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Multivalent Functions with Respect to Symmetric Conjugate Points

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Abstract. Using convolution, classes of p-valent functions with respect to symmetric conjugate points are introduced. Integral representation and closure properties under convolution of general classes with respect to $(2j,\ k)$ symmetric points are investigated.

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1. Introduction, Definitions And Preliminaries

Let $\mathcal{A}_p$ be the class of functions analytic in the open unit disc $\mathcal{U} = \{z : | z |< 1\}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \geq 1)\,. \tag{1.1}$$

and let $\mathcal{A} = \mathcal{A}_1$.

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Our favorite references of the field are [4, 5] which covers most of the topics in a lucid and economical style.

For the functions $f(z)$ of the form (1.1) and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, the Hadamard product (or convloution) of f and g is defined by $(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n\, b_n\, z^n$.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let k be a positive integer and $j = 0, 1, 2, \ldots (k-1)$. A domain D is said to be (j, k)-fold symmetric if a rotation of D about the origin through an angle $2\pi j/k$ carries D onto itself. A function $f \in \mathcal{A}$ is said to be (j, k)-symmetrical if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{1.2}$$

where $\varepsilon = \exp(2\pi i/k)$. The family of (j, k)-symmetrical functions will be denoted by $\mathcal{F}_k^j$. For every function f defined on a symmetrical subset $\mathcal{U}$ of $\mathbb{C}$, there exits a unique sequence of (j, k)-symmetrical functions $f_{j,k}(z), j = 0, 1, \ldots, k-1$ such that

$$f = \sum_{j=0}^{k-1} f_{j,k}.$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{f(\varepsilon^\nu z)}{\varepsilon^{\nu p j}}, \quad (f \in \mathcal{A}_p;\ k = 1, 2, \ldots;\ j = 0, 1, 2, \ldots (k-1)). \tag{1.3}$$

This decomposition is a generalization the well known fact that each function defined on a symmetrical subset $\mathcal{U}$ of $\mathbb{C}$ can be uniquely represented as the sum of an even function and an odd functions (see Theorem 1 of [6]). We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively. Further, it is obvious that $f_{j,k}(z)$ is a linear operator from $\mathcal{U}$ into $\mathcal{U}$. The notion of (j, k)-symmetrical functions was first introduced and studied by P. Liczberski and J. Połubiński in [6].

The class of (j, k)-symmetrical functions was extended to the class (j, k)-symmetrical conjugate functions in [8]. For fixed positive integers j and k, let $f_{2j,k}(z)$ be defined by the following equality

$$f_{2j,k}(z) = \frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu p j} f(\varepsilon^\nu z) + \varepsilon^{\nu p j}\overline{f(\varepsilon^\nu \overline{z})}\right], \quad (f \in \mathcal{A}_p). \tag{1.4}$$

If ν is an integer, then the following identities follow directly from (1.4):

$$\begin{aligned} f'_{2j,k}(z) &= \frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu p j+\nu} f'(\varepsilon^\nu z) + \varepsilon^{\nu p j-\nu}\overline{f'(\varepsilon^\nu \overline{z})}\right] \\ f''_{2j,k}(z) &= \frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu p j+2\nu} f''(\varepsilon^\nu z) + \varepsilon^{\nu p j-2\nu}\overline{f''(\varepsilon^\nu \overline{z})}\right], \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &f_{2j,k}(\varepsilon^\nu z) = \varepsilon^{\nu p j} f_{2j,k}(z), \quad f_{2j,k}(z) = \overline{f_{2j,k}(\overline{z})} \\ &f'_{2j,k}(\varepsilon^\nu z) = \varepsilon^{\nu p j-\nu} f'_{2j,k}(z), \quad f'_{2j,k}(\overline{z}) = \overline{f'_{2j,k}(z)}. \end{aligned} \tag{1.6}$$

Motivated by the concept introduced by Sakaguchi in [10], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors (see [1, 2, 12, 13, 15, 16]). In this paper, using Hadamard product (or convolution) new classes of functions in $\mathcal{A}_p$ with respect to (j, k)-symmetric points are introduced. Throughout this paper, unless otherwise mentioned the function h is a convex univalent function with a positive real part satisfying $h(0) = 1$.

We define the following.

Definition 1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^{j,\,k}(h)$ if and only if it satisfies the condition

$$\frac{1}{p}\frac{zf'(z)}{f_{2j,\,k}(z)} \prec h(z), \tag{1.7}$$

where $f_{2j,\,k}(z) \neq 0$ and is defined by the equality (1.4). Similarly, we call the class $\mathcal{C}_p^{j,\,k}(h)$ of functions $f \in \mathcal{A}_p$ with $f'_{2j,\,k}(z) \neq 0$ satisfying the subordination condition

$$\frac{1}{p}\frac{(zf'(z))'}{f'_{2j,\,k}(z)} \prec h(z). \tag{1.8}$$

Remark 2. Since $f \in \mathcal{A}_p$, the condition $f_{2j,\,k}(z) \neq 0$ in the Definition 1 is essential as $h(z)$ is assumed to be a function with positive real part.

