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Slicing Associated to a Plurisubharmonic Function

Hedi Khedhiri
Institut préparatoire aux études d'ingénieurs de Monastir,
Université de Monastir 5019, Tunisia,
E-mail: khediri_h@yahoo.fr

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Abstract. In this paper, we study the slicing of currents, with respect to a locally bounded plurisubharmonic function. For a positive closed current and its associated Lelong-Skoda potential, we prove that, with respect to a smooth and strictly plurisubharmonic function, the slices are well defined except at points lying in a pluriplolar subset. In particular, the slices of the current of integration over an analytic set, are well defined explicitly, except at points lying in a countable family of proper analytic subsets. Furthermore, we state the analogue of the generalized slicing formula due to H. Ben Messaoud and H. El Mir.

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1. Introduction and main results

Slicing of currents was studied in [3] and [6], this is an important tool for studying global geometric problems as well as for questions related to local algebra and intersection theory. Here, we develop some approach to slicing with respect to a locally bounded plurisubharmonic function.

We consider in $\mathbb{C}^n$, the unit poly-disk Δ^n and an open subset Ω such that $\Delta^n \Subset \Omega$. Let $1 \leq k \leq p \leq n$, any point $z \in \Delta^n$, is written $z = (z_1 \ldots z_n) = (z', z'')$ and $\pi(z)$ is defined by $\pi(z) = z'$, where $z' := (z_1 \ldots z_k)$ and $z'' := (z_{k+1} \ldots z_n)$. Given a locally bounded plurisubharmonic function $\varphi = \varphi(z')$ on Δ^n. The positive measure with support S_φ, such that

$$(dd^c\varphi)^k = \mu_\varphi \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_k \wedge d\bar{z}_k \tag{1. 1}$$

is denoted μ_φ and we say that μ_φ is the trace measure of the current $(dd^c\varphi)^k$.

Slicing of a current, with respect to the function φ, is defined as the following

DEFINITION 1. *For any point $a \in S_\varphi$, we say the slice $< R, \pi, a >_\varphi$ of a current R of bidimension (p,p) on Ω, associated with $\varphi(z')$, at point a, exists if, and only if the* *following limit*

$$\lim_{\varepsilon\to 0}\frac{1}{\mu_\varphi(B_k(a,\varepsilon))}\int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} R\wedge(dd^c\varphi)^k\wedge f \tag{1. 2}$$

exists in $\mathbb{C}$ *for any test form* $f\in\mathscr{D}_{(p-k,p-k)}(\Delta^n)$*; this limit is denoted* $<R,\pi,a>_\varphi(f)$.

In case $\varphi(z')=|z'|^2$ we get the definition of the slice by Federer [3].

Being introduced, the φ- slicing of currents, is the main goal of this paper.

The first part of the paper deals with some properties of slicing with respect to a locally bounded plurisubharmonic function φ. The second part, deals with the study of the existence of slices of positive closed currents T of bidimension (p,p) on Δ^n, with respect to a smooth and strictly plurisubharmonic function φ.

We will break our study into a sequence of steps. First we reduce the problem to the case of a current having continuous coefficients. Then we study the slice of a current having integrable coefficients with respect to the measure $\mu_\varphi\otimes\lambda_{n-k}$ where μ_φ is the measure defined by (1. 1) and λ_{n-k} is the Lebesgue measure on $\mathbb{C}^{n-k}$.

Under assumption that μ_φ is given by a locally bounded function on Δ^{n-k}, we show that the slice $<R,\pi,a>_\varphi$, of a current R having locally $\mu_\varphi\otimes\lambda_{n-k}$-integrable coefficients, is well defined for μ_φ-almost every $a\in S_\varphi$, which generalizes the Federer's theorem [3] for locally plate currents.

The study of the φ-slicing of the Lelong-Skoda potential U associated with a positive closed current T, is a typical case. If the potential U is given canonically by

$$U(z)=\sum_{I,J}U_{I,J}dz_I\wedge dz_J \tag{1. 3}$$

then we prove that, every coefficient $z\mapsto U_{IJ}(z)$ of the decomposition (1. 3), is a locally $\mu_\varphi\otimes\lambda_{n-k}$-integrable function and for μ_φ-almost every $a\in S_\varphi$, the current $\langle U,\pi,a\rangle_\varphi$ is well defined.

Next, we suppose that φ is smooth and strictly psh. We establish that $<U,\pi,a>_\varphi$ is well defined for any point $a\notin E_\varphi$, where E_φ is a pluripolar subset of Δ^k. Explicitly, E_φ is given by the set of points $a\in S_\varphi$ such that the current $j_a^*(U)$ does not have a locally finite mass in Δ^{n-k}, where j_a denotes the map defined on $\mathbb{C}^{n-k}$ by $j_a(z'')=(a,z'')$.

A very important example of the φ-slicing of closed positive currents is the case of currents of integration over analytic sets. This will be done in the last section of this paper.

The followings are the main results of this paper.

Theorem 3.1 *Let* $\varphi\in\mathscr{C}^2\cap Psh(\Delta^k)$ *such that* φ *is strictly psh and let* U_j *be a sequence of smooth currents which decreases weakly to* U*. Then for any* $a\in\Delta^k$, $\lim_{j\to+\infty}\langle U_j,\pi,a\rangle_\varphi$ *exists in* $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$ *if, and only if* $a\notin E_\varphi$ *and in this case, in the weak sense of currents, we have* $\lim_{j\to+\infty}\langle U_j,\pi,a\rangle_\varphi=j_a^*(U)$.

Investigating the results of [6], we establish the formula (1. 4) called the φ-slicing formula. The origin of formula (1. 4) was due to Federer [3]. It can be seen as a generalization to the Fubini Formula. When $\varphi(z')=|z'|^2$, we get the slicing formula of [6] stated in 1995. The φ-slicing formula may provide a useful tool for studying many problems related to Monge -Ampere operators, extension of currents, intersection theory,...

Theorem 4.1 *Let* $\varphi=\varphi(z')\in\mathscr{C}^2\cap Psh(\Delta^n)$ *such that* φ *is strictly psh and* $a\in\Delta^k$. *Then we have*

(1) The slice $\langle U, \pi, a\rangle_\varphi$ is well defined in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$ if and only if $a \notin E_\varphi$ and in this case the current $\langle U, \pi, a\rangle_\varphi$ is equal to $j_a^*(U)$.

