}2^{m-5}(n-1)$,

$$\lambda(u)=\begin{cases} \alpha_2+\frac{l_1^2+1}{2}, & \text{for} \quad u=x_1^{(2)l_1^2}, \\ (\alpha_2+n+2)-\frac{l_2^2+1}{2}, & \text{for} \quad u=x_2^{(2)l_2^2}, \\ (\alpha_2+n+1)+\frac{l_3^2+1}{2}, & \text{for} \quad u=x_3^{(2)l_3^2}, \\ (\alpha_2+2n+3)-\frac{l_4^2+1}{2}, & \text{for} \quad u=x_4^{(2)l_4^2}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^2})=(\alpha_2+2n+3)+\sum_{m=5}^{i}\{2^{m-5}(n-1)+1\}-\frac{l_i^2+1}{2}$, where $5\le i\le r$.

When l_i^j is odd, $3\le l_i^j\le n_i^j$, $j=2$ and $i=r+1$:

$$\lambda(x_i^{(j)l_i^j})=(\alpha_1+2n-1)+\sum_{m=5}^{i}2^{m-5}(n-1)+1+\frac{l_i^j-1}{2}.$$

When $l_i^j = 1$, $j = 2$ and $i = r + 1$:

$$\lambda(x_i^{(j)l_i^j}) = \beta_1.$$

By the above labeling scheme, the edge-sum set forms a sequence of consecutive integers $S = \{\beta_1 + 2, \beta_1 + 3, ..., \beta_1 + e\}$, therefore by Lemma 2.1, λ can be extended to a SEMT labeling and we obtain the magic constant $a = v + e + s = 2v + \beta_1 + 1$.

Theorem 3.2. For any even $n \geq 4$, and $r \geq 5$, $G \cong T_1(n_1^1, n_2^1 - 2, n_3^1 - 2, ...n_r^1) \cup T_2(n_1^2, n_2^2 - 2, n_3^2 - 2, ...n_r^2, n_{r+1}^2 - 2)$ admits a SEMT labeling if

$$\lambda(n_i^j) = \begin{cases} n, & \text{for} \quad 1 \leq i \leq 4, \\ 2^{i-4}(n), & \text{for} \quad 5 \leq i \leq r - 1 + j, \end{cases}$$

where, $j = 1, 2$.

Proof: Consider $v = (10n - 8) + \sum_{m=5}^{r} 2^{m-4}(3n)$, $e = (10n - 10) + \sum_{m=5}^{r} 2^{m-4}(3n)$, and $\alpha_1 = (7n - 5) + \sum_{m=5}^{r} 2^{m-3}(n)$. Define $\lambda : V(G) \to \{1, 2, 3, ...v\}$ as given below:

$$\lambda(c_j) = \begin{cases} n, & \text{for} \quad j = 1, \\ \alpha_1 + n, & \text{for} \quad j = 2. \end{cases}$$

When l_i^j is even, $1 \leq l_i^j \leq n_i^j$:

For $j = 1$,

$$\lambda(u) = \begin{cases} \frac{l_1^1}{2}, & \text{for} \quad u = x_1^{(1)l_1^1}, \\ n - \frac{l_2^1}{2}, & \text{for} \quad u = x_2^{(1)l_2^1}, \\ n + \frac{l_3^1}{2}, & \text{for} \quad u = x_3^{(1)l_3^1}, \\ 2n - \frac{l_4^1}{2}, & \text{for} \quad u = x_4^{(1)l_4^1}, \end{cases}$$

$$\text{and} \quad \lambda(x_i^{(j)l_i^1}) = 2n + \textstyle\sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i^1}{2}, \quad \text{where } 5 \leq i \leq r.$$