It is interesting to note that several well known and new subclasses of analytic functions can be obtained as special cases of $\mathcal{S}_p^{j,\,k}(h)$ and $\mathcal{C}_p^{j,\,k}(h)$. Here we list a few of them.

1. If we let $p = j = 1$ in definition 1, then the classes $\mathcal{S}_p^{j,\,k}(h)$ and $\mathcal{C}_p^{j,\,k}(h)$ reduces to $\mathcal{S}_{sc}^k(h)$ and $\mathcal{C}_{sc}^k(h)$ respectively. The function classes $\mathcal{S}_{sc}^k(h)$ and $\mathcal{C}_{sc}^k(h)$ were introduced by Wang in [14].
2. If $p = j = k = 1$ and $h(z) = \frac{1+\beta z}{1-\alpha\beta z}$ in definition 1, then the classes $\mathcal{S}_p^{j,\,k}(h)$ and $\mathcal{C}_p^{j,\,k}(h)$ reduces to

$$\mathcal{S}_c^*(\alpha,\,\beta) = \left\{ f:\ f \in \mathcal{A},\ \left| \frac{zf'(z)}{f(z)+\overline{f(\overline{z})}} - 1 \right| < \beta \left| \frac{\alpha zf'(z)}{f(z)+\overline{f(\overline{z})}} + 1 \right|,\ z \in \mathcal{U} \right\},$$

and

$$\mathcal{C}_c^*(\alpha,\,\beta) = \left\{ f:\ f \in \mathcal{A},\ \left| \frac{(zf'(z))'}{\left(f(z)+\overline{f(\overline{z})}\right)'} - 1 \right| < \beta \left| \frac{\alpha zf'(z)}{\left(f(z)+\overline{f(\overline{z})}\right)'} + 1 \right|,\ z \in \mathcal{U} \right\}$$

respectively. The class $\mathcal{S}_c^*(\alpha,\,\beta)$ was introduced by Sudharsan et. al. in [11].

3. If $p = j = k = 1$ and $h(z) = \frac{1+z}{1-z}$ in definition 1, then the class $\mathcal{S}_p^{j,\,k}(h)$ reduces to the class $\mathcal{S}_c^*$ investigated by EL Ashwa and Thomas in [3].

Definition 3. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p^{j,\,k}(h)$ if and only if it satisfies the condition

$$\frac{1}{p}\frac{zf'(z)}{\phi_{2j,\,k}(z)} \prec h(z),$$

where $\phi_{2j,\,k}(z) \in \mathcal{S}_p^{j,\,k}(h)$ with $\phi_{2j,\,k}(z) \neq 0$ in $\mathcal{U}$.

Similarly, the class $\mathcal{QC}_p^{j,\,k}(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination condition

$$\frac{1}{p}\frac{(zf'(z))'}{\phi'_{2j,\,k}(z)} \prec h(z),$$

for some $\phi_{2j,\,k}(z) \in \mathcal{S}_p^{j,\,k}(h)$ with $\phi'_{2j,\,k}(z) \neq 0$.

The general classes $\mathcal{S}_p^{j,\,k}(g,\,h)$, $\mathcal{C}_p^{j,\,k}(g,\,h)$, $\mathcal{K}_p^{j,\,k}(g,\,h)$ and $\mathcal{QC}_p^{j,\,k}(g,\,h)$ consists of functions $f \in \mathcal{A}_p$ for which $f * g$ respectively belongs to $\mathcal{S}_p^{j,\,k}(h)$, $\mathcal{C}_p^{j,\,k}(h)$, $\mathcal{K}_p^{j,\,k}(h)$ and $\mathcal{QC}_p^{j,\,k}(h)$.

For a choice of the fixed function $g(z) = z^p/(1-z)$, then the classes $\mathcal{S}_p^{j,\,k}(g,\,h)$, $\mathcal{C}_p^{j,\,k}(g,\,h)$, $\mathcal{K}_p^{j,\,k}(g,\,h)$ and $\mathcal{QC}_p^{j,\,k}(g,\,h)$ reduces respectively to $\mathcal{S}_p^{j,\,k}(h)$, $\mathcal{C}_p^{j,\,k}(h)$, $\mathcal{K}_p^{j,\,k}(h)$ and $\mathcal{QC}_p^{j,\,k}(h)$.

