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New Integral Inequalities through Generalized Convex Functions

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Abstract. In this article, we founded several inequalities for some single-real-valued function, related to the famous Hermite-Hadamard's $(H-H)$ inequality for mappings who has positive values lies in the classes $K_{m,1}^{\alpha,s}$ and $K_{m,2}^{\alpha,s}$.

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1. Introduction

With the outgrowth of calculus during the 19th century, the concern of inequalities has rapidly increased. Inequalities have gained a significant importance not only in Mathematics itself but also in Engineering and nearly all areas of Sciences. Such as, in numerical analysis, the estimation of a definite integral of a real valued function over an interval $[a, b]$ is a very interesting problem. An elemental inequality that contributes error bounds for quadrature formulae of a continuous convex single-valued mappings, named Hermit-Hadamard's $(H-H)$ inequality, is set as [11, p. 53]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

whence $f : [a, b] \to \mathbb{R}$ is convex single-valued function. Both inequalities turned back for f to be concave.
The impression of quasi-convex single-valued function infer generally the picture of convex single-valued function. To a greater extent, exactly a single-valued map $f : [a, b] \to \mathbb{R}$ is quasi-convex on $[a, b]$ if

$$f(\lambda u + (1-\lambda)v) \le \max\{f(u), f(v)\},$$

holds for any $u, v \in [a, b]$ and $\lambda \in [0, 1]$. Intelligibly, a single-valued convex function may be considered as a quasi-convex function. Moreover, quasi-convex single-valued functions might be convex exactly (see [5]). In [12], Özdemir et al. established several integral inequalities respecting some kinds of convexity. Especially, they discussed the following result connecting with quasi-convex functions:

Theorem 1. *Let a continuous map $f : [a,b] \neq \phi \subset [0,\infty) \to \mathbb{R}$ so that $f \in L^1([a,b])$. If f is quasi-convex on $[a,b]$ for $p,q > 0$, induces*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = (b-a)^{p+q+1}\beta(p+1,q+1)\max\{f(a),f(b)\},$$

where $\beta(x,y)$ is the Euler Beta function.

Recently, Liu [7] gave close to new integral inequalities for quasi-convex functions as comes:

Theorem 2. *Let a continuous map $f : [a,b] \neq \phi \subset [0,\infty) \to \mathbb{R}$ so that $f \in L^1([a,b])$. If for any $k > 1$, $|f|^{\frac{k}{k-1}}$ is quasi-convex on $[a,b]$ for $p,q > 0$, induces*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = (b-a)^{p+q+1}\left(\beta(kp+1,kq+1)\right)^{\frac{1}{k}} \times \left(\max\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\}\right)^{\frac{k-1}{k}}.$$

Theorem 3. *Let a continuous map $f : [a,b] \neq \phi \subset [0,\infty) \to \mathbb{R}$ so that $f \in L^1([a,b])$ and let $l \geq 1$. If $|f|^l$ is quasi-convex on $[a,b]$ for some fixed $p,q > 0$, induces*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = (b-a)^{p+q+1}\beta(p+1,q+1)\left(\max\{|f(a)|^l, |f(b)|^l\}\right)^{\frac{1}{l}}.$$

That is, this study is a further continuation of [8], where we generalise the results discussed in [8] by weaken the condition of convexity discussed in [10].

2. Principle Outcomes

In this segment, we generalize the above theorems and produce some more results using the following lemma described in [12].

Lemma 4. *Let $f : I = [a,b] \neq \phi \subset [0,\infty) \to \mathbb{R}$ is a continuous map on $[a,b]$ so that $f \in L^1([a,b])$, induces equality*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \tag{2.1}$$

holds for some fixed $p,q > 0$.

Here we recall the following definitions from [10] by Muddassar et al named as $s-(\alpha,m)$-convex functions as reproduced below;

Definition 5. A function $f : [0,\infty) \to [0,\infty)$ is supposed to $s-(\alpha,m)$-convex function in the first sense or $f \in K_{m,1}^{\alpha,s}$, if $\forall\ u,v \in [0,\infty) \bigwedge \beta \in [0,1]$ the coming inequality agrees:

$$f(\beta u + (1-\beta)v) \leq \left(\beta^{\alpha^s}\right) f(u) + m\left(1-\beta^{\alpha^s}\right) f\left(\frac{v}{m}\right),$$

where $(\alpha,m) \in [0,1]^2$ for $s \in (0,1]$.

