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Dimension and Depth of Monomial Edge Ideals of Line Graphs and Cycles by using Local Cohomology

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Abstract. We use local cohomology to compute dimension and depth of monomial edge ideals of line and cycle graphs. In both cases we computed projective dimension as an application.

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1. INTRODUCTION AND MAIN RESULTS

Let K denote a field. Let G denote a connected, simple and undirected graph over the vertices labeled by $[n] = \{1, 2, \ldots, n\}$. The monomial edge ideal $I_G \subseteq S_n = K[x_1, \ldots, x_n]$ is an ideal generated by all monomials $x_i x_j$, $i < j$, such that $\{i, j\}$ is an edge of G. It was introduced by Villarreal in [5]. The algebraic properties of monomial edge ideals in terms of combinatorial properties of graphs (and vice versa) were studied by many authors in [3], [4] and [6].
The main goal of this paper is to study some algebraic invariants of monomial edge ideals by using the technique of local cohomology. Nowadays local cohomology becomes an

essential tool to solve many problems not only in algebraic geometry but also in combinatorial commutative algebra. The first two authors of the paper used local cohomology in case of binomial edge ideals (see [8]).
The paper is structured as follows:
In Section 2, we give some preliminary definitions and results that we need in the rest of the paper. In particular we give a short summary on monomial edge ideal and its primary decomposition. In Section 3, we compute dimension, depth and projective dimension of the monomial edge ideal associated to a line graph. In Section 4, we do the same for cycle graph as we did for the line graph in Section 3.

2. PRELIMINARIES

First of all we will introduce the notation used in what follows. Moreover we summarize a few auxiliary results that we need. We denote by G a connected undirected graph on n vertices labeled by $[n] = \{1, 2, \ldots, n\}$. For an arbitrary field K let $S_n = K[x_1, \ldots, x_n]$ denote the polynomial ring in the n variables $x_1, \ldots, x_n$. To the graph G one can associate an ideal $I_G \subset S_n$ generated by all monomials $x_i x_j$ for all $1 \leq i < j \leq n$ such that $\{i, j\}$ is an edge of G. This construction was invented by Villarreal in [5]. To begin with, let us recall some of their definitions.

DEFINITION 1. *Let G be a graph with vertex set $[n]$. A subset $M \subset V$ is said to be a* ***minimal vertex cover*** *for G if:*

(a) *Every edge of G is incident with one vertex in M.*
(b) *There is no proper subset of M with the first property.*

The set M satisfying the condition (a) only is called a **vertex cover** of G. Now the next result tells us the importance of minimal vertex cover.

PROPOSITION 2.1. *Let $S_n = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring and G a graph with vertex set $[n]$. Let I be an ideal in S_n generated by $M = \{x_{i_1}, \ldots, x_{i_t}\}$, then the following conditions are equivalent:*

(a) *M is a minimal vertex cover of G.*
(b) *I is a minimal prime of I_G.*

Proof. For the proof see [7, Proposition 6.1.16]. □

Let M denote a finitely generated graded S_n-module. In the paper we shall use also the local cohomology modules of M with respect to S_+, denoted by $H^i(M), i \in \mathbb{Z}$ where $S_+ = \oplus_{d \geq 1} S^d$, S^d denotes the $d-$th homogeneous component of S_n. Note that they are graded Artinian S_n-modules. We refer to the textbook of Brodmann and Sharp (see [1]) for the basics on it. In particular the **dimension** and **depth** of M is defined as

$$\dim(M) = \max\{i : H^i(M) \neq 0\} \text{ and}$$
$$\operatorname{depth}(M) = \min\{i : H^i(M) \neq 0\}.$$

Every S_n-module M has a finite minimal graded free resolution:

$$A_\bullet : 0 \to A_p \to \cdots \to A_1 \to A_0 \to M \to 0$$

where A_i are free S_n modules for $i \geq 0$ and p is called the **projective dimension** of M. For more details of minimal free resolution we refer the book of Eisenbud [2]. The following theorem relate the depth and the projective dimension of the module. This is also known as **Auslander-Buchsbaum formula**.