(2) For any $\Psi \in \mathscr{D}_{(p-k+1,p-k+1)}(\Delta^n)$ and for any $v_1, \ldots, v_k \in L^\infty_{loc} \cap Psh(\Delta^k)$, we have the following slicing formula with respect to φ

$$\int_{\Delta^n} U \wedge dd^c\tilde{v_1} \wedge \cdots \wedge dd^c\tilde{v_k} \wedge \Psi = \int_{a\in S_\varphi} \langle U, \pi, a\rangle_\varphi(\Psi) dd^c v_1 \wedge \cdots \wedge dd^c v_k \quad (1.\ 4)$$

where $\tilde{v_j} = v_j \circ \pi, \quad j = 1, \ldots, k.$

Formula (1. 4) holds not only for the Lelong-Skoda potential but it is also available for closed positive currents:

Theorem 4.2 *Let $\varphi = \varphi(z') \in \mathscr{C}^2 \cap Psh(\Delta^n)$ such that φ is strictly psh and let $a \in \Delta^k$. Then we have*

(1) For any $a \notin E_\varphi$, the slice $\langle T, \pi, a\rangle_\varphi$ is well defined.

(2) For any $\Psi \in \mathscr{D}_{(p-k+1,p-k+1)}(\Delta^n)$ and for any $v_1, \ldots, v_k \in L^\infty_{loc} \cap Psh(\Delta^k)$, we have

$$\int_{\Delta^n} T \wedge dd^c\tilde{v_1} \wedge \cdots \wedge dd^c\tilde{v_k} \wedge \Psi = \int_{a\in S_\varphi} \langle T, \pi, a\rangle_\varphi(\Psi) dd^c v_1 \wedge \cdots \wedge dd^c v_k \quad (1.\ 5)$$

where $\tilde{v_j} = v_j \circ \pi, \quad j = 1, \ldots, k.$

A natural problem arises through the work of this paper, can these results hold with respect to a given locally bounded plurisubharmonic function without smoothness assumption. The study of the general case may be very subtile. We would like to define this, in a next paper, in some cases when the potential U is associated with $(1, 1)$-closed positive currents.

Let now review some notions and notations. We denote $\mathscr{D}_{(s,t)}(\Omega)$ the space of smooth compactly supported-differential forms of bidegree (s, t) on Ω. The dual $\mathscr{D}'_{(s,t)}(\Omega)$ is the space of currents of bidimension (s, t) or of bidegree $(n - s, n - t)$. A current R of bidimension (p, p) on Ω, is said to be positive if for all $\gamma_1, \ldots, \gamma_p$ in $\mathscr{D}_{(1,0)}(\Omega)$, the distribution $R \wedge i\gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge i\gamma_p \wedge \bar{\gamma}_p$ is a positive measure.

We denote by $Psh(\Omega)$ the set of plurisubharmonic functions on Ω and $L^\infty_{loc} \cap Psh(\Omega)$ the subset of elements in $Psh(\Omega)$ which are locally bounded. We use the standard notations for the operators $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$ and $dd^c = 2i\partial\bar{\partial}$. The Kähler form on $\mathbb{C}^n$ is denoted by $\beta(t) = dd^c|t|^2$ and can be written as $\beta(t) = \beta'(t') + \beta''(t'')$ where β' and β'' are Kähler forms on $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$ respectively.

Let $\varphi = \varphi(z') \in L^\infty_{loc} \cap Psh(\Delta^n)$, following [1], $dd^c\varphi$ and its exterior powers $(dd^c\varphi)^j$ are well defined currents on Δ^n. In particular, the positive closed current $(dd^c\varphi)^k$ satisfies the equality

$$(dd^c\varphi)^k \wedge \beta''^{n-k} = \mu_\varphi \beta^n. \quad (1.\ 6)$$

In case φ is smooth, the measure μ_φ may be considered as the Lebesgue measure with density a continuous function denoted also $\mu_\varphi(z')$. In local coordinates, the equation (1. 6) gives the following explicit formula of $\mu_\varphi(z')$:

$$\mu_\varphi(z') = c_k \det(\frac{\partial^2\varphi}{\partial z_s \partial \bar{z}_t})_{1\le s,t\le k}, \quad (1.\ 7)$$

where c_k is a positive constant. Furthermore, according to [1], if $\mu_\varphi \not\equiv 0$, then the support S_φ of the measure μ_φ is not pluripolar in Δ^k. The function φ is said to be strictly plurisubharmonic on Δ^n if, it is locally integrable on Δ^n and if, for every point $z_0 \in \Delta^n$, there exists a neighbor-hood ω of z_0 and $c > 0$, such that $\varphi(z) - c|z|^2$ is plurisubharmonic on ω. Finally, $B_k(r)$ and $B_{n-k}(r)$ are the balls centered at the origin and of radius r respectively in $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$.

2. φ-Slicing of a current with coefficients in $L^1_{loc}(\mu_\varphi \otimes \lambda_{n-k})$

We begin this study by the case of a current R having coefficients separately continuous with respect to variables z' and z''.

2.1. φ-Slicing of a current with continuous coefficients.

It is well known that the Lelong skoda potential U associated with a closed positive current T, satisfies the equality of currents $dd^cU = T + R$ where R is a smooth form. Hence, to study the φ-slicing of T it is sufficient to study the φ-slicing of the associated Lelong-Skoda potential U. So we begin this paragraph by the following proposition which will be useful in the proof of Theorem 4.2.

PROPOSITION 2.1. *Let $R \in \mathscr{D}'_{(p,p)}(\Delta^n)$, $\varphi = \varphi(z') \in L^\infty_{loc} \cap Psh(\Delta^n)$ and $a \in S_\varphi$ such that $\langle R, \pi, a\rangle_\varphi$ is well defined. Then*

(1) *$\langle R, \pi, a\rangle_\varphi$ is supported by $S_\varphi \cap Supp\, R \cap \pi^{-1}\{a\}$.*
(2) *$d\langle R, \pi, a\rangle_\varphi$ and $dd^c\langle R, \pi, a\rangle_\varphi$ are well defined, moreover we have $d\langle R, \pi, a\rangle_\varphi = \langle dR, \pi, a\rangle_\varphi$ and $dd^c\langle R, \pi, a\rangle_\varphi = \langle dd^cR, \pi, a\rangle_\varphi$.*

Proof. We verify at once that these statements come from the definition of $\langle R, \pi, a\rangle_\varphi$ and the weak continuity of the operators d and dd^c. □

We shall now prove that for all $\varphi = \varphi(z') \in L^\infty_{loc} \cap Psh(\Delta^n)$, the slice $\langle R, \pi, a\rangle_\varphi$ of a continuous current, is well defined for all $a \in S_\varphi$. Note that the study of the $\varphi-$slicing of continuous currents will be useful in the study of the $\varphi-$slicing of the Lelong-Skoda potential. The case of the potential will be studied in section 3 where a regularization procedure will be used.