For $j = 2$,

$$\lambda(u) = \begin{cases} \alpha_1 + \frac{l_1^2}{2}, & \text{for} \quad u = x_1^{(2)l_1^2}, \\ \alpha_1 + n - \frac{l_2^2}{2}, & \text{for} \quad u = x_2^{(2)l_2^2}, \\ \alpha_1 + n + \frac{l_3^2}{2}, & \text{for} \quad u = x_3^{(2)l_3^2}, \\ \alpha_1 + 2n - \frac{l_4^2}{2}, & \text{for} \quad u = x_4^{(2)l_4^2}, \end{cases}$$

$$\lambda(x_i^{(j)l_i^2}) = (\alpha_1 + 2n) + \sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i^2}{2}, \text{ where }, 5 \leq i \leq r,$$

$$\text{and} \quad \lambda(x_i^{(j)l_i^2}) = (\alpha_2 + 2n - 2) + \textstyle\sum_{m=5}^{i} 2^{m-5}(n) + \frac{l_i^2}{2}, \text{ where } i = r + 1.$$

When l_i^j is odd, $1 \leq l_i^j \leq n_i^j$:

For $j = 1$, and $\beta_1 = (5n - 3) + \sum_{m=5}^{r} 2^{m-5}(3n)$,

$$\lambda(u)=\begin{cases}\beta_1+\frac{l_1^1+1}{2}, & \text{for} \quad u=x_1^{(1)l_1^1},\\ (\beta_1+n)-\frac{l_2^1+1}{2}, & \text{for} \quad u=x_2^{(1)l_2^1},\\ (\beta_1+n-1)+\frac{l_3^1+1}{2}, & \text{for} \quad u=x_3^{(1)l_3^1},\\ (\beta_1+2n-1)-\frac{l_4^1+1}{2}, & \text{for} \quad u=x_4^{(1)l_4^1},\end{cases}$$

and $\quad \lambda(x_i^{(j)l_i^1})=(\beta_1+2n-1)+\sum_{m=5}^{i}2^{m-5}(n)-\frac{l_i^1+1}{2}$, where $5\le i\le r$.

For $j=2$, and $\alpha_2=(2n-1)+\sum_{m=5}^{r}2^{m-5}(n)$,

$$\lambda(u)=\begin{cases}\alpha_2+\frac{l_1^1+1}{2}, & \text{for} \quad u=x_1^{(2)l_1^2},\\ (\alpha_2+n)-\frac{l_2^2+1}{2}, & \text{for} \quad u=x_2^{(2)l_2^2},\\ (\alpha_2+n-1)+\frac{l_3^2+1}{2}, & \text{for} \quad u=x_3^{(2)l_3^2},\\ (\alpha_2+2n-1)-\frac{l_4^2+1}{2}, & \text{for} \quad u=x_4^{(2)l_4^2},\end{cases}$$

and $\quad \lambda(x_i^{(j)l_i^2})=(\alpha_2+2n-1)+\sum_{m=5}^{i}2^{m-5}(n)-\frac{l_i^2+1}{2}$, where $5\le i\le r$.

When l_i^j is odd, $3\le l_i^j\le n_i^j$, $j=2$, and $i=r+1$:

$$\lambda(x_i^{(j)l_i^j})=(\alpha_1+2n-1)+\sum_{m=5}^{i}2^{m-5}(n)+\frac{l_i^j-1}{2}.$$

When $l_i^j=1$, $j=2$, and $i=r+1$: $\lambda(x_i^{(j)l_i^j})=\beta_1$.
By the above labeling scheme, the edge-sum set forms a sequence of consecutive integers $S=\{\beta_1+2,\beta_1+3,...,\beta_1+e\}$, therefore by Lemma 2.1, λ can be extended to a SEMT labeling and we obtain the magic constant $a=v+e+s=2v+\beta_1+1$.

