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A Subclass of Univalent Functions with Fixed Second Coefficient

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Abstract. In this paper, we define a new subclass of univalent functions with negative coefficients. Further, we obtain the coefficient estimates, distortion bounds and extreme points by fixing the second coefficient. Also, the class we extend the study by fixing finitely many coefficients.

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1. INTRODUCTION

Let $\mathcal{S}$ be the class of analytic, univalent and normalized ($f(0) = 0 = f'(0) - 1$) functions in $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ of the form :

$$f(z) = z + \sum_{n \geqq 2} a_n z^n.$$

Let

$$\mathcal{T} := \left\{ f : f(z) = z - \sum_{n \geqq 2} |a_n| z^n, \quad z \in \mathbb{U} \right\} \tag{1. 1}$$

be the subclass of $\mathcal{S}$ (see [13]).

For $0 \leqq \mu < 1$, $0 \leqq \eta \leqq 1$ and $f \in \mathcal{T}$ the Rafid operator [4] is defined by

$$\begin{aligned} \mathcal{R}_\mu^\eta(f(z)) &= \frac{1}{(1-\mu)^{1+\eta}\Gamma(\eta+1)} \int_0^\eta t^{\eta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)\,dt \\ &= z - \sum_{n \geqq 2} \frac{(1-\mu)^{n-1}\Gamma(\eta+n)}{\Gamma(\eta+1)} |a_n| z^n. \end{aligned}$$

Definition 1. A function f defined in (1. 1) is said to be in $\mathcal{W}_\mu^\eta$ if

$$\left|\frac{\frac{zF_\lambda^{'}(z)}{F_\lambda(z)}-1}{(B-A)\beta\left[\frac{zF_\lambda^{'}(z)}{F_\lambda(z)}-\nu\right]-B\left[\frac{zF_\lambda^{'}(z)}{F_\lambda(z)}-1\right]}\right|<\kappa,$$
$$(0\leqq\nu\leqq 1,\ 0<\kappa\leqq 1,\ -1\leqq B<A\leqq 1,\ 0\leqq\beta\leqq 1,\ z\in\mathbb{U}),$$

where

$$\frac{zF_\lambda^{'}(z)}{F_\lambda(z)}=\frac{z\left(\mathcal{R}_\mu^\eta f(z)\right)^{'}+\lambda z^2\left(\mathcal{R}_\mu^\eta f(z)\right)^{''}}{(1-\lambda)\mathcal{R}_\mu^\eta f(z)+\lambda z\left(\mathcal{R}_\mu^\eta f(z)\right)^{'}},\qquad 0\leqq\lambda\leqq 1.$$

The class $\mathcal{W}_\mu^\eta$ was considered by Vijaya et al. [14] and for $f\in\mathcal{W}_\mu^\eta$ they endowed the following necessary and sufficient conditions.

Theorem 2. [14] *Let f be given in (1. 1). Then $f\in\mathcal{W}_\mu^\eta$ iff*

$$\sum_{n\geqq 2}\mathcal{R}_n|a_n|\leqq(1-\nu)(B-A)\kappa\beta, \tag{1. 2}$$

where

$$\mathcal{R}_n=(1+n\lambda-\lambda)[(n-1)(1-\kappa B)+\kappa\beta(B-A)(n-\nu)]\frac{(1-\mu)^{n-1}\Gamma(\eta+n)}{\Gamma(\eta+1)}. \tag{1. 3}$$

We derive $\mathcal{R}_2$ and $\mathcal{R}_3$ from (1. 3) as given below:

$$\mathcal{R}_2=(1+\lambda)[(1-\kappa B)+\kappa\beta(B-A)(2-\nu)](1-\mu)(\eta+1)$$

and

$$\mathcal{R}_3=(1+2\lambda)[2(1-\kappa B)+\kappa\beta(B-A)(3-\nu)](1-\mu)^2(\eta+1)(\eta+2).$$

Further, we consider the values of $\mathcal{R}_n$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are aforementioned right through one or otherwise specified.

