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Solution of the Diophantine Equation $x_1x_2x_3 \cdots x_{m-1} = z^n$

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Abstract. This work determines the entire family of positive integer solutions of the considered Diophantine equation. The solution is described in terms of $\frac{(m-1)(m+n-2)}{2}$ or $\frac{(m-1)(m+n-1)}{2}$ positive parameters depending on the parity of n. The solution of a system of Diophantine equations is also determined with the help of the solution of this Diophantine equation. All the results of the paper [5] are generalized in this paper.

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Key Words: Divisor; Diophantine equation; Diophantine system of equations; the greatest common divisor.

1. Introduction and Preliminaries

All solutions of a Diophantine equation of the form

$$ax - by = c,$$

have been found. But the theory on this equation in the literature can not apply on a Diophantine equation of the form

$$x_1x_2x_3 \cdots x_{m-1} = z^n. \qquad (1.\ 1)$$

So, we achieved the solutions of the equation 1. 1 . In [5], the author worked on the Diophantine equation 1. 1 for $m = 3$ with $n = 2, 3, 4, 5, 6,$ and $m = 4$ with $n = 2$; furthermore, he also worked on a Diophantine system of $2-$equations of $5-$variables. We extent all those results to the general case that is for all $m \geq 3$ and $n \geq 2$ and use it to find the solution of a system of s-Diophantine equations in t variables. The authors have not been able to find material on the equations of this paper in the literature.

- An integer b is called divisible by an other integer $a \neq 0$, if there exist some integer c such that
$$b = ac.$$
Symbolically,
$$a \mid b.$$
- Let a and b be given integer, with at least one of them different from zero. A positive integer d is called the greatest common divisor of the integers a and b.
$\diamondsuit$ If $d \mid a$ and $d \mid b$.
$\diamondsuit$ Whenever there is c such that $c \mid a$ and $c \mid b$, then $c \leq d$.
Symbolically, $d = \gcd(a, b)$.
- Two integer a, b are said to be relatively prime if $\gcd(a, b) = 1$.
- Whenever $\gcd(a, b) = d$; then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.

The following Theorem can be proved by using the fundamental theorem of arithmetic. It can also be proved without the use of the fundamental theorem (see [2]).

THEOREM 1.1. *Consider α, β and n to be positive integers. If $\alpha^n \mid \beta^n$, then $\alpha \mid \beta$.*

THEOREM 1.2. *If $\eta, \alpha, \beta, \gamma$ and n are positive integer such that*
$$\alpha\beta = \eta\gamma^n \quad where \quad \gcd(\alpha, \beta) = 1,$$
then
$$\alpha = \sigma{\alpha'_1}^n, \;\; \beta = \varsigma{\beta'_1}^n, \quad \eta = \varsigma\sigma, \quad and \quad \gamma = \alpha'_1\beta'_1$$
where, $\gcd(\alpha'_1, \beta'_1) = 1 = \gcd(\sigma, \varsigma)$.

THEOREM 1.3. *Let k be a positive integer, and the 3-variables Diophantine equation*
$$xy = kz^n.$$
Then all positive integer solutions can be described by 2-parametric formulas;
$x = k_1{t_1}^n, \;\; y = k_2{t_2}^n, \;\; z = t_1t_2$ *where,* $\gcd(k_1, k_2) = 1 = \gcd(t_1, t_2)$.

In reference [2], the reader can find a proof of Theorem 1.2 that makes use of Theorem 1.1; but not the factorization theorem of positive integer into prime powers.

2. MAIN RESULT

THEOREM 2.1. *The Diophantine equation in m-variables*
$$x_1x_2x_3 \cdots x_{m-1} = z^n$$
is equivalent to the $m + 3$-variables system of equations
$$X_{m-1}X_{m-2}x_1x_2 \cdots x_{m-3} = v{Z_0}^n, \tag{2. 2}$$
$$w^{n-2} = vd^2, \tag{2. 3}$$
where $d, X_{m-1}X_{m-2}, Z_0, w, v$ are positive integer variables such that
$x_{m-1} = \theta X_{m-1}, \quad x_{m-2} = \theta X_{m-2}, \quad z = wZ_0, \quad \theta = wd,$ *and*
$\theta = \gcd(x_{m-1}, x_{m-2}), \quad w = \gcd(z, \theta), \gcd(X_{m-1}, X_{m-2}) = 1 = \gcd(Z_0, d)$.