THEOREM 2.1. *Let $M \neq 0$ be finitely generated S_n-module then*

$$\operatorname{Projdim} M + \operatorname{depth} M = \operatorname{depth} S_n.$$

Proof. For the proof see [7, Theorem 2.5.13]. □

The following lemma is also important for us.

LEMMA 2.1. *Let $S_n = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring and I be an ideal in S_n then $\operatorname{depth}(S_n/I) \leq \dim(S_n/p)$ for all $p \in \operatorname{Ass}(I)$.*

3. THE MONOMIAL EDGE IDEAL OF THE LINE GRAPH

Let G be a line graph with n vertices (of length n). Let $S_n = K[x_1, x_2, \ldots, x_{n-1}, x_n]$ be a polynomial ring, $I_n = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n)$ be the monomial edge ideal for line graph and let $\lceil t \rceil$ denotes the smallest integer not less than t.

THEOREM 3.1. *With the notation above we have,*

(a) $\dim S_n/I_n = \lceil \frac{n}{2} \rceil$.
(b) $\operatorname{depth} S_n/I_n = \lceil \frac{n}{3} \rceil$.

Proof. **(a):** We use induction on n. For $n = 2$ and 3 it is trivial. Let the statement is true for all $l \leq n$. Note that $I_{n+1} = (I_{n-1}, x_n) \cap (I_n, x_{n+1})$, so we get the exact sequence

$$0 \to S_{n+1}/I_{n+1} \to S_{n-1}/I_{n-1}[x_{n+1}] \oplus S_n/I_n \to S_{n-1}/I_{n-1} \to 0. \qquad (3.\ 1)$$

Consider the case $n = 2k$. Apply local cohomology to the exact sequence 3. 1 and we get a long exact sequence of local cohomology modules

$$\cdots \to H^k(S_{2k+1}/I_{2k+1}) \to H^k(S_{2k-1}/I_{2k-1}[x_{2k+1}]) \oplus H^k(S_{2k}/I_{2k}) \to$$
$$H^k(S_{2k-1}/I_{2k-1}) \to H^{k+1}(S_{2k+1}/I_{2k+1}) \to H^{k+1}(S_{2k-1}/I_{2k-1}[x_{2k+1}]) \to 0.$$

By induction hypothesis we have $\dim S_{2k}/I_{2k} = k$ and $\dim S_{2k-1}/I_{2k-1} = k$ implies $\dim S_{2k-1}/I_{2k-1}[x_{2k+1}] = k + 1$, which further implies that $H^{k+1}(S_{2k+1}/I_{2k+1}) \neq 0$. Hence $\dim S_{2k+1}/I_{2k+1} = k + 1$.
Similar arguments will work for the case when $n = 2k + 1$.
(b): Again we will use induction on n. For $n = 2$, 3 and 4 it is trivial. Let the statement is true for all $l \leq n$. Consider the exact sequence

$$0 \to S_{n+1}/I_n : x_nx_{n+1}(-2) \overset{x_nx_{n+1}}{\to} S_{n+1}/I_n \to S_{n+1}/I_{n+1} \to 0.$$

Now $I_n : x_nx_{n+1} = (I_{n-2}, x_{n-1})$ therefore the above exact sequence becomes,

$$0 \to (S_{n-2}/I_{n-2})[x_n, x_{n+1}](-2) \overset{x_nx_{n+1}}{\to} (S_n/I_n)[x_{n+1}] \to S_{n+1}/I_{n+1} \to 0. \qquad (3.\ 2)$$

Let $n = 3k$, by induction we have $\operatorname{depth} S_{3k}/I_{3k} = k$ and $\operatorname{depth} S_{3k-2}/I_{3k-2} = k$. Therefore $\operatorname{depth}(S_{3k}/I_{3k})[x_{3k+1}] = k+1$ and $\operatorname{depth}(S_{3k-2}/I_{3k-2})[x_{3k}, x_{3k+1}] = k+2$. Applying local cohomology to the exact sequence 3. 2 , we get

$$0 \to H^{k+1}(S_{3k}/I_{3k})[x_{3k+1}] \to H^{k+1}(S_{3k+1}/I_{3k+1}) \to$$
$$H^{k+2}(S_{3k-2}/I_{3k-2}[x_{3k}, x_{3k+1}](-2) \to \ldots$$

Hence $\operatorname{depth} S_{3k+1}/I_{3k+1} = k + 1$.