Theorem 3.3. For any even $n\ge 4$, and $r\ge 5$, $G\cong T_1(n_1^1,n_2^1,2n_3^1-2,n_4^1.....n_r^1)\cup T_2(n_1^2,n_2^2-2,2n_3^2-2,n_4^2.....n_r^2)\cup T_3(n_1^3,n_2^3-2,2n_3^3-2,n_4^3.....n_r^3-2)$ admits a SEMT labeling if

$$n_i^j=\begin{cases}\frac{n+4}{j}+\sum_{m=5}^{r}2^{m-5}(\frac{n-4}{j}), & \text{for} \quad 1\le i\le 2, j=1,2,\\ (2n+2)+\sum_{m=5}^{r}2^{m-4}(n-3), & \text{for} \quad 1\le i\le 2, j=3,\end{cases}$$

and

$$n_i^j=\begin{cases}2^{i-4}(5n+j-7), & \text{for} \quad 3\le i\le r, j=1,3,\\ 2^{i-4}(n), & \text{for} \quad 3\le i\le r, j=2.\end{cases}$$

Proof: Consider $v=(29n-13)+\sum_{m=5}^{r}2^{m-5}(\frac{58n-88}{2})$, and $e=(29n-16)+\sum_{m=5}^{r}2^{m-5}(\frac{58n-88}{2})$. Define the vertex-labeling $\lambda:V(G)\to\{1,2,3,.....v\}$ as given below:

$$\lambda(c_j)=\begin{cases}2n, & \text{for} \quad j=1,\\ (21n-7)+\sum_{m=5}^{r}2^{m-5}(21n-34), & \text{for} \quad j=2,\\ (24n-6)+\sum_{m=5}^{r}2^{m-5}(24n-40), & \text{for} \quad j=3.\end{cases}$$

When $j=1$, and $\eta=(\frac{29n-14}{2})+\sum_{m=5}^{r}2^{m-5}(\frac{29n-44}{2})$:

For $l_1^1 = 1$, $\lambda(x_1^{(1)l_1^1}) = \eta + 1$,

For l_1^1 is odd, $3 \leq l_1^1 \leq n_1^1 - 1$,

$\lambda(x_1^{(1)l_1^1}) = \frac{n+4}{2} + \sum_{m=5}^{r} 2^{m-5}(\frac{n-4}{2}) - \frac{l_1^1-1}{2}$.

When l_i^j is odd, $1 \leq l_i^j \leq n_i^j - 1$:

For $j = 1$, and $\alpha_1 = (\frac{31n-2}{2}) + \sum_{m=5}^{r} 2^{m-5}(\frac{31n-52}{2})$,

$$\lambda(u) = \begin{cases} \alpha_1 - \frac{l_2^1+1}{2}, & \text{for} \quad u = x_2^{(1)l_2^1} \\ (\alpha_1 - 1) + \frac{l_3^1+1}{2}, & \text{for} \quad u = x_3^{(1)l_3^1} \\ (\alpha_1 + 5n - 7) - \frac{l_4^1+1}{2}, & \text{for} \quad u = x_4^{(1)l_4^1}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^1}) = (\alpha_1 + 5n - 7) + \sum_{m=5}^{i} 2^{m-5}(5n-6) - \frac{l_i^1+1}{2}$, where $5 \leq i \leq r$.

For $j = 2$, and $\beta_1 = (\frac{13n-2}{2}) + \sum_{m=5}^{r} 2^{m-5}(\frac{13n-24}{2})$,

$$\lambda(u) = \begin{cases} (6n-3) + \sum_{m=5}^{r} 2^{m-5}(6n-10) + \frac{l_1^2+1}{2}, & \text{for} \quad u = x_1^{(2)l_1^2} \\ \beta_1 - \frac{l_2^2+1}{2}, & \text{for} \quad u = x_2^{(2)l_2^2} \\ (\beta_1 - 1) + \frac{l_3^2+1}{2}, & \text{for} \quad u = x_3^{(2)l_3^2} \\ (\beta_1 + n - 1) - \frac{l_4^2+1}{2}, & \text{for} \quad u = x_4^{(2)l_4^2} \end{cases}$$

and $\lambda(x_i^{(j)l_i^2}) = (\beta_1 + n - 1) + \sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i^2+1}{2}$, where $5 \leq i \leq r$.

