Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 47(2)(2015) pp. 65-76

Some Hermite-Hadamard Type Inequalities for r-Times Differentiable (σ, t)-Convex Functions

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Received: 08 May, 2015 / Accepted: 31 August, 2015 / Published online: 26 October, 2015

Abstract. In the present paper, the notion of (σ, t)-convex functions is used to proved some new results on inequalities for r-times differntiable $c$onvex functions, which are much same as famous Hermite Hadamard's integral inequality for convex functions, by applying these inequalities we have constructed inequalities for special means for two positive numbers.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Hermite-Hadamard's inequality, Convex function, t-convex function, (σ, t)-convex function.

1. INTRODUCTION AND MAIN RESULTS

A function g is called convex, if is defined by $g : \mathfrak{J} \subseteq \mathbb{R} \to \mathbb{R}$ and

$$g(zu + (1-z)v) \leq zg(u) + (1-z)g(v),$$

holds for every $u, v \in \mathfrak{J}$ and $z \in [0, 1]$.

Then the following double inequality

$$g\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q g(u)du \leq \frac{g(p)+g(q)}{2}. \tag{1. 1}$$

holds for convex functions and titled as the Hermite Hadamard Inequality(H-H). (see [5]). Both the inequalities in (1. 1) if holds in reversed direction then g is concave. Inequalities (1. 1) are famous in mathematical literature due to their variety of applications and rich geometrical significance.

For several results which gives generalization, improvement and extension of inequalities (1. 1), we refer the interested reader to [2, 7, 6].

To give estimations to the difference between the middle and the rightmost terms in (1.1), Dragomir and Agarwal [2], established these inequalities for differentiable functions.

THEOREM 1.1. [4]
Let $g : \mathfrak{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathfrak{J}^{\circ}$, and p, $q \in \mathfrak{J}$ with $p < q$ and $g' \in L^1[p,q]$ then:

$$\left|\frac{g\left(p\right)+g(q)}{2}-\frac{1}{q-p}\int_p^q g(u)du\right| \leq \frac{q-p}{2}\int_0^1 \left(1-2z\right)g'\left(zp+(1-z)q\right)dz. \quad (1.\ 2)$$

THEOREM 1.2. [1] *Let $g : \mathfrak{J}^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping, and p, $q \in \mathfrak{J}^{\circ}$ with $p < q$ and $|g'(u)|$ is convex on $[p,q]$ then:*

$$\left|\frac{g(p)+g(q)}{2}-\frac{1}{q-p}\int_p^q g(u)du\right| \leq \frac{(q-p)\left(|g'(p)|+|g'(q)|\right)}{8}. \quad (1.\ 3)$$

THEOREM 1.3. [9] *Let $g : \mathfrak{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathfrak{J}^{\circ}$. If $\left|g^{'}\right|^r$ is a convex function on $[p,q]$, for some $r \geq 1$, with p, $q \in \mathfrak{J}$ and $p < q$, then the following inequality holds:*

$$\left|\frac{g(p)+g(q)}{2}-\frac{1}{q-p}\int_p^q g(u)du\right| \leq \frac{q-p}{4}\left(\frac{\left|g^{'}\left(p\right)\right|^r+\left|g^{'}\left(q\right)\right|^r}{2}\right)^{\frac{1}{r}}, \quad (1.\ 4)$$

and

$$\left|\frac{g(p+q)}{2}-\frac{1}{q-p}\int_p^q g(u)du\right| \leq \frac{q-p}{4}\left(\frac{\left|g^{'}\left(p\right)\right|^r+\left|g^{'}\left(q\right)\right|^r}{2}\right)^{\frac{1}{r}}. \quad (1.\ 5)$$

DEFINITION 1. [1] *A function $g : [0,q] \to \mathbb{R}$, $q > 0$, forenamed as t-convex, if*

$$g(zu+t(1-z)v) \leq zg(u)+t(1-z)g(v)$$

holds, $\forall\, u,v \in [0,q]$, where $z \in [0,1]$ and $t \in (0,1]$. If the function $-g$ is t-convex then the function g is called t-concave.
The class of all t-convex functions defined on $[0,q]$ is denoted by $K_t(q)$, for which $g(0) \leq 0$.

Obviously, if we take $t = 1$ definition 1 take back the concept of standard convex functions on $[0,q]$, and for the value $t = 0$ it gives the concept of star shaped functions.