Corollary 3. *Let f be given in (1. 1) and $f\in\mathcal{W}_\mu^\eta$. Then*

$$|a_n|\leqq\frac{(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n},\ \ n\geqq 2.$$

Taking into consideration of Theorem 2, for f as given in (1. 1) and $f\in\mathcal{W}_\mu^\eta$ then

$$|a_2|=\frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2},\ 0\leqq d\leqq 1.$$

We observe that it is a task to fixing the second coefficient in Taylor series and discussing the distortion theorems, growth theorems and similar other properteis (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12]) and the references therein. Attracted by aforecited works a new subclass $\mathcal{W}_\mu^\eta(d)$ of $\mathcal{W}_\mu^\eta$ is considered as given below:

$$\mathcal{W}_\mu^\eta(d):=\left\{f\in\mathcal{W}_\mu^\eta : f(z)=z-\frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2}z^2-\sum_{n\geqq 3}|a_n|z^n\right\}. \tag{1. 4}$$

2. Characterization Properties

Theorem 4. *Let f be given in (1. 4). Then $f \in \mathcal{W}_{\mu}^{\eta}(d)$ iff*

$$\sum_{n \geqq 3} \mathcal{R}_n \, |a_n| \leqq (1-d)(1-\nu)(B-A)\kappa\beta \, .$$

Proof. Taking $|a_2| = \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2}$ in (1. 2) we have the required one. □

Corollary 5. *Let f be given in (1. 4) and in $\mathcal{W}_{\mu}^{\eta}(d)$. Then*

$$|a_n| \leqq \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n}, \;\; n \geqq 3. \tag{2. 5}$$

Theorem 6. *The class $\mathcal{W}_{\mu}^{\eta}(d)$ is closed under convex linear combination.*

Proof. Let f as in (1. 4) and

$$g(z) = z - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} z^2 - \sum_{n \geqq 3} |b_n| z^n, \text{ and } 0 \leqq d \leqq 1$$

be in $\mathcal{W}_{\mu}^{\eta}(d)$. It is enough to show that

$$\Psi(z) = \delta f(z) + (1-\delta) g(z), \;\; 0 \leqq \delta \leqq 1$$

is also in $\mathcal{W}_{\mu}^{\eta}(d)$. In view of the fact that

$$\Psi(z) = z - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} z^2 - \sum_{n \geqq 3} (\delta |a_n| + (1-\delta)|b_n|) z^n, \;\; 0 \leqq d \leqq 1.$$

We notice that

$$\sum_{n \geqq 3} \mathcal{R}_n (\delta |a_n| + (1-\delta)|b_n|) \leqq (1-d)(1-\nu)(B-A)\kappa\beta.$$

It is evident from Theorem 4 that $\Psi \in \mathcal{W}_{\mu}^{\eta}(d)$. □

Theorem 7. *Let*

$$f_j(z) = z - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} z^2 - \sum_{n \geqq 3} |a_{n,\,j}| z^n, \;\; j = 1, \dots k.$$

be the functions in $\mathcal{W}_{\mu}^{\eta}(d)$. Then Φ is defined by

$$\Phi(z) = \sum_{j=1}^{k} \lambda_j f_j(z),$$

is also in $\mathcal{W}_{\mu}^{\eta}(d)$, where $\sum\limits_{j=1}^{k} \lambda_j = 1$.

Proof. From the hypothesis of theorem we get

$$\Phi(z) = z - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} z^2 - \sum_{n \geqq 3} \left(\sum_{j=1}^{k} \lambda_j |a_{n,j}| \right) z^n.$$

In view of fact that $f_j \in \mathcal{W}_{\mu}^{\eta}(d)$ $j = 1, \dots, k$, Theorem 4 gives

$$\sum_{n \geqq 3} \mathcal{R}_n \, |a_{n,j}| \leqq (1-d)(1-\nu)(B-A)\kappa\beta, \text{ for } j = 1, \dots, k.$$

which implies

$$\sum_{n\geqq 3}\mathcal{R}_n\left(\sum_{j=1}^{k}\lambda_j|a_{n,j}|\right)=\sum_{j=1}^{k}\lambda_j\left(\sum_{n\geqq 3}\mathcal{R}_n|a_{n,j}|\right)\leqq(1-d)(1-\nu)(B-A)\kappa\beta.$$

It is evident from Theorem 4 that $\Phi\in\mathcal{W}_{\mu}^{\eta}(d)$. □

Theorem 8. *Let*

$$f_2(z)=z-\frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2}z^2 \qquad (2.\ 6)$$

and

$$f_n(z)=z-\frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2}z^2-\frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n}z^n,\ n\geqq 3. \qquad (2.\ 7)$$

Then $f\in\mathcal{W}_{\mu}^{\eta}(d)$ *iff*

$$f(z)=\sum_{n\geqq 2}\sigma_n f_n(z),\ \sigma_n\geqq 0 \text{ and } \sum_{n\geqq 2}\sigma_n=1. \qquad (2.\ 8)$$