For $j = 3$ and $\gamma_1 = (\frac{15n-6}{2}) + \sum_{m=5}^{r} 2^{m-5}(\frac{15n-24}{2})$,

$$\lambda(u) = \begin{cases} \gamma_1 + \frac{l_1^3+1}{2}, & \text{for} \quad u = x_1^{(3)l_1^3} \\ (\gamma_1 + 2n + 2) + \sum_{m=5}^{r} 2^{m-5}(2n-6) - \frac{l_2^3+1}{2}, & \text{for} \quad u = x_2^{(3)l_2^3} \\ (\gamma_1 + 2n + 1) + \sum_{m=5}^{r} 2^{m-5}(2n-6) + \frac{l_3^3+1}{2}, & \text{for} \quad u = x_3^{(3)l_3^3} \\ (\gamma_1 + 7n - 3) + \sum_{m=5}^{r} 2^{m-5}(2n-6) - \frac{l_4^3+1}{2}, & \text{for} \quad u = x_4^{(3)l_4^3} \end{cases}$$

and $\lambda(x_i^{(j)l_i^3}) = (\gamma_1 + 7n - 3) + \sum_{m=5}^{i} 2^{m-5}(5n-4) + \sum_{m=5}^{i+1} 2^{m-5}(2n-6) - \frac{l_i^3+1}{2}$, where $5 \leq i \leq r - 1$ and $r \geq 6$.

When l_i^j is odd, $3 \leq l_i^j \leq n_i^j - 1$, $j = 3$, and $i = r$:

$\lambda(x_i^{(j)l_i^j}) = (\frac{53n-20}{2}) + \sum_{m=5}^{r} 2^{m-5}(\frac{53n-84}{2}) + \frac{l_i^j+1}{2}$.

When $l_i^j = 1$, $j = 3$ and $i = r$: $\lambda(x_i^{(j)l_i^j}) = (\frac{29n-12}{2}) + \sum_{m=5}^{r} 2^{m-5}(\frac{29n-44}{2}) - \frac{l_i^j+1}{2}$.

When l_i^j is even, $1 \leq l_i^j \leq n_i^j$: For $j = 1$,

$$\lambda(u)=\begin{cases}(15n-3)+\sum_{m=5}^{r}(15n-24)-\frac{l_1^1}{2}, & \text{for} \quad u=x_1^{(1)l_1^1},\\ (n+4)+\sum_{m=5}^{r}2^{m-5}(n-4)-\frac{l_2^1}{2}, & \text{for} \quad u=x_2^{(1)l_2^1},\\ (n+4)+\sum_{m=5}^{r}2^{m-5}(n-4)+\frac{l_3^1}{2}, & \text{for} \quad u=x_3^{(1)l_3^1},\\ (6n-2)+\sum_{m=5}^{r}2^{m-5}(n-4)-\frac{l_4^1}{2}, & \text{for} \quad u=x_4^{(1)l_4^1},\end{cases}$$

and $\lambda(x_i^{(j)l_i^1})=(6n-2)+\sum_{m=5}^{r}2^{m-4}(3n-5)-\frac{l_i^1}{2}$, where $5\le i\le r$.

For $j=2$, and $\beta_2=(\frac{42n-14}{2})+\sum_{m=5}^{r}2^{m-5}(\frac{42n-68}{2})$,

$$\lambda(u)=\begin{cases}(\frac{41n-18}{2})+\sum_{m=5}^{r}2^{m-5}(\frac{41n-64}{2})+\frac{l_1^2}{2}, & \text{for} \quad u=x_1^{(2)l_1^2}\\ \beta_2-\frac{l_2^2}{2}, & \text{for} \quad u=x_2^{(2)l_2^2},\\ \beta_2+\frac{l_3^2}{2}, & \text{for} \quad u=x_3^{(2)l_3^2},\\ (\beta_2+n)-\frac{l_4^2}{2}, & \text{for} \quad u=x_4^{(2)l_4^2},\end{cases}$$

and $\lambda(x_i^{(j)l_i^2})=(\beta_2+n)+\sum_{m=5}^{i}2^{m-5}(n)-\frac{l_i^2}{2}$, where $5\le i\le r$.