PROPOSITION 2.2. *Let $\varphi = \varphi(z') \in L^\infty_{loc} \cap Psh(\Delta^n)$ and $R \in \mathscr{D}'_{(p,p)}(\Delta^n)$ such that the coefficients are continuous separately with respect to z', z''. Then we have the following*

(1) *for any $a \in S_\varphi$ the slice $\langle R, \pi, a\rangle_\varphi$ is well defined. Further, for any test form $\Psi \in \mathscr{D}_{(p-k,p-k)}(\Delta^n)$, we have*

$$\langle R, \pi, a\rangle_\varphi(\Psi) = \int_{z'' \in \Delta^{n-k}} j_a^* R \wedge j_a^* \Psi.$$

(2) *For any $\Psi \in \mathscr{D}_{(p-k,p-k)}(\Delta^n)$ and for any $v_1, \ldots, v_k \in L^\infty_{loc} \cap Psh(\Delta^k)$, we have*

$$\int_{\Delta^n} R \wedge dd^c\tilde{v_1} \wedge \cdots \wedge dd^c\tilde{v_k} \wedge \Psi = \int_{a \in S_\varphi} \langle R, \pi, a\rangle_\varphi(\Psi) dd^c v_1 \wedge \cdots \wedge dd^c v_k.$$

where $\tilde{v_j} = v_j \circ \pi, \quad j = 1, \ldots, k$.

Proof. We may assume the current R takes the form $R = f\sigma_{n-p} dz_I \wedge d\bar{z}_J$ where, f is a function continuous separately with respect to z', z''. Let $\sigma_s = i^{s^2} 2^{-s},\ i^2 = -1\ (s \in \mathbb{N}^*)$

and let $\Psi = \psi\sigma_{p-k} dz_L \wedge d\bar{z}_K$ be a test form such that $I \cap K \cap \{1, \dots, k\} \neq \emptyset$ and $J \cap L \cap \{1, \dots, k\} \neq \emptyset$; which means that $I, J, K, L \subset \{k+1, \dots, n\}$. Put

$$\Gamma_\delta = \frac{1}{\mu_\varphi(B_k(a,\delta))} \int_{B_k(a,\delta)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge \Psi, \tag{2. 8}$$

an easy computation of the second member of (2. 8) yields

$$\Gamma_\delta = \frac{1}{\mu_\varphi(B_k(a,\delta))} \int_{\{|z'-a|<\delta\}} d\mu_\varphi(z') \int_{z''\in\Delta^{n-k}} f(z', z'')\psi(z', z'') d\lambda_{n-k}(z'') \tag{2. 9}$$

Let $g(z') = \int_{z''\in\Delta^{n-k}} f(z', z'')\psi(z', z'') d\lambda_{n-k}(z'')$, then (2. 9) can written as

$$\Gamma_\delta - g(a) = \frac{1}{\mu_\varphi(B_k(a,\delta))} \int_{\{|z'-a|<\delta\}} (g(z') - g(a)) d\mu_\varphi(z') \tag{2. 10}$$

Since for each fixed $z'' \in \Delta^{n-k}$ the function $z' \mapsto f(z', z'')\Psi(z', z'')$ is continuous on Δ^k, then the function g is continuous on $\bar{\Delta}^k$ and hence it is uniformly continuous on $\bar{\Delta}^k$, so for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, we have by (2. 10), $|\Gamma_\delta - g(a)| < \varepsilon$. Hence we get

$$\begin{array}{rcl} \langle R, \pi, a\rangle_\varphi(\Psi) & = & \lim_{\delta\to 0} \Gamma_\delta \\ & = & g(a) \\ & = & \int_{z''\in\Delta^{n-k}} f(a, z'')\psi(a, z'') d\lambda_{n-k} \end{array}$$

this means that

$$\langle R, \pi, a\rangle_\varphi(\Psi) = \int_{z''\in\mathbb{C}^{n-k}} j_a^*(R) \wedge j_a^*(\Psi). \tag{2. 11}$$

The equality (2. 11) is equivalent to $\langle R, \pi, a\rangle_\varphi(\Psi) = \pi_*(R \wedge \Psi)$ and this achieves the proof of the first statement. To prove the second statement, we observe that

$$\int_{\Delta^n} R \wedge dd^c\tilde{v_1} \wedge \dots \wedge dd^c\tilde{v_k} \wedge \Psi = \langle dd^c v_1 \wedge \dots \wedge dd^c v_k\ ,\ \pi_*(R \wedge \Psi)\rangle. \tag{2. 12}$$

The equality (2. 12) holds true since $\pi_*(R\wedge\Psi)$ is the function $a \mapsto \langle R, \pi, a\rangle_\varphi(\Psi)$ which is continuous and compactly supported in S_φ. □

REMARK 1. *If the slice $\langle R, \pi, a\rangle_\varphi$ is well defined then it depends only on S_φ, more precisely if $\varphi_1, \varphi_2 \in L^\infty_{loc} \cap Psh(\Delta^n)$ such that $S_{\varphi_1} = S_{\varphi_2}$ and if $\langle R, \pi, a\rangle_{\varphi_i}$, $(i = 1, 2)$, is well defined then we have $\langle R, \pi, a\rangle_{\varphi_1} = \langle R, \pi, a\rangle_{\varphi_2}$ in the weak sense of continuous currents.*

2.2. **φ-Slicing of a current with coefficients in $L^1_{loc}(\mu_\varphi \otimes \lambda_{n-k})$.** Let $\varphi = \varphi(z') \in L^\infty_{loc} \cap Psh(\Delta^n)$ and assume that $0 \in S_\varphi$. For technical reasons, we need to use a regularization method that can be related to the definition (1. 2) of the slice associated with φ, so we define the following convolution procedure: for any $\varepsilon > 0$ we denote by $\alpha_{1,\varepsilon}(.)$ the function in $L^1_{loc}(\pi(\Omega), \mu_\varphi)$ defined by the quotient $\alpha_{1,\varepsilon}(t) = \dfrac{1\!\!1_{B_k(\varepsilon)}(t)}{\mu_{\varphi(B_k(\varepsilon))}}$; if $f \in L^1_{loc}(\pi(\Omega), \mu_\varphi)$, then the convolution of the function $z' \mapsto f(z')$ by the measure $\alpha_{1,\varepsilon}\mu_\varphi$ is given by

$$(f * \alpha_{1,\varepsilon}\mu_\varphi)(z') = \frac{1}{\mu_\varphi(B_k(\varepsilon))} \int_{B_k(\varepsilon)} f(z' - t') d\mu_\varphi(t'). \tag{2. 13}$$