Definition 6. A function $f:[0,\infty)\to[0,\infty)$ is supposed to $s-(\alpha,m)$-convex function in the second sense or $f\in K^{\alpha,s}_{m,2}$, if $\forall\ u,v\in[0,\infty)\ \bigwedge\ \beta\in[0,1]$ the coming inequality agrees:

$$f(\beta u+(1-\beta)v)\leq(\beta^{\alpha})^{s}f(u)+m\left(1-\beta^{\alpha}\right)^{s}f\left(\frac{v}{m}\right),$$

where $(\alpha,m)\in[0,1]^2$ for $s\in(0,1]$.

Note that for $s=1$, we get $K^{\alpha}_{m}(I)$ class of convex functions and for $\alpha=1$ and $m=1$, we get $K^1_s(I)$ and $K^2_s(I)$ class of convex functions.

Theorem 7. *Let $f:I=[a,b]\neq\phi\subset[0,\infty)\to\mathbb{R}$ is a continuous map on $[a,b]$ so that $f\in L^1([a,b])$. If $|f|\in K^{\alpha,s}_{m,1}$ on $[a,b]$ for $p,q>0$,*

$$\int_a^b(x-a)^p(b-x)^qf(x)dx\leq(b-a)^{p+q+1}\Bigg\{\beta(q+\alpha s+1,p+1)\left(|f(a)|-m\left|f\left(\frac{b}{m}\right)\right|\right)+m\beta(q+1,p+1)\left|f\left(\frac{b}{m}\right)\right|\Bigg\}\quad(2.2)$$

Proof. Taking absolute value of Lemma 4,

$$\int_a^b(x-a)^p(b-x)^qf(x)dx\leq(b-a)^{p+q+1}\int_0^1(1-t)^pt^q\left|f(ta+(1-t)b)\right|dt.\quad(2.3)$$

Since $|f|\in K^{\alpha,s}_{m,1}$ on $[a,b]$, then the inequality (2.3) can be written as

$$\int_0^1(1-t)^pt^q\left|f(ta+(1-t)b)\right|dt\leq\int_0^1(1-t)^pt^q\left(t^{\alpha s}|f(a)|+m(1-t^{\alpha s})|f(b)|\right)dt,\quad(2.4)$$

As,

$$\int_0^1(1-t)^pt^{q+\alpha s}dt=\beta(q+\alpha s+1,p+1)\quad(2.5)$$

and

$$\int_0^1(1-t)^pt^q(1-t^{\alpha s})dt=\beta(q+1,p+1)-\beta(q+\alpha s+1,p+1).\quad(2.6)$$

Using (2.4), (2.5) and (2.6) in (2.3), we get (2.2). □

Theorem 8. *Let $f:I=[a,b]\neq\phi\subset[0,\infty)\to\mathbb{R}$ is a continuous map on $[a,b]$ so that $f\in L^1([a,b])$ and let $k>1$. If $|f|^{\frac{k}{k-1}}\in K^{\alpha,s}_{m,1}$ on $[a,b]$ for $p,q>0$,*

$$\int_a^b(x-a)^p(b-x)^qf(x)dx\leq(b-a)^{p+q+1}\left(\beta(\alpha s+1,1)\right)^{\frac{k-1}{k}}\left(\beta(qk+1,pk+1)\right)^{\frac{1}{k}}\left[|f(a)|^{\frac{k}{k-1}}+m\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}\quad(2.7)$$

Proof. Applying the Hölder's Inequality on (2.3), implies

$$\int_0^1(1-t)^pt^q\left|f(ta+(1-t)b)\right|dt\leq\left[\int_0^1\left((1-t)^pt^q\right)^kdt\right]^{\frac{1}{k}}\times\left[\int_0^1\left|f(ta+(1-t)b)\right|^{\frac{k}{k-1}}dt\right]^{1-\frac{1}{k}}\quad(2.8)$$

here,
$$\int_0^1 (1-t)^{pk} t^{qk} dt = \beta(qk+1, pk+1). \tag{2.9}$$
Since $|f|^{\frac{k}{k-1}} \in K_{m,1}^{\alpha,s}$ on $[a,b]$ for $k>1$, therefore
$$\int_0^1 |f(ta+(1-t)b)|^{\frac{k}{k-1}}\, dt \le \int_0^1 \left(t^{\alpha s}|f(a)|^{\frac{k}{k-1}} + m(1-t^{\alpha s})|f(b)|^{\frac{k}{k-1}}\right) dt, \tag{2.10}$$
furthermore,
$$\int_0^1 t^{\alpha s} dt = \int_0^1 (1-t)^{\alpha s} dt = \beta(\alpha s+1, 1). \tag{2.11}$$
Inequalities (2.3), (2.8), (2.10) and equations (2.9),(2.11) together implies (2.7). □