THEOREM 1.4. [1] *Let $g : \mathfrak{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a t-convex function, where $t \in (0,1]$ for $g \in L\left([p,q]\right)$ and $0 \leq p < q < \infty$, then:*

$$\frac{1}{q-p}\int_p^q g(u)du \leq \min\left\{\frac{g(p)+tg(\frac{q}{t})}{2}, \frac{g(\frac{p}{t})+tg(q)}{2}\right\}. \quad (1.\ 6)$$

2. (σ,t)-CONVEX FUNCTIONS

DEFINITION 2. [6] *The function g defined as $g : [0,q] \to \mathbb{R}$, where $q > 0$, is called as (σ,t)-convex, for $(\sigma,t) \in [0,1]^2$, if we have the inequality*

$$g(zu+t(1-z)v) \leq z^{\sigma}g(u)+t(1-z^{\sigma})g(v)$$

where $z \in [0,1]$, and $u,v \in [0,q]$.
The class of all (σ,t)-convex functions defined on $[0,q]$ are denoted by $K_t^{\sigma}(q)$, for which $g(0) \leq 0$.

We can easily obtain the following classes of functions: increasing, t-convex, convex, σ-convex functions, σ-starshaped and starshaped, respectively, for $(\sigma, t) \in \{(1,0), (1,t), (1,1), (\sigma,1), (0,0), (\sigma,0)\}$.
It is obvious that in the class $K_1^1(q)$ are the only convex functions $g : [0,q] \to \mathbb{R}$ for which $g(0) \leq 0$, i.e $K_1^1(q)$ is a proper subclass of the class of all convex functions defined on $[0,q]$.
For several recent results on inequalities for (σ,t)-convex and t-convex functions, we refer the interested readers to [4, 7, 8] and [1].
The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities for functions whose rth derivatives in absolute value are convex. We believe that the results presented in this paper are better than those established in [2] and hence better than those given in [3]. Applications of our results to special means are given in Section 4.

3. Main Results

The following Lemma is essential in establishing our main results in this section:

LEMMA 3.1. [5] *Suppose $\mathfrak{J} \subseteq \mathbb{R} \to \mathbb{R}$ be function such that $g^{(r)}$ exists on $\mathfrak{J}^\circ$ for $r \in \mathbb{N}$ and $g^{(r)} \in L[p,q]$, where p, $q \in \mathfrak{J}^\circ$ with $p < q$, we have resulting inequality of the form:*

$$\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du - \sum_{k=1}^{r-1} \frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{p+q}{2}\right)$$
$$= \frac{(q-p)^r}{2^{r+1}r!} \int_0^1 (1-z)^{r-1}(r-1+z) g^{(r)}\left(\frac{1-z}{2}p + \frac{1+z}{2}q\right) dz$$
$$+ \frac{(-1)^r(q-p)^r}{2^{r+1}r!} \int_0^1 (1-z)^{r-1}(r-1+z) g^{(r)}\left(\frac{1-z}{2}q + \frac{1+z}{2}p\right) dz, \quad (3.\ 7)$$

where an empty sum is understood to be nil.

THEOREM 3.1. *Suppose $g : \mathfrak{J} \subset \mathbb{R}_0 \to \mathbb{R}$ be a function with $g^{(r)}$ exists on $\mathfrak{J}^\circ$ for some $r \in \mathbb{N}$ and let $p, q \in \mathfrak{J}^\circ$ with $0 \leq p < q < \infty$ and σ, $t \in (0,1]$. If $g^{(r)} \in L([p, \frac{q}{t}])$ and $|g^{(r)}|^n$ is (σ,t)-convex on $[p, \frac{q}{t}]$ for $n \geq 1$, then we have the inequality as:*

$$\Bigg| \frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du - \sum_{k=1}^{r-1} \frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{p+q}{2}\right)\Bigg|$$
$$\leq \frac{(q-p)^r}{2^{r+\frac{\sigma}{n}+1}r!}\left(\frac{r}{r+1}\right)^{1-\frac{1}{n}} \Bigg\{ \Bigg[\frac{r^2+\sigma r-\sigma}{(\sigma+r)(\sigma+r+1)} \left(|g^{(r)}(p)|^n - t\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right)$$
$$+ \frac{2^\sigma tr}{r+1}\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\Bigg]^{\frac{1}{n}} + \Bigg[\frac{2^\sigma tr}{r+1}\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n$$
$$+ \frac{r^2+\sigma r-\sigma}{(\sigma+r)(\sigma+r+1)}\left(|g^{(r)}(q)|^n - t\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right)\Bigg]^{\frac{1}{n}}\Bigg\}. \quad (3.\ 8)$$