Proof. If we state f as of the form (2. 8), then we get

$$\begin{aligned} f(z) &= z-\frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2}z^2-\sum_{n\geqq 3}\sigma_n\frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n}z^n \\ &= z-\sum_{n\geqq 2}A_n z^n, \end{aligned}$$

where

$$A_2=\frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2}$$

and

$$A_n=\frac{\sigma_n(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n},\quad n\geqq 3.$$

Thus

$$\begin{aligned} \sum_{n\geqq 2}\mathcal{R}_n A_n &= d(1-\nu)(B-A)\kappa\beta+\sum_{n\geqq 3}\sigma_n(1-d)(1-\nu)(B-A)\kappa\beta \\ &= (1-\nu)[d+(1-\sigma_2)(1-d)](B-A)\kappa\beta \\ &\leqq (1-\nu)(B-A)\kappa\beta. \end{aligned}$$

From Theorem 2 and Theorem 4, we observe that $f\in\mathcal{W}_{\mu}^{\eta}(d)$. Conversely, let us consider f of the form (1. 4) in $\mathcal{W}_{\mu}^{\eta}(d)$. Applying (2. 5), we have

$$|a_n|\leqq\frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n},\ \ n\geqq 3.$$

Taking

$$\sigma_n=\frac{\mathcal{R}_n}{(1-d)(1-\nu)(B-A)\kappa\beta}|a_n|,\quad n\geqq 3$$

and

$$\sigma_2=1-\sum_{n\geqq 3}\sigma_n,$$

we reach (2. 8). □

Lemma 9. *Let*

$$f_3(z) = z - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} z^2 - \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} z^3 \,. \tag{2. 9}$$

Then,

$$|f_3(re^{i\vartheta})| \geqq r - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r^2 - \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} r^3, \tag{2. 10}$$
$$(0 \leqq r < 1 \text{ and } 0 \leqq d \leqq 1,)$$

equivalent when $\vartheta = 0$.

$$|f_3(re^{i\vartheta})| \leqq r + \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r^2 - \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} r^3, \tag{2. 11}$$
$$(\text{either } 0 \leqq d < d_0 \text{ and } 0 \leqq r \leqq r_0 \text{ or } d_0 \leqq d \leqq 1,)$$

equivalent when $\vartheta = \pi$. *The* d_0 *and* r_0 *are given below:*

$$\begin{aligned} d_0 = & \frac{1}{2(1-\nu)(B-A)\kappa\beta} \times \\ & \{(1-\nu)(B-A)\kappa\beta - 4\mathcal{R}_2 - \mathcal{R}_3 \\ & +[((1-\nu)(B-A)\kappa\beta - 4\mathcal{R}_2 - \mathcal{R}_3)^2 + 16\mathcal{R}_2(1-\nu)(B-A)\kappa\beta]^{1/2}\} \end{aligned}$$

and

$$r_0 = \frac{-4(1-d)\mathcal{R}_2 + [16(1-d)^2\mathcal{R}_2^2 + 4d^2(1-d)(1-\nu)(B-A)\kappa\beta\mathcal{R}_3]^{1/2}}{2d(1-d)(1-\nu)(B-A)\kappa\beta}.$$

Proof. In view of fact that

$$\begin{aligned} \frac{\partial|f_3(re^{i\vartheta})|^2}{\partial\vartheta} = & \; 2(1-\nu)(B-A)\kappa\beta r^3 \sin\vartheta \\ & \left(\frac{d}{\mathcal{R}_2} + \frac{4(1-d)\cos\vartheta}{\mathcal{R}_3} r - \frac{d(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2\mathcal{R}_3} r^2\right), \end{aligned} \tag{2. 12}$$

we observe

$$\frac{\partial|f_3(re^{i\vartheta})|^2}{\partial\vartheta} = 0, \;\; \vartheta_1 = 0, \text{ and } \vartheta_2 = \pi$$

and

$$\vartheta_3 = \cos^{-1}\left(\frac{d[(1-d)(1-\nu)(B-A)\kappa\beta r^2 - \mathcal{R}_3]}{4r(1-d)\mathcal{R}_2}\right),$$

In fact ϑ_3 is suitable only when $-1 \leqq \cos\vartheta_3 \leqq 1$. Thus, third one will appear if and only if $r_0 \leqq r < 1$ and $0 \leqq d \leqq d_0$. Therefore matching up to the maximum and minimum $|f_3(re^{i\vartheta_m})|$, $m = 1, 2, 3$ on the correct periods, we get preferred results. □