Theorem 9. *Let $f : I = [a,b] \neq \phi \subset [0,\infty) \to \mathbb{R}$ is a continuous map on $[a,b]$ so that $f \in L^1([a,b])$ and let $l \ge 1$. If $|f|^l \in K_{m,1}^{\alpha,s}$ on $[a,b]$ for $p,q>0$,*
$$\int_a^b (x-a)^p(b-x)^q f(x)dx \le (b-a)^{p+q+1} \left(\beta(q+1,p+1)\right)^{\frac{l-1}{l}} \Bigg[\beta(q+\alpha s+1, p+1) \left\{|f(a)|^l - m\left|f\left(\frac{b}{m}\right)\right|^l\right\} + m\beta(q+1,p+1)\left|f\left(\frac{b}{m}\right)\right|^l\Bigg]^{\frac{1}{l}} \tag{2.12}$$

Proof. Now applying the Hölder's Inequality on (2.3), we get
$$\int_0^1 (1-t)^p t^q \left|f(ta+(1-t)b)\right| dt \le \left[\int_0^1 (1-t)^p t^q dt\right]^{1-\frac{1}{l}} \times \left[\int_0^1 (1-t)^p t^q \left|f(ta+(1-t)b)\right|^l dt\right]^{\frac{1}{l}} \tag{2.13}$$
here,
$$\int_0^1 (1-t)^p t^q dt = \beta(q+1, p+1). \tag{2.14}$$
Since $|f|^l \in K_{m,1}^{\alpha,s}$ on $[a,b]$ for $l \ge 1$, therefore
$$\int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|^l dt \le \int_0^1 (1-t)^p t^q \left(t^{\alpha s}|f(a)|^l + m(1-t^{\alpha s})|f(b)|^l\right) dt \tag{2.15}$$
which completes the proof. □

Some more integral inequalities can be found using $K_{m,2}^{\alpha,s}$ class of convex functions in similar way.

3. Conclusion

It is long-familiar that the convexity has been bringing a key role in mathematical programming, engineering, and optimisation theory. The generalisation of convexity is one of the most significant panorama in mathematical programming and optimisation theory. There have been many efforts to weaken the convexity presumption in the literature. A substantial generalisation of convex functions is that of $s-(\alpha,m)$ functions brought in by Muddassar et al in [10]. In [12], Özdemir et al talked about some integral inequalities for different kinds of convexity. In this paper we developed some more results on hermite-Hadamard's type inequalities by weaken the condition of convexity discussed in [10].

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REFERENCES

[1] S.S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math. **33**, No. 1 (2002) 55-65.

[2] S.S. Dragomir and C.E.M. Pearce: *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[3] S.S. Dragomir and C.E.M. Pearce, *Quasi-convex functions and Hadamards inequality*, Bull. Austral. Math. Soc. **57**, No. 3 (1998) 377-385.

[4] S.S. Dragomir, J. Pečarić and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21**, No. 3 (1995) 335-341.

[5] D.A. Ion, *Some estimates on the Hermite-Hadamard inequality through quasiconvex functions*, An. Univ. Craiova Ser. Mat. Inform. **34**, (2007) 83-88.

[6] U.S. Kirmaci and M.E. Ozedemir, *on some inequalities for differentiable mappings and applicatioons to special means of real numbers and to midpoint formula*, App.Math.Comp, **153**, (2004) 361-368.

[7] W.J. Liu, *New integral inequalities via (α, m)-convexity and quasi-convexity*, arXiv:1201.6226v1 [math.FA]

[8] W. Liu, *Some New Integral Inequalities via P-convexity*, arxiv:1202.0127v1 [math.FA] 1 FEB 2012, http://arxiv.org/abs/1202.0127

[9] M. Muddassar, M.I. Bhatti and M.Iqbal. *Some new s-Hermite Hamadard Type Inequalities for differentiable functions and their Applications*, Proceedings of Pakistan Academy of Science **49**, No. 1 (2012) 9-17.

[10] M. Muddassar, M.I. Bhatti and Wajeeha Irshad, *Generalization of integral inequalities of the type of Hermit-Hamadard through convexity*, Bull. Austral. Math. Soc. **88**, (2013) 320-330.

[11] C. Niculescu and L. E. Persson, *Convex functions and their applications, Springer*, Berlin Heidelberg NewYork, 2004.

[12] M.E. Özdemir, E. Set and M. Alomari, *Integral inequalities via several kinds of convexity*, Creat. Math. Inform. **20**, No. 1 (2011) 62-73.

[13] C.E.M. Pearce and J. Pečarić, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula* App. Math. Lett, **13**, No. 2 (2000) 51-55.