Proof. Consider the equation 1. 1 and let $\theta = \gcd(x_{m-1}, x_{m-2})$. Then

$$\Big\{x_{m-1} = \theta X_{m-1}; x_{m-2} = \theta X_{m-2}, \gcd(X_{m-1}, X_{m-2}) = 1\Big\}. \qquad (2.\ 4)$$

Now, from equations 1. 1 and 2. 4 we obtain,

$$\theta^2 X_{m-1} X_{m-2} x_1 x_2 \cdots x_{m-3} = z^n. \qquad (2.\ 5)$$

Assume that $w = \gcd(z, \theta)$, then

$$\Big\{z = wZ_0;\ \theta = wd,\ \gcd(Z_0, d) = 1\Big\} \qquad (2.\ 6)$$

and from equations 2. 12 and 2. 5 we further have,
$w^2 d^2 X_{m-1} X_{m-2} x_1 x_2 \cdots x_{m-1} = Z_0^n w^n$,
$d^2 X_{m-1} X_{m-2} x_1 \cdots x_{m-3} = Z_0^n w^{n-2}$. Thus

$$d^2 X_{m-1} X_{m-2} x_3 \cdots x_{m-3} = Z_0^n w^{n-2}, \quad n \geq 2 \qquad (2.\ 7)$$

since $\gcd(d, Z_0) = 1$, it follows that $\gcd(d^2, Z_0^n) = 1$ which together with equation 2. 5 and Theorem 1.1; implies that d^2 must be a divisor of w^{n-2} that is

$$w^{n-2} = vd^2, \quad \text{for some} \quad v \in \mathbb{Z} \qquad (2.\ 8)$$

from equations 2. 8 and 2. 7 , we have the desired result. □

THEOREM 2.2. *All positive solutions of the equation 2. 3 can be described by the parametric formulas:*

i: *If n is odd, then*

$$w = \prod_{i=0}^{\frac{n-3}{2}} r_{2i+1} g^2,\ \ d = \prod_{i=0}^{\frac{n-3}{2}} (r_{2i+1})^i g^{n-2},\ \ v = \prod_{i=0}^{\frac{n-3}{2}} (r_{2i+1})^{n-2i-2}.$$

ii: *If n is even, then*

$$w = \prod_{i=0}^{\frac{n-4}{2}} r_{2i+1} h,\ \ d = \prod_{i=0}^{\frac{n-4}{2}} (r_{2i+1})^i h^{\frac{n-2}{2}},\ \ v = \prod_{i=0}^{\frac{n-4}{2}} (r_{2i+1})^{n-2i-2}.$$

Proof. **i**: Let $D_0 = \gcd(w, d)$. Then

$$w = D_0.r_1 \text{ and } d = D_0.r_0 \text{ such that } \gcd(r_1, r_0) = 1. \qquad (2.\ 9)$$

So from equations 2. 2 and 2. 3 , we have $D_0^{n-2} r_1^{n-2} = vD^2 r_0^2$

$$D_0^{n-4} {r_1}^{n-2} = vr_0^2 \qquad (2.\ 10)$$

Since $\gcd(r_1, r_0) = 1$, so by using Theorem 1.1, we have $r_0^2 | D_0^{n-4}$. Consider $D_1 = \gcd(D_0, r_0)$, then

$$D_0 = D_1 r_3 \text{ and } r_0 = D_1 r_2 \text{ such that } \gcd(r_3, r_2) = 1. \qquad (2.\ 11)$$

Thus ${D_1}^{n-4} {r_3}^{n-4} r_1^{n-2} = v{D_1}^2 {r_2}^2$

$${D_1}^{n-6} {r_3}^{n-4} r_1^{n-2} = v{r_2}^2 \qquad (2.\ 12)$$

where $\gcd(r_2, r_1 r_3) = 1$, and using Theorem1.1, we get ${r_2}^2 \mid {D_1}^{n-6}$. Continued in this way and assume that $D_i = \gcd(r_{2i-2}, D_{i-1})$. Then

$$D_{i-1} = D_i r_{2i+1} \text{ and } r_{2i-2} = D_i r_{2i} \text{ such that } \gcd(r_{2i+1}, r_{2i}) = 1, \qquad (2.\ 13)$$

and we obtain $D_i^{n-2i} {r_{2i+1}}^{n-2i} {r_{2i-1}}^{n-2i-6} \cdots {r_3}^{n-2} r_1^{n-2} = v{D_i}^2 . {r_{2i}}^2$

$${D_i}^{n-2i-2} {r_{2i+1}}^{n-2i} {r_{2i-1}}^{n-4i-6} \cdots {r_3}^{n-2} r_1^{n-2} = v{r_{2i}}^2. \qquad (2.\ 14)$$