Proof. From Lemma 3.1, the Power mean's inequality and (σ, t)-convexity of $|g^{(r)}|^n$ on $[p, \frac{q}{t}]$, we get

$$\begin{aligned}
&\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du \right.\\
&\qquad \left. - \sum_{k=1}^{r-1} \frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{p+q}{2}\right)\right|\\
&\leq \frac{(q-p)^r}{2^{r+1}r!}\left(\int_0^1 (1-z)^{r-1}(r-1+z)dz\right)^{1-\frac{1}{n}}\\
&\times \left\{ \left(\int_0^1 (1-z)^{r-1}(r-1+z)\left[\left(\frac{1-z}{2}\right)^{\sigma}\left|g^{(r)}(p)\right|^n \right.\right.\right.\\
&\qquad \left.\left. + t\left(1-\left(\frac{1-z}{2}\right)^{\sigma}\right)\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right] dz\right)^{\frac{1}{n}}\\
&+ \left(\int_0^1 (1-z)^{r-1}(r-1+z)\left[\left(\frac{1-z}{2}\right)^{\sigma}\left|g^{(r)}(q)\right|^n \right.\right.\\
&\qquad \left.\left.\left. + t\left(1-\left(\frac{1-z}{2}\right)^{\sigma}\right)\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right] dz\right)^{\frac{1}{n}} \right\}. \qquad (3.\ 9)
\end{aligned}$$

Since

$$\int_0^1 (1-z)^{\sigma+r-1}(r-1+z)dz = \frac{r^2+\sigma r-\sigma}{(\sigma+r)(\sigma+r+1)} \qquad (3.\ 10)$$

and

$$\int_0^1 (1-z)^{r-1}(r-1+z)dz = \frac{r}{(r+1)}. \qquad (3.\ 11)$$

Hence, the inequality (3. 8) follows by using (3. 10) and (3. 11) in (3. 9). Which is required. □

COROLLARY 3.1. *In Theorem 3.1, if we take $n = 1$, then (3. 8) becomes the inequality:*

$$\begin{aligned}
&\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du \right.\\
&\qquad \left. - \sum_{k=1}^{r-1} \frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{p+q}{2}\right)\right|\\
&\leq \frac{(q-p)^r}{2^{r+\sigma+1}r!}\left\{ t\left(\frac{2^{\sigma}r}{r+1} - \frac{r^2+\sigma r-\sigma}{(\sigma+r)(\sigma+r+1)}\right)\left[\left|g^{(r)}\left(\frac{q}{t}\right)\right| + \left|g^{(r)}\left(\frac{p}{t}\right)\right|\right] \right.\\
&\qquad \left. + \frac{r^2+\sigma r-\sigma}{(\sigma+r)(\sigma+r+1)}\left[\left|g^{(r)}(p)\right| + \left|g^{(r)}(q)\right|\right]\right\}. \qquad (3.\ 12)
\end{aligned}$$

COROLLARY 3.2. *If $\sigma = t = 1$ in Corollary 3.1, then:*

$$\left| \frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du - \sum_{k=1}^{r-1} \frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{p+q}{2}\right) \right| \leq \frac{r(q-p)^r}{2^{r+1}(r+1)!} \left[\left|g^{(r)}(q)\right| + \left|g^{(r)}(p)\right|\right]. \quad (3.\ 13)$$

COROLLARY 3.3. *If $r = 1$ in Theorem 3.1, then (3. 8) reduces to the inequality:*

$$\left| \frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du \right| \leq \frac{(q-p)}{2^{\frac{\sigma}{n}+2}} \left(\frac{1}{2}\right)^{1-\frac{1}{n}} \times \left\{ \left[\frac{1}{(\sigma+1)(\sigma+2)} \left(|g'(p)|^n - t\left|g'\left(\frac{q}{t}\right)\right|^n \right) + 2^{\sigma-1} t \left|g'\left(\frac{q}{t}\right)\right|^n \right]^{\frac{1}{n}} + \left[\frac{1}{(\sigma+1)(\sigma+2)} \left(|g'(q)|^n - t\left|g'\left(\frac{p}{t}\right)\right|^n \right) + 2^{\sigma-1} t \left|g'\left(\frac{p}{t}\right)\right|^n \right]^{\frac{1}{n}} \right\}. \quad (3.\ 14)$$

COROLLARY 3.4. *If $\sigma = t = 1$ in Corollary 3.3, then:*

$$\left| \frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du \right| \leq \frac{(q-p)}{8} \times \left\{ \left[\frac{1}{6}|g'(p)|^n + \frac{5}{6}|g'(q)|^n \right]^{\frac{1}{n}} + \left[\frac{1}{6}|g'(q)|^n + \frac{5}{6}|g'(p)|^n \right]^{\frac{1}{n}} \right\}. \quad (3.\ 15)$$