One can ask if the family $(f * \alpha_{1,\varepsilon}\mu_\varphi)$, given by formula (2. 13), is continuous and converges to f pointwise as $\varepsilon \to 0$. We Know by the Lebesgue's theorem that for any locally λ_k-integrable function on $\mathbb{C}^k$ and for λ_k-almost every $a \in \mathbb{C}^k$, we have

$$\frac{1}{\omega_{2k}\varepsilon^{2k}} \int_{B_k(a,\varepsilon)} f(z')d\lambda_k(z') \longrightarrow f(a) \quad \text{as} \quad \varepsilon \to 0$$

where ω_{2k} is the volume of the unitary ball in $\mathbb{C}^k$. It is not evident to justify this result for the family $(f * \alpha_{1,\varepsilon}\mu_\varphi)_\varepsilon$. In addition, the function $z' \mapsto f * \alpha_{1,\varepsilon}\mu_\varphi(z')$ do not need to be continuous, indeed when the measure μ_φ takes the form $\sigma + \delta_0$ where σ is a positive measure having support without holes and δ_0 is the Dirac measure at point 0, then we get $f * \alpha_{1,\varepsilon}\mu_\varphi = f * \alpha_{1,\varepsilon}\sigma + f$. In order to surmount these difficulties, it will be convenient in this section, to assume that φ satisfies the following assumption:

> there exists a locally bounded function m on $\pi(\Omega)$ and a constant $c_0 > 0$ such that
>
> $$\begin{array}{l} (dd^c\varphi)^k \equiv m\beta'^k \\ \text{for } \lambda_k - \text{almost every } z' \in \pi(\Omega), \quad m(z') \geq c_0. \end{array} \tag{2. 14}$$

PROPOSITION 2.3. *Let $\varphi \in L^\infty_{loc} \cap Psh(\Delta^k)$ satisfying (2. 14) and $R \in \mathscr{D}'_{p,p}(\Delta^n)$ be a positive current with coefficients in $L^1_{loc}(\Delta^n, \mu_\varphi \otimes \lambda_{n-k})$. Then*

$$\lim_{\varepsilon\to 0} \frac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge \Psi$$

exists for all $\Psi(z) \in \mathscr{D}_{(p-k,p-k)}(\Delta^n)$ if, and only if,

$$\lim_{\varepsilon\to 0} \frac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge \Psi$$

exists for all $\Psi(z'') \in \mathscr{D}_{(p-k,p-k)}(\Delta^{n-k})$.

Proof. We have only to prove the converse statement. Take $\Psi = \sum_{I,J} \Psi_{I,J} dz_I \wedge d\bar{z}_J \in \mathscr{D}_{(p-k,p-k)}(\Delta^n)$ and put $||\Psi||(z) = \sum_{I,J} |\Psi_{I,J}|$. By the rest integral formula, there exists $\Psi_0 \in \mathscr{D}(\Delta^n)$ positive such that

$$||\Psi - j_a^*(\Psi)|| \leq |z' - a|\Psi_0(z).$$

Let $\Psi_1 \in \mathscr{D}(\Delta^{n-k})$ be a smooth function with compact support such that $\Psi_0(z) \leq \Psi_1(z'')$ for all $z \in \Delta^n$. Put

$$I_\delta = \frac{1}{\mu_\varphi(B_k(a,\delta))} \int_{B_k(a,\delta)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge [\Psi - j_a^*(\Psi)],$$

by the assumption that $(dd^c\varphi)^k \equiv m(z')\beta'^k$, we have

$$\begin{aligned} |I_\delta| &\leq ||R \wedge \frac{m 1\!\!1_{B_k(a,\delta)\times\Delta^{n-k}}}{\mu_\varphi(B_k(a,\delta))} \beta'^k \wedge \beta''^{p-k}|| \,(||\Psi - j_a^*(\Psi)||) \\ &\leq \frac{1}{\mu_\varphi(B_k(a,\delta))} \int_{B_k(a,\delta)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge |z' - a|\Psi_0(z)\beta''^{p-k} \\ &\leq \frac{\delta}{\mu_\varphi(B_k(a,\delta))} \int_{B_k(a,\delta)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge \Psi_1(z'')\beta''^{p-k} \end{aligned}$$

the last quantity goes to 0 when $\delta \to 0$, and this proves the statement. $\square$

In [6], it was proved that, for a current R having locally $\lambda_k \otimes \lambda_{n-k}$-integrable coefficient, the slice $\langle R, \pi, a\rangle$ is well defined for λ_k-almost every $a \in \Delta^k$. We want to improve this result by proving that, for any current R with $\mu_\varphi \otimes \lambda_{n-k}$-integrable coefficients, the slice $\langle R, \pi, a\rangle_\varphi$ is well defined except for a in a μ_φ-negligible subset of Δ^k. More precisely we have:

THEOREM 2.1. *Let $\varphi = \varphi(z') \in L^\infty_{loc} \cap Psh(\Delta^n)$ satisfying (2. 14) and $R \in \mathscr{D}'_{p,p}(\Delta^n)$ be a current with coefficients in $L^1_{loc}(\Delta^n, \mu_\varphi \otimes \lambda_{n-k})$. Then for μ_φ-almost every $a \in S_\varphi$ the slice $\langle R, \pi, a\rangle_\varphi$ is well defined and is equal to $j_a^*(R)$.*

Proof. Assume $R = f\beta^{n-p}$ where $f \in L^1_{loc}(\Delta^n, \mu_\varphi \otimes \lambda_{n-k})$. Take a test function $\psi(z'') \in \mathscr{D}(\Delta^{n-k})$ and put

$$\Gamma_\delta = \frac{1}{\mu_\varphi(B_k(a,\delta))}\int_{B_k(a,\delta)\times\Delta^{n-k}} R \wedge (dd^c\varphi)^k \wedge \psi\beta''^{p-k}.$$

For $\mu_\varphi-$almost every $a \in S_\varphi$ we have

$$\begin{aligned}
\Gamma_\delta &= \tfrac{1}{\mu_\varphi(B_k(a,\delta))}\int_{\{|z'-a|<\delta\}} d\mu_\varphi(z')\int_{z''\in\Delta^{n-k}} f(z',z'')\psi(z'')d\lambda_{n-k}(z'')\\
&= \tfrac{1}{\mu_\varphi(B_k(a,\delta))}\int_{\{|z'-a|<\delta\}} g(z')d\mu_\varphi(z')\\
&= \tfrac{1}{\mu_\varphi(B_k(a,\delta))}\int_{\{|z'-a|<\delta\}} (g(z')-g(a))\, m(z')d\lambda_k(z') + g(a)
\end{aligned}$$

where g is the function defined by

$$g(z') = \int_{z''\in\Delta^{n-k}} f(z',z'')\psi(z'')d\lambda_{n-k}(z''). \tag{2. 15}$$