Since $\gcd(r_{2i}, r_1r_3\dots r_{2i+1}) = 1$ and using Theorem 1.1, we have ${r_{2i}}^2 \mid {D_i}^{n-2i-4}$ where $i = 0, 1, 2, \dots, \frac{n-5}{2}$. Finally we have ${r_{n-5}}^2 \mid D_{\frac{n-5}{2}}$ as n is odd so
$D_{\frac{n-5}{2}} = r_{n-2}{r_{n-5}}^2, v = r_{n-2}r_{n-4}^3r_{n-6}^5\cdots r_3^{n-4}r_1^{n-2}\ D_{\frac{n-5}{2}} = r_{n-2}{r_{n-5}}^2$ and
$r_{n-7} = r_{n-2}{r_{n-5}}^3$ substituting backward we get the required result, where $g = r_{n-5}$ and $h = r_{n-6}$. □

REMARK 1. *The parameter θ is given as*

$$\theta = \begin{cases} \prod_{i=0}^{\frac{n-3}{2}} (r_{2i+1})^{i+1} g^n, & \text{if } n \text{ is odd;} \\ \prod_{i=0}^{\frac{n-4}{2}} (r_{2i+1})^{i+1} h^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

THEOREM 2.3. *Consider the m-variables Diophantine equation 1. 1 .*

i: *If n is odd, then all the positive solution of this equation can be described in terms of the parametric formulas as:*

$$x_j = \prod_{i=0}^{\frac{n-3}{2}} \left(k_{2i+1}^{j-1}\right)^{n-2i-2} \left(\prod_{t=1}^{j-1}\left(\gamma_t^{j-t}\right)\prod_{t=1}^{j}\left(\gamma_j^{j-t}\right)\right)^{n-1}\gamma_j^{j-t}\eta_j^{m-2-j}$$

$j = 1, 2, \dots, m-3$

$$x_{m-2} = (\prod_{t=1}^{m-3}(\prod_{i=0}^{\frac{n-3}{2}}((k_{2i+1}^{t-1}l_{2i+1}^{m-3})^{i+1})(k_{2i+1}^{m-3})^{n-2i-2})(\gamma_t^{m-2-t})^{n-1})s_2^n g^n$$

$$x_{m-1} = (\prod_{t=1}^{m-3}(\prod_{i=0}^{\frac{n-3}{2}}((k_{2i+1}^{t-1}l_{2i+1}^{m-3})^{i+1})l_{2i+1}^{m-3\,n-2i-2})(\eta_t^{m-2-t})^{n-1})s_1^n g^n$$

$$z = \prod_{\lambda=1}^{m-3}\prod_{i=0}^{\frac{n-3}{2}}(k_{2i+1}^{\lambda-1}l_{2i+1})(\prod_{i=1}^{m-3}\prod_{\lambda=0}^{m-2-i}\eta_i\gamma_i^{\lambda})s_1s_2g^2$$

ii: *If n is even, then all the positive solution of this equation can be described in terms of the parametric formulas as:*

$$x_j = \prod_{i=0}^{\frac{n-4}{2}} \left(k_{2i+1}^{j-1}\right)^{n-2i-2} \left(\prod_{t=1}^{j-1}\left(\gamma_t^{j-t}\right)\prod_{t=1}^{j}\left(\gamma_j^{j-t}\right)\right)^{n-1}\gamma_j^{j-t}\eta_j^{m-2-j}$$

for all $j = 1, 2, \dots, m-3$

$$x_{m-2} = (\prod_{t=1}^{m-3}(\prod_{i=0}^{\frac{n-3}{2}}((k_{2i+1}^{t-1}l_{2i+1}^{m-3})^{i+1})(k_{2i+1}^{m-3})^{n-2i-2})(\gamma_t^{m-2-t})^{n-1})s_2^n h^{\frac{n}{2}}$$

$$x_{m-1} = (\prod_{t=1}^{m-3}(\prod_{i=0}^{\frac{n-4}{2}}((k_{2i+1}^{t-1}l_{2i+1}^{m-3})^{i+1})l_{2i+1}^{m-3\,n-2i-2})(\eta_t^{m-2-t})^{n-1})s_1^n h^{\frac{n}{2}}$$

$$z = \prod_{\lambda=1}^{m-3}\prod_{i=0}^{\frac{n-3}{2}}(k_{2i+1}^{\lambda-1}l_{2i+1}^{m-3})(\prod_{i=1}^{m-3}\prod_{\lambda=0}^{m-2-i}\eta_i^{\lambda}\gamma_i^{\lambda})s_1s_2g^2$$

where $k_{2i+1}^t, l_{2i+1}^{m-3}, b_i^t a_i^t, s_1, s_2, h$ and g are positive integers such that $1 = \gcd(s_1, s_2), \gcd(k_{2i-1}, k_{2i-1}^t l_{2i-1}^{m-3}) = 1 = \gcd(\gamma_i^t\eta_i, \gamma_i) = 1$ $\forall\ i = 1, 2, \dots, \frac{n-6}{2},\ t = 1, 2, \dots, m-3.$