COROLLARY 3.5. *If $r = 1$, $n = 1$, in Theorem 3.1, then (3. 8) reduces to the inequality:*

$$\left| \frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du \right| \leq \frac{(q-p)}{2^{\sigma+2}} \times \left\{ t\left(2^{\sigma-1} - \frac{1}{(\sigma+1)(\sigma+2)}\right)\left(\left|g'\left(\frac{p}{t}\right)\right| + \left|g'\left(\frac{q}{t}\right)\right|\right) + \frac{1}{(\sigma+1)(\sigma+2)}\left(|g'(p)| + |g'(q)|\right) \right\}. \quad (3.\ 16)$$

COROLLARY 3.6. *If $r = 2$ in Theorem 3.1, then (3. 8) becomes:*

$$\left| \frac{g(p)+g(q)}{2} - \frac{1}{(q-p)} \int_p^q g(u)du \right| \leq \frac{(q-p)^2}{2^{\frac{\sigma}{n}+4}} \left(\frac{2}{3}\right)^{1-\frac{1}{n}} \times \left\{ \left[\frac{4+\sigma}{(\sigma+2)(\sigma+3)} \left(|g''(p)|^n - t\left|g''\left(\frac{q}{t}\right)\right|^n \right) + \frac{2^{\sigma+1}t}{3} \left|g''\left(\frac{q}{t}\right)\right|^n \right]^{\frac{1}{n}} + \left[\frac{4+\sigma}{(\sigma+2)(\sigma+3)} \left(|g''(q)|^n - t\left|g''\left(\frac{p}{t}\right)\right|^n \right) + \frac{2^{\sigma+1}t}{3} \left|g''\left(\frac{p}{t}\right)\right|^n \right]^{\frac{1}{n}} \right\}. \quad (3.\ 17)$$

COROLLARY 3.7. *If $n = 1$ in Corollary 3.6, then:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right| \leq \frac{(q-p)^2}{2^{\sigma+4}} \times \left\{ t\left(\frac{2^{\sigma+1}}{3} - \frac{4+\sigma}{(\sigma+2)(\sigma+3)}\right)\left(\left|g''\left(\frac{p}{t}\right)\right| + \left|g''\left(\frac{q}{t}\right)\right|\right) + \frac{(4+\sigma)}{(\sigma+2)(\sigma+3)}\left(|g''(p)| + |g''(q)|\right)\right\}. \quad (3.\ 18)$$

THEOREM 3.2. *Suppose $g : \mathfrak{J} \subset \mathbb{R}_0 \to \mathbb{R}$ be a function, and $g^{(r)}$ exists on $\mathfrak{J}^\circ$ for $r \in \mathbb{N}$, and let $p, q \in \mathfrak{J}^\circ$ with $0 \leq p < q < \infty$ and $\sigma, t \in (0, 1]$. If $g^{(r)} \in L([p, \frac{q}{t}])$ and $|g^{(r)}|^n$ is (σ, t)-convex on $[p, \frac{q}{t}]$ for $n > 1$, then we have:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du - \sum_{k=1}^{r-1}\frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!}g^{(k)}\left(\frac{p+q}{2}\right)\right|$$
$$\leq \frac{(q-p)^r}{2^{r+\frac{\sigma}{n}+1}r!}\left(\frac{n-1}{2n-1}\right)^{1-\frac{1}{n}}\left(r^{\frac{2n-1}{n-1}} - (r-1)^{\frac{2n-1}{n-1}}\right)^{1-\frac{1}{n}} \times \left\{\left[\frac{1}{nr+\sigma-n+1}\left(|g^{(r)}(p)|^n - t\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right) + \frac{2^\sigma t}{rn-n+1}\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}} + \left[\frac{2^\sigma t}{nr-n+1}\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n + \frac{1}{rn+\sigma-n+1}\left(|g^{(r)}(q)|^n - t\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right)\right]^{\frac{1}{n}}\right\}. \quad (3.\ 19)$$

Proof. From Lemma 3.1, the Hölder inequality and (σ, t)-convexity of $|g^{(r)}|^n$ on $[p, \frac{q}{t}]$, we have

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du - \sum_{k=1}^{r-1}\frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!}g^{(k)}\left(\frac{p+q}{2}\right)\right|$$
$$\leq \frac{(q-p)^r}{2^{r+1}r!}\left(\int_0^1(r-1+z)^{n/(n-1)}dz\right)^{1-\frac{1}{n}}\left\{\left(\int_0^1(1-z)^{n(r-1)}\left[\left(\frac{1-z}{2}\right)^\sigma|g^{(r)}(p)|^n + t\left(1-\left(\frac{1-z}{2}\right)^\sigma\right)|g^{(r)}(q)|^n\right]dz\right)^{\frac{1}{n}} + \left(\int_0^1(1-z)^{n(r-1)}\left[\left(\frac{1-z}{2}\right)^\sigma|g^{(r)}(q)|^n + t\left(1-\left(\frac{1-z}{2}\right)^\sigma\right)|g^{(r)}(p)|^n\right]dz\right)^{\frac{1}{n}}\right\}. \quad (3.\ 20)$$