For $j=3$, and $\gamma_2=(22n-8)+\sum_{m=5}^{r}2^{m-5}(22n-34)$,

$$\lambda(u)=\begin{cases}\gamma_2+\frac{l_1^3}{2}, & \text{for} \quad u=x_1^{(3)l_1^3}\\ (\gamma_2+2n+2)+\sum_{m=5}^{r}2^{m-5}(2n-6)-\frac{l_2^3}{2}, & \text{for} \quad u=x_2^{(3)l_2^3},\\ (\gamma_2+2n+2)+\sum_{m=5}^{r}2^{m-5}(2n-6)+\frac{l_3^3}{2}, & \text{for} \quad u=x_3^{(3)l_3^3},\\ (\gamma_2+7n-2)+\sum_{m=5}^{r}2^{m-5}(2n-6)-\frac{l_4^3}{2}, & \text{for} \quad u=x_4^{(3)l_4^3},\end{cases}$$

and $\lambda(x_i^{(j)l_i^3})=(\gamma_2+7n-2)+\sum_{m=5}^{i}2^{m-5}(5n-4)+\sum_{m=5}^{r}2^{m-5}(2n-6)-\frac{l_i^3}{2}$, where $5\le i\le r-1$.

For $j=3$ and $i=r=5$, $\lambda(x_i^{(j)l_i^j})=(\gamma_1+7n-4)+\sum_{m=5}^{r}2^{m-5}(2n-6)+\frac{l_i^j}{2}$.

For $j=3$ and $i=r\ge 6$,

$$\lambda(x_i^{(j)l_i^j})=(\gamma_1+7n-4)+\sum_{m=5}^{i-1}2^{m-5}(5n-4)+\sum_{m=5}^{i}2^{m-5}(2n-6)+\frac{l_i^j}{2}.$$

By the above labeling scheme, the edge-sum set forms a sequence of consecutive integers $S=\{\eta+3,\eta+4,...,\eta+e\}$, then by Lemma 2.1, λ can be extended to a SEMT labeling and we obtain the magic constant $a=v+e+s=2v+\eta+2$.

Theorem 3.4. For any even $n\ge 4$, and $r\ge 5$, $G\cong T_1(n_1^1,n_2^1-2,2n_3^1-2,n_4^1.....n_r^1)\cup T_2(n_1^2,n_2^2-2,2n_3^2-2,n_4^2.....n_r^2,n_{r+1}^2-2)\cup\ T_3(n_1^3,n_2^3,2n_3^3-2,n_4^3.....n_r^3,n_{r+1}^3,n_{r+2}^3)$ admits a SEMT labeling if

$$n_i^j=\begin{cases}\frac{n+2}{2}+\sum_{m=5}^{r+1}2^{m-5}(\frac{n-2}{2}) & \text{for} \quad 1\le i\le 2, j=1,\\ (2n-1)+\sum_{m=5}^{r+1}2^{m-5}(2n-3) & \text{for} \quad 1\le i\le 2, j=2\\ (2n-4)+\sum_{m=5}^{r+1}2^{m-4}(n-2) & \text{for} \quad 1\le i\le 2, j=3\end{cases}$$

and

$$n_i^j = \begin{cases} 2^{i-4}(n) & \text{for} \quad 3 \le i \le r-1+j, j=1,3 \\ 2^{i-3}(n) & \text{for} \quad 3 \le i \le r-1+j, j=2 \end{cases}$$

Proof: Consider $v = (18n-17) + \sum_{m=5}^{r+1} 2^{m-5}(18n-16)$, and $e = (18n-20) + \sum_{m=5}^{r+1} 2^{m-5}(18n-16)$. Define $\lambda : V(G) \to \{1, 2, 3, ...v\}$ as given below:

$$\lambda(c_j) = \begin{cases} (\frac{27n-24}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{27n-26}{2}), & \text{for} \quad j=1, \\ (16n-14) + \sum_{m=5}^{r+1} 2^{m-5}(16n-16), & \text{for} \quad j=2, \\ (2n-4) + \sum_{m=5}^{r+1} 2^{m-5}(2n-4), & \text{for} \quad j=3. \end{cases}$$