Since $f \in L^1_{loc}(\Delta^n, \mu_\varphi \otimes \lambda_{n-k})$ and since g is given by (2. 15), then by the Fubini theorem we may affirm that $g \in L^1_{loc}(\mu_\varphi, \pi(\Omega))$. By hypothesis on φ, we have $\mu_\varphi \equiv m\lambda_k$ where the function m is positive and locally bounded. As a consequence, the function $z' \mapsto (g(z') - g(a))m(z')$ lies in $L^1_{loc}(\lambda_k, \pi(\Omega))$. Put $D(\delta) = \Gamma_\delta - g(a)$, we have

$$\begin{aligned}
|D(\delta)| &\le \tfrac{1}{c_0\omega_{2k}\delta^{2k}}\int_{\{|z'-a|<\delta\}} |g(z')-g(a)|m(z')d\lambda_k(z')\\
&= \tfrac{1}{c_0\omega_{2k}}\int_{\{|z'|<1\}} |g(\delta z'+a)-g(a)|m(\delta z'+a)d\lambda_k(z').
\end{aligned}$$

Since the function $z' \mapsto |g(\delta z'+a) - g(a)|m(\delta z'+a)$ is finite λ_k- almost everywhere on $B_k(0,1)$ and for almost every $a \in \Delta^k$, it tends to the zero function, as $\delta \to 0$, then we can conclude, by the dominated convergence theorem, that $\lim_{\delta\to 0} D(\delta) = 0$. This shows that, for $\mu_\varphi-$almost every $a \in \Delta^k$, the slice $\langle R, \pi, a\rangle_\varphi$ is well defined and is equal to $j_a^*(R)$. □

REMARK 2. *In case $\varphi(z') = |z'|^2$ we find the theorem of Federer* [3] *for the locally plate currents.*

3. φ-SLICING OF THE LELONG-SKODA POTENTIAL

In this section we study the existence of the slice $\langle U, \pi, a\rangle_\varphi$ of the Lelong-Skoda potential associated with a closed positive current T, defined by

$$U(z) = \int_{x\in\mathbb{C}^n} \eta(x)N(z-x)T(x)\wedge\beta^{n-1}(z-x) \tag{3. 16}$$

where η is a function in $\mathscr{D}(\Omega)$ such that $0 \leq \eta \leq 1, \eta \equiv 1$ on a neighborhood of Δ^n, and $N(t) = \frac{-c_n}{|t|^{2n-2}}$ is the Newton Kernel in $\mathbb{C}^n$. We start this study, by the following Lemma

LEMMA 3.1. *Let T be a positive closed current of bidimension (p,p) on Ω and U be the potential of T such that $dd^cU = T$ on a neighborhood of Δ^n. Let $v \in$Psh$(\Delta^n) \cap L^\infty_{loc}(\partial\Delta^n)$, then we have*

$$\int_{\Delta^n} T \wedge (dd^c v)^k \wedge \beta^{p-k} < \infty \quad \textit{and} \quad \int_{\Delta^n} -U \wedge (dd^c v)^k \wedge \beta^{p-k+1} < \infty.$$

Proof. Let $\omega \subset \Omega$ be a neighborhood of Δ^n. Without loss of generalities we may assume that $v \equiv ||z||^2$ on $\omega \setminus \Delta^n$. Let g be a function in $\mathscr{D}(\omega)$ such that $0 \leq g \leq 1$ and $g = 1$ on Δ^n. By Stokes theorem we have

$$\begin{aligned} \int_{\Delta^n} T \wedge (dd^c v)^k \wedge \beta^{p-k} &= \int_{\partial\Delta^n} T \wedge d^c v \wedge (dd^c v)^{k-1} \wedge \beta^{p-k} \\ &= \int_{\Delta^n} T \wedge \beta^p < \infty. \end{aligned}$$

For the second integral we have

$$I = \int_\omega U \wedge (dd^c v)^k \wedge dd^c(g||z||^2)^{p-k+1} = \int_{\Delta^n} + \int_{\omega \setminus \Delta^n} \tag{3. 17}$$

Since, by Stokes theorem, we have

$$I = \int_\omega g||z||^2 T \wedge (dd^c v)^k \wedge dd^c(g||z||^2)^{p-k}$$

we observe that the first term of the second hand right of (3. 17) is bounded. Furthermore since g and v are smooth on $\omega \setminus \Delta^n$ then I is bounded. Hence

$$\int_{\Delta^n} U \wedge (dd^c v)^k \wedge \beta^{p-k+1} = \int_{\Delta^n} U \wedge (dd^c v)^k \wedge (dd^c g||z||^2)^{p-k+1} > -\infty.$$

$\square$

For $I \cup J \subset \{k+1, \ldots, n\}$, let $U = \sum_{|I|=|J|=n-p-1} U_{I,J} dz_I \wedge d\bar{z}_J$ be the canonical decomposition of the potential U. Following Lemma 3.1, if $v \equiv \varphi$ then we have $\int_{\Delta^k \times \Delta^{n-k}} -U \wedge (dd^c\varphi)^k \wedge \beta''^{p-k+1} < \infty$. Hence we get

$$\int_{\Delta^k \times \Delta^{n-k}} \sum_I -U_{II}(z) d\mu_\varphi(z') \otimes d\lambda_{n-k}(z'') < \infty. \tag{3. 18}$$

From (3. 18) we deduce that $z \mapsto \sum_I -U_{II}(z)$ is a locally integrable function with respect to the measure $\mu_\varphi \otimes \lambda_{n-k}$. According to [2], for all I, J we have

$$|U_{IJ}| \leq c \sum_I -U_{II}$$

where $c > 0$ is a fixed constant. Hence each $z \mapsto U_{IJ}(z)$ is a $\mu_\varphi \otimes \lambda_{n-k}$-integrable function and we have

$$\int_{\Delta^k} d\mu_\varphi(z') \int_{\Delta^{n-k}} |U_{IJ}(z', z'')| d\lambda_{n-k}(z'') < \infty, \tag{3. 19}$$

(3. 19) implies that for μ_φ-almost every $a \in \Delta^k$ the function $z'' \mapsto U_{IJ}(a, z'')$ is locally λ_{n-k}-integrable on Δ^{n-k}. Then for any point $a \in \Delta^k$, we set

$$j_a^*(U) = \sum_{|I|=|J|=n-p-1,\ I\cup J \subset \{k+1,\ldots,n\}} U_{I,J}(a, z'') dz_I \wedge d\bar{z}_J;$$

and we denote by E_φ the set of points $a \in S_\varphi$ such that $j_a^*(U)$ does not have locally, a finite mass in Δ^{n-k}. It is clear that, $a \in E_\varphi$, means that at least one coefficient $U_{I,J}(a, z'')$ of the current $j_a^*(U)$ is not locally λ_{n-k}-integrable on Δ^{n-k}. In addition, we can easily see that

$$E_\varphi = \{a \in S_\varphi;\ z'' \mapsto \sum_I U_{II}(a, z'') \notin L^1_{loc}(\Delta^{n-k}, \lambda_{n-k}).$$

REMARK 3. *In the particular case $\varphi(z) = |z'|^2$, it is well known, by* [6], *that*

$$E_\varphi = E = \{a \in \Delta^k;\ z'' \mapsto \sum_{|I|=n-p-1} U_{I,I}(a, z'') \notin L^1_{loc}(\Delta^{n-k}, \lambda_{n-k})\}.$$

It was proved by [6] *that E is a pluripolar subset of Δ^k.*

Now we suppose that φ is smooth of class $\mathscr{C}^2$ and strictly psh, the measure μ_φ is then considered as the Lebesgue measure with density given by the continuous function $z' \mapsto \mu_\varphi(z')$ defined in local coordinates by formula (1. 7).