Since

$$\left(\int_0^1 (r-1+z)^{n/(n-1)}du\right)^{1-\frac{1}{n}} = \left(\frac{n-1}{2n-1}\right)^{1-\frac{1}{n}} \left(r^{\left(\frac{2n-1}{n-1}\right)} - (r-1)^{\left(\frac{2n-1}{n-1}\right)}\right)^{1-\frac{1}{n}}, \tag{3. 21}$$

$$\int_0^1 (1-z)^{n(r-1)+\sigma}dz = \frac{1}{rn-n+\sigma+1} \tag{3. 22}$$

and

$$\int_0^1 (1-z)^{n(r-1)}dz = \frac{1}{nr-n+1}. \tag{3. 23}$$

Using (3. 21),(3. 22) and (3. 23) in (3. 20), we got result. □

COROLLARY 3.8. *If $r = 1$ in Theorem 3.2, then (3. 19) reduces to:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right| \le \frac{(q-p)}{2^{2+\frac{\sigma}{n}}}\left(\frac{n-1}{2n-1}\right)^{1-\frac{1}{n}}$$
$$\times\left\{\left[\frac{|g'(p)|^n - t\left|g'\left(\frac{q}{t}\right)\right|^n}{\sigma+1} + 2^\sigma t\left|g'(\frac{q}{t})\right|^n\right]^{\frac{1}{n}} + \left[\frac{|g'(q)|^n - t\left|g'\left(\frac{p}{t}\right)\right|^n}{\sigma+1} + 2^\sigma t\left|g'(\frac{p}{t})\right|^n\right]^{\frac{1}{n}}\right\}. \tag{3. 24}$$

COROLLARY 3.9. *If $\sigma = 1$, $t = 1$ in corollary 3.8, then (3. 19) becomes:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right|$$
$$\le \frac{(q-p)}{2^{2+\frac{1}{n}}}\left(\frac{n-1}{2n-1}\right)^{1-\frac{1}{n}}\left[\frac{1}{2}|g'(p)|^n + \frac{3}{2}\left|g'(q)\right|^n\right]^{\frac{1}{n}}$$
$$+\left[\frac{1}{2}|g'(q)|^n + \frac{3}{2}\left|g'(p)\right|^n\right]^{\frac{1}{n}}. \tag{3. 25}$$

COROLLARY 3.10. *If $r = 2$ in Theorem 3.2, then (3. 19) becomes:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right| \le \frac{(q-p)^2\left[2^{\frac{2n-1}{n-1}}-1\right]^{1-\frac{1}{n}}}{2^{4+\frac{\sigma}{n}}}\left(\frac{n-1}{2n-1}\right)^{1-\frac{1}{n}}$$
$$\times\left\{\left[\frac{1}{\sigma+n+1}\left(|g''(p)|^n - t\left|g''\left(\frac{q}{t}\right)\right|^n\right) + \frac{2^\sigma t}{n+1}\left|g''\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}}\right.$$
$$\left.+\left[\frac{1}{\sigma+n+1}\left(|g''(q)|^n - t\left|g''\left(\frac{p}{t}\right)\right|^n\right) + \frac{2^\sigma t}{n+1}\left|g''\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\}. \tag{3. 26}$$

THEOREM 3.3. *Let $g : \mathfrak{I} \subset \mathbb{R}_0 \to \mathbb{R}$ be a function, and $g^{(r)}$ exists on $\mathfrak{I}^\circ$ for $r \in \mathbb{N}$, and let $p, q \in \mathfrak{I}^\circ$ with $0 \le p < q < \infty$ and $\sigma, t \in (0, 1]$. If $g^{(r)} \in L([p, \frac{q}{t}])$ and $|g^{(r)}|^n$ is*

(σ,t)-convex on $[p,\frac{q}{t}]$ for $n>1$, then we get:

$$\begin{aligned}&\Bigg|\frac{g(p)+g(q)}{2}-\frac{1}{(q-p)}\int_p^q g(u)du\\&\qquad-\sum_{k=1}^{r-1}\frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!}g^{(k)}\left(\frac{p+q}{2}\right)\Bigg|\\&\leq\frac{(q-p)^r}{2^{r+1}r!}\left(\frac{n-1}{rn-1}\right)^{1-\frac{1}{n}}\left\{\left[\frac{1}{2^\sigma}r^{\sigma+n+1}\beta\left(\frac{1}{r},1+\sigma,1+n\right)\right.\right.\\&\times\left(|g^{(r)}(p)|^n-t\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right)+t\left(\frac{r^{n+1}-(r-1)^{n+1}}{n+1}\right)\left.\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}}\\&+\left[\frac{1}{2^\sigma}r^{\sigma+n+1}\beta\left(\frac{1}{r};1+\sigma,1+n\right)\left(|g^{(r)}(q)|^n-t\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right)\right.\\&\left.\left.+t\left(\frac{r^{n+1}-(r-1)^{n+1}}{n+1}\right)\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\},\end{aligned}\tag{3. 27}$$

where

$$\beta(u;l,s)=\int_0^u z^{l-1}(1-z)^{s-1}dz,0\leq u\leq 1,l>0,s>0.\tag{3. 28}$$

is the incomplete beta function.