Now let $n = 3k + 1$, by induction hypothesis, $\operatorname{depth} S_{3k}/I_{3k} = k$, which implies $\operatorname{depth} S_{3k}/I_{3k}[x_{3k+2}] = k + 1$. Applying local cohomology to the exact sequence 3. 1 for the case $n = 3k + 1$, we get

$$0 \to H^k(S_{3k}/I_{3k}) \to H^{k+1}(S_{3k+2}/I_{3k+2}) \to \\ H^{k+1}(S_{3k}/I_{3k}[x_{3k+2}]) \oplus H^{k+1}(S_{3k+1}/I_{3k+1}) \to \dots$$

Which shows $H^{k+1}(S_{3k+2}/I_{3k+2}) \neq 0$. Therefore $\operatorname{depth} S_{3k+2}/I_{3k+2} = k + 1$.
At the end consider $n = 3k + 2$. By induction hypothesis, $\operatorname{depth} S_{3k}/I_{3k} = k$ implies $\operatorname{depth} S_{3k}/I_{3k}[x_{3k+2}, x_{3k+3}] = k + 2$. Also $\operatorname{depth} S_{3k+2}/I_{3k+2} = k + 1$ implies $\operatorname{depth} S_{3k+2}/I_{3k+2}[x_{3k+3}] = k + 2$. Now by using local cohomology on the exact sequence 3. 2 , we get

$$0 \to H^{k+1}(S_{3k+3}/I_{3k+3}) \to H^{k+2}(S_{3k}/I_{3k}[x_{3k+2}, x_{3k+3}](-2)) \to \\ H^{k+2}(S_{3k+2}/I_{3k+2}[x_{3k+3}]) \to H^{k+2}(S_{3k+3}/I_{3k+3}) \to \dots$$

Which shows that $\operatorname{depth} S_{3k+3}/I_{3k+3} \geqslant k + 1$.
For the other inequality, let $A = \{1, 3, 4, 6, 7, 9, 10, \dots, 3k, 3k + 1, 3k + 3\}$ be a subset of vertex set $[3k + 3]$. It is easy to see that A is a vertex cover because it covers all the edges. Now If we remove either 1 or $3t$ for some $t = 1, 2, \dots, k + 1$ from A then the resulting set is not a vertex cover because either $\{1, 2\}$ or $\{3t - 1, 3t\}$ will not covered. Similarly by removing $3t + 1$ for $t = 1, 2, \dots, k$ from A the edge $\{3t + 1, 3t + 2\}$ is not covered. Which shows that A is a minimal vertex cover. Further if we add one vertex say β in A and consider $B = A \cup \{\beta\}$ then $B \setminus \{\beta - 1, \beta + 1\}$ is a minimal vertex cover contained in B. Which shows that B is not a minimal vertex cover. Hence, A is a minimal vertex cover of maximal cardinality. Note that the number of elements in A are $2k + 2$. Thus $\operatorname{depth} S_{3k+3}/I_{3k+3} \leq k + 1$ by Lemma 2.1. Which completes the proof. □

It should be noted that one can give an alternative proof of Theorem 3.1 by using Depth Lemma (see [7, Lemma 1.3.9] together with the last argument of our proof.

COROLLARY 3.1. *With the notation above we have,*

$$\operatorname{Projdim} S_n/I_n = n - \lceil \frac{n}{3} \rceil$$

Proof. It is easily seen from Theorems 3.1 (b) and 2.1. □

4. THE MONOMIAL EDGE IDEAL OF THE CYCLE GRAPH

Let $S_n = K[x_1, x_2, ..., x_n]$ be a polynomial ring and let $J_n = (I_n, x_n x_1)$ be the monomial edge ideal of a cycle graph on n vertices where I_n is monomial edge ideal of a line graph. Let $\lfloor t \rfloor$ denotes the largest integer not greater than t.