When l_i^j is odd, $1 \le l_i^j \le n_i^j$:

For $j = 1$ and $\alpha_1 = (4n-5) + \sum_{m=5}^{r+1} 2^{m-5}(4n-4)$,

$$\lambda(u) = \begin{cases} \alpha_1 + \frac{l_1^1+1}{2}, & \text{for} \quad u = x_1^{(1)l_1^1}, \\ (\alpha_1 + \frac{n+2}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) - \frac{l_2^1+1}{2}, & \text{for} \quad u = x_2^{(1)l_2^1}, \\ (\alpha_1 + \frac{n}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) + \frac{l_3^1+1}{2}, & \text{for} \quad u = x_3^{(1)l_3^1}, \\ (\alpha_1 + \frac{3n}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) - \frac{l_4^1+1}{2}, & \text{for} \quad u = x_4^{(1)l_4^1}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^1}) = (\alpha_1 + \frac{3n}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) + \sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i^1+1}{2}$, where $5 \le i \le r$.

For $j = 2$, and $\beta_1 = (7n-7) + \sum_{m=5}^{r+1} 2^{m-5}(7n-8)$,

$$\lambda(u) = \begin{cases} (5n-6) + \sum_{m=5}^{r+1} 2^{m-5}(5n-5) + \frac{l_1^2+1}{2}, & \text{for} \quad u = x_1^{(2)l_1^2}, \\ \beta_1 - \frac{l_2^2+1}{2}, & \text{for} \quad u = x_2^{(2)l_2^2}, \\ (\beta_1 - 1) + \frac{l_3^2+1}{2}, & \text{for} \quad u = x_3^{(2)l_3^2}, \\ (\beta_1 + 2n - 1) - \frac{l_4^2+1}{2}, & \text{for} \quad u = x_4^{(2)l_4^2}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^2}) = (\beta_1 + 2n - 1) + \sum_{m=5}^{i} 2^{m-5}(2n) - \frac{l_i^2+1}{2}$, where $5 \le i \le r$.

For $j = 2$, $u = x_i^{(2)l_i^2}$, and $i = r+1$,

$$\lambda(u) = \begin{cases} (9n-10) + \sum_{m=5}^{r+1} 2^{m-5}(9n-8) + \frac{l_i^2+1}{2}, & \text{for} \quad l_i^2 = 1, \\ (17n-15) + \sum_{m=5}^{r+1} 2^{m-5}(17n-16) + \frac{l_i^2-1}{2}, & \text{for} \quad 3 \le l_i^2 \le n_i^2 - 1. \end{cases}$$

For $j = 3$, $\gamma_1 = (11n-11) + \sum_{m=5}^{r+1} 2^{m-5}(11n-12)$, and $\eta = (9n-9) + \sum_{m=5}^{r+1} 2^{m-5}(9n-8)$,

$$\lambda(u) = \begin{cases} \eta + \frac{l_1^3+1}{2}, & \text{for} \quad u = x_1^{(3)l_1^3} \; l_1 = 1, \\ (n-2) + \sum_{m=5}^{r+1} 2^{m-5}(n-2) - \frac{l_1^3-1}{2} & \text{for} \quad u = x_1^{(3)l_1} \; 3 \le l_1^3 \le n_1^3 - 1, \\ \gamma_1 - \frac{l_2^3+1}{2}, & \text{for} \quad u = x_2^{(3)l_2^3}, \\ (\gamma_1 - 1) + \frac{l_3^3+1}{2}, & \text{for} \quad u = x_3^{(3)l_3^3}, \\ (\gamma_1 + n - 1) - \frac{l_4^3+1}{2}, & \text{for} \quad u = x_4^{(3)l_4^3}, \end{cases}$$

and $\lambda(x_i^{(j)l_i^3}) = (\gamma_1 + n - 1) + \sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i^3+1}{2}$ where $5 \leq i \leq r+2$.