Proof. From Lemma 3.1, the Hölder inequality and (σ,t)-convexity of $|g^{(r)}|^n$ on $[p,\frac{q}{t}]$, we obtain

$$\begin{aligned}&\Bigg|\frac{g(p)+g(q)}{2}-\frac{1}{(q-p)}\int_p^q g(u)du\\&\qquad-\sum_{k=1}^{r-1}\frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!}g^{(k)}\left(\frac{p+q}{2}\right)\Bigg|\\&\leq\frac{(q-p)^r}{2^{r+1}r!}\left(\int_0^1(1-z)^{n(r-1)/(n-1)}dz\right)^{1-\frac{1}{n}}\left\{\left[\int_0^1(r-1+z)^n\left[\left(\frac{1-z}{2}\right)^\sigma|g^{(r)}(p)|^n\right.\right.\right.\\&\left.\left.+t\left(1-\left(\frac{1-z}{2}\right)^\sigma\right)\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right]dz\right]^{\frac{1}{n}}+\left[\int_0^1(r-1+z)^n\left[\left(\frac{1-z}{2}\right)^\sigma|g^{(r)}(q)|^n\right.\right.\\&\left.\left.\left.+t\left(1-\left(\frac{1-z}{2}\right)^\sigma\right)\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right]dz\right]^{\frac{1}{n}}\right\}.\end{aligned}\tag{3. 29}$$

Since

$$\left(\int_0^1(1-z)^{n(r-1)/(n-1)}dz\right)^{1-\frac{1}{n}}=\left(\frac{n-1}{rn-1}\right)^{1-\frac{1}{n}}\tag{3. 30}$$

and

$$\int_0^1(r-1+z)^n(1-z)^\sigma dz=r^{\sigma+n+1}\beta\left(\frac{1}{r};1+\sigma,1+n\right).\tag{3. 31}$$

Hence, the inequality (3. 27) follows by using(3. 30) and (3. 31) in (3. 29). Which is required. □

COROLLARY 3.11. *If $r = 1$ in Theorem 3.3, by (3. 27) we get:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right| \leq \frac{(q-p)}{2^2}$$
$$\times\left\{\left[\frac{1}{2^\sigma}\beta(1;1+\sigma,1+n)\left(|g'(p)|^n - t\left|g'\left(\frac{q}{t}\right)\right|^n\right) + \frac{t}{n+1}\left|g'\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}}\right.$$
$$\left.+\left[\frac{1}{2^\sigma}\beta(1;1+\sigma,1+n)\left(|g'(q)|^n - t\left|g'\left(\frac{p}{t}\right)\right|^n\right) + \frac{t}{n+1}\left|g'\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\}. \quad (3.\ 32)$$

COROLLARY 3.12. *If $r = 2$ in Theorem 3.3, by (3. 27) we have:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right| \leq \frac{(q-p)^2}{2^4}\left(\frac{n-1}{2n-1}\right)^{1-\frac{1}{n}}$$
$$\times\left\{\left[\frac{1}{2^\sigma}\xi\left(|g''(p)|^n - t\left|g''\left(\frac{q}{t}\right)\right|^n\right) + t\,\lambda\left|g''\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}}\right.$$
$$\left.+\left[\frac{1}{2^\sigma}\xi\left(|g''(q)|^n - t\left|g''\left(\frac{p}{t}\right)\right|^n\right) + t\lambda\left|g''\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\}, \quad (3.\ 33)$$