For any smooth regularization kernel $(\chi_j)_j$ depending only on $|z|^2$, we let

$$U_j(z) = U * \chi_j(z) := \int_{\mathbb{C}^n} \eta(x).(N * \chi_j)(z - x)T(x) \wedge \beta^{n-1}(z - x) \qquad (3.\ 20)$$

where U is defined as in (3. 16).

PROPOSITION 3.1. *Let $\varphi = \varphi(z') \in \mathscr{C}^2 \cap Psh(\Delta^n)$ such that φ is strictly psh, let $a \in \Delta^k$, and $U_j = U * \chi_j$. If $a \notin E_\varphi$, then we have $\lim_{j\to\infty} j_a^*(U_j) = j_a^*(U)$ weakly.*

Proof. By Proposition 2.2 and Lemma 3.1, since the coefficients of the potential U_j are continuous, then the slice $\langle U_j, \pi, a\rangle$ is well defined except for points a lying in a μ_φ-negligible subset of Δ^k, furthermore $\langle U_j, \pi, a\rangle_\varphi = j_a^*(U_j)$. As $a \notin E_\varphi$, then every coefficient $z'' \mapsto U_{IJ}(a, z'')$ of the current $j_a^*(U)$ is a locally integrable function on Δ^{n-k}. Consider a strongly positive test form $g \in \mathscr{D}_{(n-p+k-1,n-p+k-1)}(\mathbb{C}^{n-k})$ and define $I_j = \int_{\mathbb{C}^{n-k}} j_a^*(U_j)(z'') \wedge g(z'')$. It is sufficient to prove that

$$\lim_{j\to+\infty} I_j = \int_{\mathbb{C}^{n-k}} j_a^* U(z'') \wedge g(z'').$$

Using (3. 20), we have

$$j_a^*(U_j)(z'') = \int_{x\in\mathbb{C}^n} \eta(x).(N * \chi_j)((a, z'') - x)T(x) \wedge \beta^{n-1}((a, z'') - x),$$

then we get

$$I_j = \int_{(x,z'')} \eta(x).(N * \chi_j)((a, z'') - x)T(x) \wedge \beta^{n-1}((a, z'') - x) \wedge g(z'').$$

Since $\eta(x)T(x) \wedge \beta^{n-1}((a, z'') - x) \wedge g(z'')$ is a positive measure compactly supported in $\mathbb{C}^n \times \mathbb{C}^{n-k}$, and since

$$\lim_{j\to+\infty} (N * \chi_j)((a, z'') - x) = N((a, z'') - x)$$

pointwise, then by Egorof's theorem, for any $\varepsilon > 0$, there exists a set $A \subset \mathbb{C}^n \times \mathbb{C}^{n-k}$ such that

$$[\eta(x)T(x) \wedge \beta^{n-1}((a, z'') - x) \wedge g(z'')](A) \le \varepsilon$$

and

$$\lim_{j\to+\infty} (N * \chi_j)((a, z'') - x) = N((a, z'') - x)$$

uniformly on $\complement A$. Hence, for $\varepsilon > 0$ and for $j \in \mathbb{N}$ big enough, there is a constant $C > 0$ such that

$$|I_j - \int_{\mathbb{C}^{n-k}} j_a^* U \wedge g| \le C \sup_{(x,z'')\in \complement A} |N * \chi_j((a,z'') - x) - N((a,z'') - x)| \le \varepsilon.$$

$\square$

THEOREM 3.1. *Let $\varphi = \varphi(z') \in \mathscr{C}^2 \cap Psh(\Delta^n)$ such that φ is strictly psh and let $U_j = U * \chi_j$. Then for all $a \in \Delta^k$, $\lim_{j\to+\infty} \langle U_j, \pi, a\rangle_\varphi$ exists in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$ if, and only if $a \notin E_\varphi$, and in this case we have $\lim_{j\to+\infty} \langle U_j, \pi, a\rangle_\varphi = j_a^*(U)$ weakly.*

Proof. Let $a \in S_\varphi$. By Proposition 2.2, we have $\langle U_j, \pi, a\rangle_\varphi = j_a^*(U_j)$. For any positive test function h in $\mathscr{D}(\Delta^{n-k})$, we put

$$\begin{aligned} I(\varepsilon, j) &:= \tfrac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} U_j \wedge (dd^c\varphi)^k \wedge h\beta''^{p-k+1} \\ &= \tfrac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)} d\mu_\varphi(z') \int_{\mathbb{C}^{n-k}} u_j(z',z'')h(z'')d\lambda_{n-k}(z'') \\ &= \tfrac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)} w_j(z')d\mu_\varphi(z') \end{aligned}$$

where the function w_j is defined on Δ^k by

$$w_j(z') := \int_{\mathbb{C}^{n-k}} u_j(z',z'')h(z'')d\lambda_{n-k}(z'') \qquad (3.\ 21)$$

and the function u_j is defined on Δ^n by

$$u_j(z) = \sum_{|I|=n-p-1} U_{II} * \chi_j(z) \qquad (3.\ 22)$$

We know by [5] that $(u_j)_j$ given by (3. 22) is a sequence of negative subharmonic functions which decreases to the subharmonic function u defined by $u(z) = \sum_I U_{II}(z)$. Since h is positive, then the sequence $(w_j)_j$ given by (3. 21) decreases pointwise to the function w defined by

$$w(z') := \int_{\mathbb{C}^{n-k}} u(z',z'')h(z'')d\lambda_{n-k}(z''). \qquad (3.\ 23)$$

Since, by Lemma 3.4.1, for $\mu_\varphi-$almost every $a \in \Delta^k$, the function w defined by (3. 23) satisfies

$$\begin{aligned} \int_{z'\in B_k(a,\varepsilon)} |w(z')|d\mu_\varphi &\le \int_{z'\in B_k(a,\varepsilon)} d\mu_\varphi \int_{z''\in\mathbb{C}^{n-k}} \sum_{|I|=|J|} |U_{IJ}(z',z'')|h(z'')d\lambda_{n-k} \\ &< \infty \end{aligned}$$

which implies that w is μ_φ-integrable on $B_k(a,\varepsilon)$. Recall that, since φ is smooth and stricly psh, then the measure μ_φ is the Lebesgue measure on $\mathbb{C}^k$ with density the continuous function $z' \mapsto \mu_\varphi(z')$ given, in local coordinates, by formula (1. 7). Then, for λ_k-almost every $a \in \Delta^k$, the function $w\mu_\varphi$ is λ_k-integrable on $B_k(a,\varepsilon)$.