When l_i^j is even, $1 \leq l_i^j \leq n_i^j$:

For $j = 1$, and $\alpha_2 = (13n - 13) + \sum_{m=5}^{r+1} 2^{m-5}(13n - 12)$,

$$\lambda(u) = \begin{cases} \alpha_2 + \frac{l_1^1}{2}, & \text{for } \quad u = x_1^{(1)l_1^1} \\ (\alpha_2 + \frac{n+2}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) - \frac{l_2^1}{2}, & \text{for } \quad u = x_2^{(1)l_2^1} \\ (\alpha_2 + \frac{n+2}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) + \frac{l_3^1}{2}, & \text{for } \quad u = x_3^{(1)l_3^1} \\ (\alpha_2 + \frac{3n+2}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) - \frac{l_4^1}{2}, & \text{for } \quad u = x_4^{(1)l_4^1} \end{cases}$$

and $\lambda(x_i^{(j)l_i^1}) = (\alpha_2 + \frac{3n+2}{2}) + \sum_{m=5}^{r+1} 2^{m-5}(\frac{n-2}{2}) + \sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i^1}{2}$, where $5 \leq i \leq r$.

For $j = 2$, and $\beta_2 = (16n - 14) + \sum_{m=5}^{r+1} 2^{m-5}(16n - 16)$

$$\lambda(u) = \begin{cases} (14n - 13) + \sum_{m=5}^{r+1} 2^{m-5}(4n - 13) + \frac{l_1^2}{2}, & \text{for } \quad u = x_1^{(2)l_1^2} \\ \beta_2 - \frac{l_2^2}{2}, & \text{for } \quad u = x_2^{(2)l_2^2} \\ (\beta_2) + \frac{l_3^2}{2}, & \text{for } \quad u = x_3^{(2)l_3^2} \\ (\beta_2 + 2n) - \frac{l_4^2}{2}, & \text{for } \quad u = x_4^{(2)l_4^2} \end{cases}$$

$$\lambda(u) = \begin{cases} (\beta_2 + 2n) + \sum_{m=5}^{i} 2^{m-5}(2n) - \frac{l_i}{2}, & \text{for } \quad u = x_i^{(2)l_i} \\ & \qquad i = r, \\ (\beta_1 + 2n - 2) + \sum_{m=5}^{i} 2^{m-5}(2n) + \frac{l_i}{2}, & \text{for } \quad u = x_i^{(2)l_i} \\ & \qquad i = r+1. \end{cases}$$

For $j = 3$, and $\gamma_2 = (2n - 4) + \sum_{m=5}^{r} 2^{m-5}(2n - 4)$

$$\lambda(u) = \begin{cases} (10n - 9) + \sum_{m=5}^{r+1} 2^{m-5}(10n - 10) - \frac{l_1^3}{2} & \text{for } \quad u = x_1^{(3)l_1^3} \\ \gamma_2 - \frac{l_2}{2}, & \text{for } \quad u = x_2^{(3)l_2^3} \\ \gamma_2 + \frac{l_3}{2}, & \text{for } \quad u = x_3^{(3)l_3^3} \\ (\gamma_2 + n) - \frac{l_4^3}{2}, & \text{for } \quad u = x_4^{(3)l_4^3}. \end{cases}$$

and $\lambda(x_i^{(j)l_i}) = (\gamma_2 + n) + \sum_{m=5}^{i} 2^{m-5}(n) - \frac{l_i}{2}$, where $5 \leq i \leq r+2$.
By the above labeling scheme, the edge-sum set forms a sequence of consecutive integers $S = \{\eta + 3, \eta + 4, ..., \eta + e\}$, therefore by Lemma 2.1, λ can be extended to a SEMT labeling and we obtain the magic constant $a = v + e + s = 2v + \eta + 2$.

4. Conclusion

In this paper, we proved the existence of the SEMT labeling for the disjoint unions of the subdivided stars with at most three copies. However, the problem is open to study the existence of the SEMT labeling for the disjoint unions of the subdivided stars with arbitrary number of copies.

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