Proof. Since $n \geq 2$, then by Theorem 2.1, the given equation is equivalent to the Diophantine system, $v = a_0, X_{m-1}X_{m-2}x_1x_2x_3\dots x_{m-3} = Z_0^n a_0$ and $w^{n-2} = vd^2$.

Assume that $P_1 = \gcd(x_1, Z_0)$. Then $x_1 = P_1.X_1$ and $Z_0 = P_1.Z_1$, so $X_{m-1}X_{m-2}X_1x_2\cdots x_{m-3} = Z_1^n P_1^{n-1}a_0$. Again let $\gcd(Z_1, X_1) = 1$. Then, we have $X_1 \mid a_0P_1^{n-1} \Rightarrow X_1.a_1 = a_0P_1^{n-1}$ and by Theorem 1.2, $X_1 = \alpha_1\gamma_1^{n-1}$ and $a_1 = \beta_1\eta_1^{n-1}$ such that $\alpha_1\beta_1 = a_0$ and $\gamma_1\eta_1 = P_1$ where $\gcd(\alpha_1, \beta_1) = 1 = \gcd(\gamma_1, \eta_1)$.

We have $X_{m-1}X_{m-2}x_2x_3....x_{m-3} = Z_0^n a_1$. Let $P_i = \gcd(x_i, Z_{i-1})$. Then $x_i = P_i.X_i$ and $Z_{i-1} = P_i.Z_i$. Thus $X_{m-1}X_{m-2}X_i \prod_{j=i+1}^{m-3} x_j = Z_i^n P_i^{n-1}a_{i-1}$. Since $\gcd(Z_i, X_i) = 1$ thus, $X_i \mid a_{i-1}P_i^{n-1} \Rightarrow X_i.a_i = a_{i-1}P_i^{n-1}$ so by Theorem 1.2, we get $X_i = \alpha_i\gamma_i^{n-1}$ and $a_i = \beta_i\eta_i^{n-1}$ such that $\alpha_i\beta_i = a_i$ and $\gamma_i\eta_i = P_i$ where $\gcd(\alpha_i, \beta_i) = 1 = \gcd(\gamma_i, \eta_i)$.

So, $X_{m-1}X_{m-2} \prod_{j=i+1}^{m-3} x_j = Z_{i-1}^n a_i \ \forall\ i = 1, 2, 3, \ldots, m-3$ and at last we get $X_{m-1}X_{m-2} = Z_{m-3}^n a_{m-3}$ then by Theorem 1.2 $X_{m-2} = \alpha_{m-2}s_1^n$ and $X_{m-1} = \beta_{m-2}s_2^n$ such that $\alpha_{m-2}\beta_{m-2} = a_{m-3}$ and $s_1s_2 = Z_{m-4}$ with $\gcd(\alpha_{m-2}, \beta_{m-2}) = 1 = \gcd(s_1, s_1)$. Now by using Theorem 1.2 and technique in proof of Theorem 2.2, we have for all $j = 1, 2, 3, \ldots, m-2$, $\alpha_j = (\prod_{i=1}^{m-3}(k_{2i+1}^{j-1})^{n-2i-2})(\prod_{t=1}^{j}\gamma_t^{j-t})^{n-1}$, and $\beta_j = (\prod_{i=1}^{m-3}(l_{2i+1}^{j-1})^{n-2i-2})(\prod_{t=1}^{j}\eta_t^{j-t})^{n-1}$.