where

$$\xi = 2^{\sigma+n+1}\beta\left(\frac{1}{2};1+\sigma,1+n\right)$$

and

$$\lambda = \frac{2^{n+1}-1}{n+1}.$$

THEOREM 3.4. *$g : \mathfrak{J} \subset \mathbb{R}_0 \to \mathbb{R}$ be a function, such that $g^{(r)}$ exists on $\mathfrak{J}^\circ$ for $r \in \mathbb{N}$, and let $p, q \in \mathfrak{J}^\circ$ with $0 \leq p < q < \infty$ and $\sigma, t \in (0,1]$. If $g^{(r)} \in L([p, \frac{q}{t}])$ and $|g^{(r)}|^n$ is (σ, t)-convex on $[p, \frac{q}{t}]$ for $n > 1$, then we have:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du - \sum_{k=1}^{r-1}\frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!}g^{(k)}\left(\frac{p+q}{2}\right)\right|$$
$$\leq \frac{r^{r+1-1/n}(q-p)^r}{2^{r+\frac{\alpha}{n}+1}r!}\left[\beta\left(\frac{1}{r},\frac{rn-1}{n-1},\frac{2n-1}{n-1}\right)\right]^{1-\frac{1}{n}}$$
$$\times\left\{\left[\frac{1}{\sigma+1}\left(|g^{(r)}(p)|^n - t\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right) + t\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}}\right.$$
$$\left.+\left[\frac{1}{\alpha+1}\left(|g^{(r)}(q)|^n - t\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right) + t\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\}, \quad (3.\ 34)$$

where

$$\beta(u;l,s) = \int_0^u z^{l-1}(1-z)^{s-1}dz, 0 \leq u \leq 1, l > 0, s > 0.$$

is the incomplete beta function.

Proof. From Lemma 3.1, the Hölder inequality and (σ, t)-convexity of $|g^{(r)}|^n$ on $[p, \frac{q}{t}]$, we obtain

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du - \sum_{k=1}^{r-1}\frac{k\left[1+(-1)^k\right](q-p)^k}{2^{k+1}(k+1)!}h^{(k)}\left(\frac{p+q}{2}\right)\right|$$
$$\leq \frac{(q-p)^r}{2^{r+1}r!}\left(\int_0^1 (1-z)^{n(r-1)/(n-1)}(r-1+z)^{n/(n-1)}dz\right)^{1-\frac{1}{n}}$$
$$\times\left\{\left(\int_0^1\left[\left(\frac{1-z}{2}\right)^\sigma |g^{(r)}(p)|^n + t\left(1-\left(\frac{1-z}{2}\right)^\sigma\right)\left|g^{(r)}\left(\frac{q}{t}\right)\right|^n\right]dz\right)^{\frac{1}{n}}\right.$$
$$\left.+\left(\int_0^1\left[\left(\frac{1-z}{2}\right)^\sigma |g^{(r)}(q)|^n + t\left(1-\left(\frac{1-z}{2}\right)^\sigma\right)\left|g^{(r)}\left(\frac{p}{t}\right)\right|^n\right]dz\right)^{\frac{1}{n}}\right\}. \quad (3.\ 35)$$

Since

$$\int_0^1 (1-z)^{n(r-1)/(n-1)}(r-1+z)^{n/(n-1)}dz = r^{\frac{rn+n-1}{n-1}}\beta\left(\frac{1}{r}; \frac{rn-1}{n-1}, \frac{2n-1}{n-1}\right). \quad (3.\ 36)$$

Using (3. 36) in (3. 35), we got required result. □

COROLLARY 3.13. *If $r = 1$ in Theorem (3.4), since for $r = 1$, we have*

$$\beta\left(1; 1, \frac{2n-1}{n-1}\right) = \beta\left(1, \frac{2n-1}{n-1}\right) = \int_0^1 (1-z)^{\frac{2n-1}{n-1}-1}dz = \frac{n-1}{2n-1}. \quad (3.\ 37)$$

so (3. 34) becomes:

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right|$$
$$\leq \frac{(q-p)}{2^{2+\frac{\sigma}{n}}}\left[\frac{n-1}{2n-1}\right]^{1-\frac{1}{n}}$$
$$\times\left\{\left[\frac{1}{\sigma+1}\left(|g'(p)|^n - t\left|g'\left(\frac{q}{t}\right)\right|^n\right) + t\left|g'\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}}\right.$$
$$\left.+\left[\frac{1}{\alpha+1}\left(|g'(q)|^n - t\left|g'\left(\frac{p}{t}\right)\right|^n\right) + t\left|g'\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\}. \quad (3.\ 38)$$

COROLLARY 3.14. *If $r = 2$ in Theorem 3.4, then (3. 34) reduces to:*

$$\left|\frac{g(p)+g(q)}{2} - \frac{1}{(q-p)}\int_p^q g(u)du\right| \leq \frac{2^{3-1/n}(q-p)^2}{2^{4+\frac{\sigma}{n}}}\left[\beta\left(\frac{1}{2}, \frac{2n-1}{n-1}, \frac{2n-1}{n-1}\right)\right]^{1-\frac{1}{n}} \times \left\{\left[\frac{1}{\sigma+1}\left(|g''(p)|^n - t\left|g''\left(\frac{q}{t}\right)\right|^n\right) + t\left|g''\left(\frac{q}{t}\right)\right|^n\right]^{\frac{1}{n}} + \left[\frac{1}{\alpha+1}\left(|g''(q)|^n - t\left|g''\left(\frac{p}{t}\right)\right|^n\right) + t\left|g''\left(\frac{p}{t}\right)\right|^n\right]^{\frac{1}{n}}\right\}. \quad (3.\ 39)$$