In addition for small $\varepsilon > 0$, we have

$$\begin{aligned} I(\varepsilon, j) &\sim \tfrac{\varepsilon^{2k}}{\omega_{2k}\mu_\varphi(a)} \int_{B_k(a,\varepsilon)} w_j(z')\mu_\varphi(z')d\lambda_k(z') \\ &= \tfrac{1}{\omega_{2k}\mu_\varphi(a)} \int_{B_k(0,1)} w_j(a+\varepsilon t')\mu_\varphi(a+\varepsilon t')d\lambda_k(t'). \end{aligned}$$

If $a \notin E_\varphi$ then for all $j \in \mathbb{N}$ we have

$$\begin{array}{rcl} \lim_{\varepsilon\to 0} I(\varepsilon, j) & = & w_j(a) \\ & \geq & w(a) \\ & > & -\infty. \end{array}$$

Hence, by the Lebesgue's dominated convergence theorem and Proposition 3.1, if $a \notin E_\varphi$, then $\lim_{j\to+\infty}\langle U_j, \pi, a\rangle_\varphi$ exists in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$ and is equal to $j_a^*(U)$. Conversely, if for any positive test function $g \in \mathscr{D}(\Delta^{n-k})$, the limit, as $j \to +\infty$, of $\langle U_j, \pi, a\rangle_\varphi(g\beta''^{p-k+1})$ exists, and is given by

$$\lim_{j\to+\infty}\langle U_j, \pi, a\rangle_\varphi(g\beta''^{p-k+1}) = \int_{\Delta^{n-k}} j_a^*(U) \wedge g(z'')\beta''^{p-k+1} \tag{3. 24}$$

hence, we have

$$\lim_{j\to+\infty}\int_{z''\in\Delta^{n-k}} j_a^*(U_j) \wedge g(z'')\beta''^{p-k+1} = \int_{z''\in\Delta^{n-k}} j_a^*(U) \wedge g(z'')\beta''^{p-k+1}.$$

By taking an increasing sequence (g_s) of smooth functions compactly supported in Δ^{n-k} such that $\lim_{s\to 0} g_s = 1\!\!1_{\Delta^{n-k}}$, the equality (3. 24) implies that $j_a^*(U)$ has a locally finite mass on Δ^{n-k} and this means that $a \notin E_\varphi$. □

REMARK 4. *Since $E_\varphi = S_\varphi \cap E$ where $E = E_{z'\mapsto|z'|^2}$ is the exceptional subset introduced in* [6] *and since it was proved in* [6] *that E is a pluripolar subset of Δ^k, then it is clear that E_φ is also a pluripolar subset of Δ^k.*

4. GENERALIZED φ-SLICING FORMULA

Now, we give the proof of the formula (4. 25) which is an amelioration of the slicing formula of H. Ben Messaoud and H. El Mir [6], the origin of the slicing formula is due to Federer [3]. Using results of [6], we prove that formula (4. 25) holds with respect to any smooth and strictly plurisubharmonic function. Here is a question that remains open: does formula (4. 25) hold for a given locally bounded plurisubharmonic function on Δ^k?

THEOREM 4.1. *Let $\varphi = \varphi(z') \in \mathscr{C}^2 \cap Psh(\Delta^n)$ such that φ is strictly psh and let $a \in \Delta^k$. Then the following statements hold*

(1) *$\langle U, \pi, a\rangle_\varphi$ exists in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$ if and only if $a \notin E_\varphi$ and in this case we have $\langle U, \pi, a\rangle_\varphi = j_a^*(U)$.*

(2) *For any $\Psi \in \mathscr{D}_{(p-k+1,p-k+1)}(\Delta^n)$ and for any $v_1, \dots, v_k \in L^\infty_{loc}Psh(\Delta^k)$, we have*

$$\int_{\Delta^n} U \wedge dd^c\tilde{v}_1 \wedge \cdots \wedge dd^c\tilde{v}_k \wedge \Psi = \int_{a\in S_\varphi}\langle U, \pi, a\rangle_\varphi(\Psi) dd^c v_1 \wedge \cdots \wedge dd^c v_k \tag{4. 25}$$

where $\tilde{v}_j = v_j \circ \pi, \quad j = 1, \dots, k.$

Proof. Let us prove the first assertion. Suppose that the slice $\langle U, \pi, a\rangle_\varphi$ is well defined in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$, which is, by [6], equivalent to the existence of the following weak limit in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$:

$$\lim_{j\to\infty}\langle U_j, \pi, a\rangle_\varphi = j_a^*(U),$$

hence we have $a \notin E_\varphi$ (by Theorem 3.1). Conversely, let $a \in S_\varphi$ such that $a \notin E_\varphi$. It is sufficient to prove that $\langle U, \pi, a\rangle_\varphi$ is well defined in $\mathscr{D}'_{(p-k+1,p-k+1)}(\Delta^n)$. Take $h \in \mathscr{D}(\Delta^{n-k})$ a test function, then for small $\varepsilon > 0$, we have

$$\begin{aligned}\mu_\varphi(B_k(a,\varepsilon) &= \int_{B_k(a,\varepsilon)} \mu_\varphi(z')d\lambda_k(z') \\ &\sim \omega_{2k}\mu_\varphi(a)\varepsilon^{2k}.\end{aligned}$$

Put

$$I_\varphi(a,\varepsilon,h) = \frac{1}{\mu_\varphi(B_k(a,\varepsilon))}\int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} U\wedge(dd^c\varphi)^k\wedge h(z'')\beta''^{n-k}$$

using [6], we get

$$\begin{aligned}I_\varphi(a,\varepsilon,h) &\sim_{\varepsilon\to 0} \frac{1}{\omega_{2k}\mu_\varphi(a)\varepsilon^{2k}}\int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} U\wedge\beta'^k\wedge\mu_\varphi(z')h(z'')\beta''^{n-k} \\ &= \frac{1}{\mu_\varphi(a)}\langle U,\pi,a\rangle\left(\mu_\varphi(a)h\beta''^{n-k}\right) \\ &= \langle U,\pi,a\rangle\left(h\beta''^{n-k}\right) \\ &= \langle U,\pi,a\rangle_\varphi\left(h\beta''^{n-k}\right).\end{aligned}$$

This implies that $a\notin E_\varphi$. To prove the second statement, we observe that, following Proposition 5.1, the formula (4. 25) holds with U_ε. The general case may be deduced by letting $\varepsilon\to 0$. □