Put in equations 2. 3 and 2. 5 we get the required result. □

REMARK 2. *The solution is described in terms of* $\frac{(m-1)(m+n-2)}{2}$ *or* $\frac{(m-1)(m+n-1)}{2}$ *positive parameters depending on* n *even or odd.*

THEOREM 2.4. *Let* $p_1, p_2, \ldots, p_s$ *and* r *be positive integers and suppose that* $t = p_1 + p_2 + \cdots + p_s - r$. *Consider the* t*-variables Diophantine system of* s *equations,*

$x_{11}x_{12}x_{13}\cdots x_{1p_1-1} = z_1^{k_1}$,
$x_{21}x_{22}x_{23}\cdots x_{2p_2-1} = z_2^{k_2}$,
$\vdots$
$x_{i1}x_{i2}x_{i3}\cdots x_{ip_i-1} = z_i^{k_i}$,
$\vdots$
$x_{s1}x_{s2}x_{s3}\cdots x_{sp_s-1} = z_s^{k_s}$,

where $2 \leq k_j$ *are positive numbers and* s *is the number of equations and* r *is the number of repeated variables in this system. Then all the positive solutions of this system of equations can be described by using the following algorithm:*

Algorithm:

- Write solution of each equation by using Theorem 2.3
- Select one variable from r-repeated variables and find the unique d in solution by using techniques of Theorem 2.3
- Replace those values of parameters appear in selected repeated variable, in other variables.
- Do the same activity with other repeated variables.

- If all repeated variables have unique solution then substituting the values of parameter that exist in that variables in other variable.

To explain above algorithm, we give one example below:

EXAMPLE 1. *Consider the* 6*-variables Diophantine system of two equations,*

$$x_1x_2x_3 = z_1^3 \text{ and } x_3x_4 = z_2^2.$$

Step 1: We apply Theorem 2.3 on equation 1 *of the above two equations system to get the following solution:*
$x_1 = k_1{k_1'}^2 l_1' {\gamma_1'}^2 R_1^3 g^3,\ x_2 = k_1 k_1' {l_1'}^2 {\eta_1'}^2 R_2^3 g^3,\ x_3 = k_1\gamma_1^3\gamma_1'\eta_1',$
$z_1 = l_1'k_1'k_1\gamma_1\gamma_1'\eta_1'R_1R_2g^2$
and for equation 2 *of this system, we have*
$x_4 = dr_1^2,\ x_3 = r_2^2 d,\ z_2 = dr_1r_2$

Step 2: Since x_3 *is the repeated variable in the equations of the system, so*

$$r_2^2 d = x_3 = k_1\gamma_1^3\gamma_1'\eta_1'. \tag{2. 15}$$

Now we will find that unique d *for* x_3*, and suppose* $a = \gcd(d, k_1)$ *such that*

$$d = aD_1; k_1 = aK_1,\ \gcd(D_1, K_1) = 1$$

put in equation 2. 15

$$r_2^2 D_1 = \gamma_1^3\gamma_1'\eta_1'K_1 \ \Rightarrow D_1 \mid \gamma_1^3\gamma_1'\eta_1'. \tag{2. 16}$$

Then there exist a positive integer b *such that* $D_1 b = \gamma_1^3\gamma_1'\eta_1'$ *by using Theorem 2.3,* $D_1 = \beta_1 c_1^3,\ \ b = \beta_2 c_2^3,\ \ \gamma_1 = c_1c_2,\ \gamma_1'\eta_1' = \beta_1\beta_2$ *such that* $\gcd(c_2, c_1) = 1 = \gcd(\beta_1, \beta_1)$ *now equation 2. 16 becomes*

$$r_2^2 = \beta_2 c_2^3 K_1 \tag{2. 17}$$

from equation 2. 17 , we get $r_2^2 \mid c_2^3$*, then there exist a positive integer* e *such that* $r_2^2 e = c_2^3$ *then by using Theorem 2.2*

$$c_2 = \alpha_1\alpha_3 f^2,\ \ r_2 = \alpha_3 f^3,\ \ e = \alpha_1^3\alpha_3$$

substituting the values in equation 2. 17 , we get $1 = \beta_2\alpha_1^3\alpha_3K_1 \Longrightarrow \beta_2 = \alpha_1 = 1 = \alpha_3 = K_1$ *replacing the values, we get all the positive solutions of this system described by the* 11 *parametric formulas*
$x_1 = a{k_1'}^2 l_1' {\gamma_1'}^2 R_1^3 g^3,\ \ x_2 = ak_1'{l_1'}^2{\eta_1'}^2R_2^3g^3,\ \ x_3 = a\gamma_1'\eta_1'b_1^3f^6,$
$x_4 = a\gamma_1'\eta_1'b_1^3r_1^2, z_1 = al_1'k_1'\gamma_1'\eta_1'b_1f^2R_1R_2g^2,\ \ z_2 = a\gamma_1'\eta_1'b_1^3f^3r_1.$

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