Lemma 10. *Let* $f_n(z)$ *be given in (2. 7) with* $n \geqq 4$. *Then*

$$|f_n(re^{i\vartheta})| \leqq |f_4(-r)|. \tag{2. 13}$$

Proof. In view of fact that we take decreasing form $\frac{r^n}{n}$, which implies

$$f_n(z) = z - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} z^2 - \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n} z^n.$$

Thus

$$\begin{aligned} |f_n(re^{i\vartheta}| \leqq & \; r + \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r^2 + \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_4} r^4 \\ = & \; -f_4(-r) \end{aligned}$$

which proves (2. 13). □

Next, we state Theorem 11 without proof.

Theorem 11. *Let f be given in (1. 4) and in $\mathcal{W}_\mu^\eta(d)$. Then*

$$|f(re^{i\vartheta})| \geqq r - \frac{d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r^2 - \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} r^3 \;\; 0 \leqq r < 1$$

equal for $f_3(z)$ at $z = r$, and

$$|f(re^{i\vartheta})| \leqq \max\{\max_\vartheta |f_3(re^{i\vartheta})|, -f_4(-r)\},$$

where $\max_\vartheta |f_3(re^{i\vartheta})|$ is as in Lemma 9.

Lemma 12. *Let $f_3(z)$ be given in (2. 9). Then,*

$$|f_3'(re^{i\vartheta})| \geqq 1 - \frac{2d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r - \frac{3(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} r^2$$

equivalent when $\vartheta = 0$.

$$|f_3'(re^{i\vartheta})| \leqq 1 + \frac{2d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r - \frac{3(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} r^2$$
$$(\text{either } 0 \leqq d < d_1 \text{ and } 0 \leqq r \leqq r_1 \text{ or } d_1 \leqq d \leqq 1)$$

equivalent when $\vartheta = \pi$. The d_1 and r_1 are given below:

$$\begin{aligned} d_1 &= \frac{1}{6(1-\nu)(B-A)\kappa\beta}\{(3(1-\nu)(B-A)\kappa\beta - 3\mathcal{R}_2 - \mathcal{R}_3) \\ &\quad + [(3(1-\nu)(B-A)\kappa\beta - 3\mathcal{R}_2 - \mathcal{R}_3)^2 + 36(1-\nu)(B-A)\kappa\beta\mathcal{R}_2]^{1/2}\} \end{aligned}$$

and

$$\begin{aligned} r_1 &= \frac{1}{6d(1-d)(1-\nu)(B-A)\kappa\beta}\{-3(1-d)\mathcal{R}_2 \\ &\quad + [9(1-d)^2\mathcal{R}_2^2 + 12d^2(1-d)\mathcal{R}_3(1-\nu)(B-A)\kappa\beta]^{1/2}\}. \end{aligned}$$

Theorem 13. *Let f be given in (1. 4) and in $\mathcal{W}_\mu^\eta(d)$. Then*

$$|f'(re^{i\vartheta})| \geqq 1 - \frac{2d(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_2} r - \frac{3(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_3} r^2$$

equivalently for $f_3'(z)$ at $z = r$, and

$$|f'(re^{i\vartheta})| \leqq \max\{\max_\vartheta |f_3'(re^{i\vartheta})|, -f_4'(-r)\},$$

where $\max_\vartheta |f_3'(re^{i\vartheta})|$ is obtained from Lemma 12.

Next, we define a new subclass by fixing finitely many coefficients:

$$\mathcal{W}_\mu^\eta(d_n, k) := \left\{ f \in \mathcal{W}_\mu^\eta(d) : f(z) = z - \sum_{n=2}^{k} \frac{d_n(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n} z^n \right.$$
$$\left. - \sum_{n \geqq k+1} |a_n| z^n, \; 0 \leqq \sum_{n=2}^{k} d_n = d \leqq 1 \right\}.$$

For the above said class, next we state extreme points without proof. Since the proof is line similar to method used for the class $\mathcal{W}_\mu^\eta(d)$.

Theorem 14. *For $\mathcal{W}_{\mu}^{\eta}(d_n, k)$ the extreme points are*

$$f_k(z) = z - \sum_{n=2}^{k} \frac{d_n(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n} z^n$$

and

$$f_n(z) = z - \sum_{n=2}^{k} \frac{d_n(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n} z^n - \sum_{n \geqq k+1} \frac{(1-d)(1-\nu)(B-A)\kappa\beta}{\mathcal{R}_n} z^n.$$

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