4. APPLICATION TO SOME SPECIAL MEANS

Let reminiscence inequalities for any two reals which are positive. $p, q \in \mathbb{R}_+$. We take

(1) *The Harmonic mean*

$$H \equiv H(p,q) = \frac{2pq}{p+q}$$

(2) *The Arithmetic mean*

$$A \equiv A(p,q) = \frac{p+q}{2}$$

(3) *The Identric mean*

$$I \equiv I(p, q) = \begin{cases} p, & p = q, \\ \frac{q-p}{\ln q - \ln p}, & p \neq q. \end{cases}$$

(4) *The Logarithmic mean*

$$L \equiv L(p, q) = \begin{cases} p, & p = q, \\ \frac{1}{e}\left(\frac{q^q}{p^p}\right)^{\frac{1}{q-p}}, & p \neq q. \end{cases}$$

(5) *The $p-$Logarithmic mean*

$$L_p \equiv L_p(p,q) = \begin{cases} p, & p = q, \\ \left[\frac{q^{m+1}-p^{m+1}}{(m+1)(q-n)}\right]^{\frac{1}{m}}, & p \neq q \text{ and } m \neq 0, -1. \end{cases}$$

Here $p, q \in \mathbb{R}_+$, $p < q$, $m \in \mathbb{N}$ and $m > 1$ is taken all over.

THEOREM 4.1.

$$\left|A(p^{m+1}, q^{m+1}) - L_{m+1}^{m+1}(p^{m+1}, q^{m+1})\right| \leq \frac{(m+1)(q-p)}{8} \times \left\{\left[\frac{1}{3}A\left(p^{mn}, q^{mn}\right) + \frac{1}{6}q^{mn}\right]^{\frac{1}{n}} + \left[\frac{1}{3}A\left(p^{mn}, q^{mn}\right) + \frac{1}{6}p^{mn}\right]^{\frac{1}{n}}\right\}. \quad (4.\ 40)$$

Proof. For the function $g(u) = u^{m+1}$, $u \in \mathbb{R}_+$, $m \in \mathbb{N}$, then we have $|g''(u)| = m(m+1)u^{m-1}$ is a convex function on $\mathbb{R}_+$. Applying Corollary 3.4, we obtain a required result. □

THEOREM 4.2.

$$\left|H^{-1}(p,q) - L^{-1}(p,q)\right| \leq \frac{A\left(p^3, q^3\right)}{3G\left(p,q\right)\left[H(p,q)\right]^2} \tag{4. 41}$$

Proof. For the function $g(u) = \frac{1}{u}$, $u \in \mathbb{R}_+$, then we have $|g''(u)| = \frac{2}{u^3}$ is a convex function on $\mathbb{R}_+$. Applying Corollary 3.7 with $\alpha = t = 1$, we obtain a required result. □

THEOREM 4.3.

$$\left|A(p^{m+1}, q^{m+1}) - L_{m+1}^{m+1}(p^{m+1}, q^{m+1})\right| \leq \frac{m(m+1)(q-p)^2}{2^4}\left\{\frac{2}{3}\left(A\left(p^{m-1}, q^{m-1}\right)\right)\right\} \tag{4. 42}$$

Proof. For the function $g(u) = u^{m+1}$, $u \in R_+$, $m \in \mathbb{N}$, then we have $|g''(u)| = m(m+1)u^{m-1}$ is a convex function on $\mathbb{R}_+$. Applying Corollary 3.5 with $\alpha = t = 1$. We obtain required result. □

THEOREM 4.4.

$$\left|H^{-1}(p,q) - L^{-1}(p,q)\right| \leq \frac{(q-p)^2}{2^3}\left\{\frac{2}{3}\left(A\left(p^{-3}, q^{-3}\right)\right)\right\} \tag{4. 43}$$

Proof. For the function $g(u) = \frac{1}{u}$, $u \in \mathbb{R}_+$, then we have $|g''(u)| = \frac{2}{u^{-3}}$ is a convex function on $\mathbb{R}_+$. Applying $\alpha = t = 1$ in Corollary 3.7. we obtain a required result. □

5. ACKNOWLEDGMENTS

We are thankful to Abdus Salam School of Mathematical Sciences (ASSMS), GC University, Lahore-Pakistan for providing us the opportunity to avail the facilities, to get this article complete.

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