THEOREM 4.1. *With the notation above we have,*

(a) $\dim Sn/Jn = \lfloor \frac{n}{2} \rfloor$

(b)

$$\operatorname{depth} Sn/Jn = \begin{cases} k, & \textit{if} \ \ n = 3k \textit{ or } 3k - 1 \textit{;} \\ k - 1, & \textit{if} \ \ n = 3k - 2 \textit{.} \end{cases}$$

Proof. **(a):** It is obvious that

$$J_n = (I_{n-1}, x_n) \cap (I'_{n-1}, x_1), \tag{4. 3}$$

where $I_{n-1} = (x_1x_2, \ldots, x_{n-2}x_{n-1})$ and $I'_{n-1} = (x_2x_3, \ldots, x_{n-1}x_n)$. Both I_{n-1} and I'_{n-1} are monomial edge ideals of line graphs with same lengths $n-1$. Therefore we have a short exact sequence

$$0 \to S_n/J_n \to S_n/(I_{n-1}, x_n) \oplus S_n/(I'_{n-1}, x_1) \to S_n/(I''_{n-2}, x_1, x_n) \to 0. \tag{4. 4}$$

where $I''_{n-2} = (x_2x_3, \ldots, x_{n-2}x_{n-1})$ is monomial edge ideal of line graph of length $n-2$. Note that $\dim Sn/Jn = \max\{\dim S_n/(I_{n-1}, x_n), \dim S_n/(I'_{n-1}, x_1)\} = \dim S_{n-1}/I_{n-1}$. By Theorem 3.1 (a), we have the required dimension.

(b): Since $J_n = (I_n, x_nx_1)$, therefore we have a short exact sequence

$$0 \to S_n/I_n : x_nx_1(-2) \overset{x_nx_1}{\to} S_n/I_n \to S_n/J_n \to 0.$$

Which is equivalent to the following exact sequence

$$0 \to \tilde{S}_{n-4}/\tilde{I}_{n-4}[x_1, x_n](-2) \overset{x_1x_n}{\to} S_n/I_n \to S_n/J_n \to 0. \tag{4. 5}$$

where $\tilde{S}_{n-4}$ is polynomial ring over the variables $x_3, \ldots, x_{n-2}$ and $\tilde{I}_{n-4}$ is a edge ideal of line on vertex set $\{3, \ldots, n-2\}$. First consider the case $n = 3k-2$, by Theorem 3.1 (b), we have $\operatorname{depth} S_{3k-2}/I_{3k-2} = k$ and $\operatorname{depth} \tilde{S}_{3k-6}/\tilde{I}_{3k-6}[x_1, x_{3k-2}](-2) = k$. If we apply local cohomology to the exact sequence 4. 5 for the case $n = 3k-2$, we get $\operatorname{depth} S_{3k-2}/J_{3k-2} \geq k-1$. On the other hand by viewing Equation 4. 3 and by Theorem 3.1 (b) for $n = 3k-2$, both $S_{3k-3}/(I_{3k-3}, x_{3k-2})$ and $S_{3k-3}/(I'_{3k-3}, x_1)$ have depth $k-1$. Therefore $\operatorname{depth} S_{3k-2}/J_{3k-2} \leq k-1$.

Now we consider the case $n = 3k-1$, by Theorem 3.1 (b) implies $\operatorname{depth} S_{3k-1}/I_{3k-1} = k$ and $\operatorname{depth} \tilde{S}_{3k-5}/\tilde{I}_{3k-5}[x_1, x_{3k-1}](-2) = k+1$. Using local cohomology to the exact sequence 4. 5 for the case $n = 3k-1$, we get $H^k(S_{3k-1}/J_{3k-1}) \neq 0$. Which implies $\operatorname{depth} S_{3k-1}/J_{3k-1} = k$. Similar arguments will work for the case $n = 3k$. □

COROLLARY 4.1. *With the notation above we have,*

$$\operatorname{Projdim} Sn/Jn = \begin{cases} n-k, & \text{if } n = 3k \text{ or } 3k-1\ ; \\ n-k+1, & \text{if } n = 3k-2\ . \end{cases}$$

Proof. It is easily seen from Theorems 4.1 (b) and 2.1. □

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