We get the following result which can be deduced from Theorem 4.1

THEOREM 4.2. *Let $\varphi=\varphi(z')\in\mathscr{C}^2\cap Psh(\Delta^n)$ such that φ is strictly psh and let $T_j = T*\chi_j$. Then we have*

(1) *For any $a\in\Delta^k\setminus E_\varphi$, the slice $\langle U,\pi,a\rangle_\varphi$ is well defined and is equal to $j_a^*(U)$.*
(2) *For any $a\in\Delta^k\setminus E_\varphi$, the slice $\langle T,\pi,a\rangle_\varphi$ is well defined. Furthermore, in the weak sense of currents we have $\lim_{j\to+\infty}\langle T_j,\pi,a\rangle_\varphi = \langle T,\pi,a\rangle_\varphi$.*
(3) *For any $\Psi\in\mathscr{D}_{(p-k,p-k)}(\Delta^n)$ and for all $v_1,\dots,v_k\in L^\infty_{loc}\cap Psh(\Delta^k)$, we have the following slicing formula for positive closed currents*

$$\int_{\Delta^n} T\wedge dd^c\tilde{v_1}\wedge\cdots\wedge dd^c\tilde{v_k}\wedge\Psi = \int_{a\in S_\varphi}\langle T,\pi,a\rangle_\varphi(\Psi)dd^cv_1\wedge\cdots\wedge dd^cv_k \quad (4.\ 26)$$

where $\tilde{v_j}=v_j\circ\pi,\quad j=1,\dots,k$.

Proof. The first statement is a result of Theorem 4.1. The second statement is a consequence of the first statement and Proposition 2.2 since we have, in the weak sense of currents, $dd^cU = T+R$, where R is a smooth form. The third statement holds since formula (4. 26) is a consequence of formula (4. 25). □

5. φ-SLICING OF A THE CURRENT OF INTEGRATION OVER AN ANALYTIC SET

In this section we want to exprime explicitly the slice $\langle [X],\pi,a\rangle_\varphi$ where $\varphi(z')$ is a smooth and strictly plurisubharmonic function on Δ^n and $[X]$ is the current of integration over an analytic subset X of Δ^n.

In order to do this, we need the following well known proposition (for more details about the proof we can see [6]):

PROPOSITION 5.1. *Let X be an analytic subset of Δ^n and m be its complex dimension. Then the following statements hold:*

- *if $m<k$ then $\pi(X)$ is contained in a countable union of analytic subsets of Δ^k of dimensions $\leq m$.*
- *If $m\geq k$ then the set $Z=\{a\in\Delta^k\ /\ dim_{\mathbb{C}}(X\cap\pi^{-1}(X))\geq m-k+1\}$ is contained in a countable union of analytic subsets of Δ^k of dimension $\leq k-1$.*

PROPOSITION 5.2. *Let X be an analytic subset of Δ^n of complex dimension $p \geq k$ and let $\varphi(z')$ be a smooth and strictly plurisubharmonic function on Δ^n. Then, there exists a subset Z contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$ such that for all $a \in \Delta^k,\ X \cap \pi^{-1}(a)$ is an analytic set of dimension $p-k$ (otherwise is empty) and $\langle [X], \pi, a\rangle_\varphi = [X \cap \pi^{-1}(a)]$.*

Proof. Let X_{reg} be the set of regular points of X. We may assume that $0 \in X_{reg}$. Put $Z_1 = \{a \in \Delta^k \ / \ dim_{\mathbb{C}}(X \cap \pi^{-1}(X)) > p-k\}$; by Proposition 5.1, Z_1 is contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$. As the dimension m of the set X_{sing} of singular points, satisfies $m \leq p-1$, then, by Proposition 5.1, there exists a set Z_2 contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$, such that for all $a \in \Delta^k \smallsetminus Z_2$, $X_{sing} \cap \pi^{-1}(a)$ is an analytic subset of Δ^k of dimension $m-k$ (otherwise is empty). Put $Z = Z_1 \cup Z_2$ and denote $\bar{\pi} := \pi_{|X_{reg}}$. Let $\Psi \in \mathscr{D}_{(p-k,p-k)}(\Delta^n)$ of the form $f(z)\beta^{p-k}$, take $a \in \Delta^k \smallsetminus Z$ and set

$$\Gamma_\varepsilon := \frac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} [X] \wedge (dd^c\varphi)^k \wedge \Psi. \tag{5. 27}$$

Since φ is smooth, then by the definition of the current of integration over X, the equality (5. 27) can be written as

$$\Gamma_\varepsilon = \frac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{X_{reg}\smallsetminus(\bar{\pi}^{-1}(Z))\cap(B_k(a,\varepsilon)\times\mathbb{C}^{n-k})} \bar{\pi}^*[(dd^c\varphi)^k \wedge \Psi]. \tag{5. 28}$$

For small $\varepsilon > 0$, we can find local coordinates $(z_1, \ldots, z_k, w_1, \ldots, w_{n-k})$ such that

$$X_{reg} \smallsetminus (\bar{\pi}^{-1}(Z)) \cap (B_k(a,\varepsilon) \times \mathbb{C}^{n-k}) = B_k(a,\varepsilon) \times \mathbb{C}^{p-k} \times \{0\}_{\mathbb{C}^{n-p}}.$$

Since $\mu_\varphi(B_k(a,\varepsilon)) \sim \omega_{2k}\varepsilon^{2k}\mu_\varphi(a)$ as $\varepsilon \to 0$, then by an application of Fubini's theorem and by the change of variable $z' \leftrightarrow \frac{z'-a}{\varepsilon}$, when ε is small enough $(\varepsilon \to 0)$, the equality (5. 28) can be transformed to the following

$$\Gamma_\varepsilon = \frac{1}{\omega_{2k}\mu_\varphi(a)} \int_{\mathbb{C}^{p-k}} \int_{B_k(0,1)} \mu_\varphi(a+\varepsilon t) f(a+\varepsilon t, w) d\lambda_k(t) d\lambda_{p-k}(w). \tag{5. 29}$$

By letting $\varepsilon \to 0$ in (5. 29), we get

$$\begin{aligned} \lim_{\varepsilon\to 0}\Gamma_\varepsilon &= \int_{\mathbb{C}^{p-k}} f(a,w)d\lambda_{p-k}(w) \\ &= \int_{X_{reg}\cap\pi^{-1}(a)} j_a^*(\Psi) \\ &= \langle [X \cap \pi^{-1}(a)], j_a^*(\Psi)\rangle \end{aligned}$$

the last equality holds since the set $X_{sing} \cap \pi^{-1}(a)$ has dimension $< p-k$, this achieves the proof of the Proposition. □

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