For $\gamma < 1$, the class $\mathcal{R}_\gamma$ of prestarlike functions of order γ is defined by

$$\mathcal{R}_\gamma = \left\{ f \in \mathcal{A} :\ f * \frac{z}{(1-z)^{2-2\gamma}} \in \mathcal{S}^*(\gamma) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $Re\, f(z)/z > 1/2$. The well-known result that the classes of starlike functions of order γ and convex functions of order γ are closed under convolution with prestarlike functions of order γ is a consequence of the following:

Lemma 4. [9] *Let $\gamma < 1$, $\phi \in \mathcal{R}_\gamma$ and $f \in \mathcal{S}^*(\gamma)$. Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{co}(H(\mathcal{U})),$$

for any analytic function $H \in \mathcal{H}(\mathcal{U})$, where $\overline{co}(H(\mathcal{U}))$ denote the closed convex hull $H(\mathcal{U})$.

Using Lemma 4, we have the following result.

Lemma 5. *If $\phi(z)/z^{p-1} \in \mathcal{R}_\gamma$ and $f(z) \in \mathcal{S}^*(\gamma)$. Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{co}(H(\mathcal{U})),$$

for any analytic function $H \in \mathcal{H}(\mathcal{U})$.

2. INCLUSION RELATIONSHIP

Theorem 6. *Let h be a convex univalent function satisfying*

$$Re h(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \le \gamma < 1),$$

and $\phi \in \mathcal{A}_p$, with $\phi/z^{p-1} \in \mathcal{R}_\gamma$. If $f \in \mathcal{S}_p^{j,\,k}(g,\,h)$ for a fixed function g in $\mathcal{A}_p$, then $\phi * f \in \mathcal{S}_p^{j,\,k}(g,\,h)$.

Proof. From the definition of $\mathcal{S}_p^{j,\,k}(h)$, then for any fixed $z \in \mathcal{U}$ we have

$$\frac{1}{p}\frac{zf'(z)}{f_{2j,\,k}(z)} \in h(\mathcal{U})\,. \tag{2.1}$$

If we replace z by $\varepsilon^\nu z$ in (2.1), then (2.1) will be of the form

$$\frac{1}{p}\frac{\varepsilon^\nu z f'(\varepsilon^\nu z)}{f_{2j,\,k}(\varepsilon^\nu z)} \in h(\mathcal{U})\,, \quad (z \in \mathcal{U};\, \nu = 0,\,1,\,2,\,\ldots,\,k-1). \tag{2.2}$$

From (2.2), we have

$$\frac{1}{p}\frac{\overline{\varepsilon^\nu \overline{z}}\;\overline{f'(\varepsilon^\nu \overline{z})}}{\overline{f_{2j,\,k}(\varepsilon^\nu \overline{z})}} \in h(\mathcal{U})\,, \quad (z \in \mathcal{U};\, \nu = 0,\,1,\,2,\,\ldots,\,k-1). \tag{2.3}$$

Using the equality (1.6), (2.2) and (2.3) can be rewritten as

$$\frac{1}{p}\frac{\varepsilon^{\nu-\nu pj}zf'(\varepsilon^{\nu}z)}{f_{2j,\,k}(z)}\in h(\mathcal{U}), \quad (z\in\mathcal{U};\ \nu=0,\,1,\,2,\,\ldots,\,k-1), \tag{2.4}$$

and

$$\frac{1}{p}\frac{\varepsilon^{\nu pj-\nu}\ z\overline{f'(\varepsilon^{\nu}\overline{z})}}{f_{2j,\,k}(z)}\in h(\mathcal{U}), \quad (z\in\mathcal{U};\ \nu=0,\,1,\,2,\,\ldots,\,k-1). \tag{2.5}$$

Adding (2.4) and (2.5), we get

$$\frac{1}{p}\frac{z\left[\varepsilon^{\nu-\nu pj}f'(\varepsilon^{\nu}z)+\varepsilon^{\nu pj-\nu}\ \overline{f'(\varepsilon^{\nu}\overline{z})}\right]}{f_{2j,\,k}(z)}\in h(\mathcal{U}), \quad (z\in\mathcal{U};\ \nu=0,\,1,\,2,\,\ldots,\,k-1). \tag{2.6}$$

Let $\nu=0,\,1,\,2,\,\ldots,\,k-1$ in (2.6) respectively and summing them, we get

$$\frac{1}{p}\frac{z\left[\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu pj+\nu}f'(\varepsilon^{\nu}z)+\varepsilon^{\nu pj-\nu}\overline{f'(\varepsilon^{\nu}\overline{z})}\right]\right]}{f_{2j,\,k}(z)}\in h(\mathcal{U}), \quad (z\in\mathcal{U}).$$

Or equivalently,

$$\frac{1}{p}\frac{zf'_{2j,\,k}(z)}{f_{2j,\,k}(z)}\in h(\mathcal{U}), \quad (z\in\mathcal{U}),$$

that is $f_{2j,\,k}(z)\in\mathcal{S}_p^{j,\,k}(h)$.

Set $H(z)$ and $\psi(z)$ by

$$H(z)=\frac{zf'(z)}{pf_{2j,\,k}(z)} \quad \text{and} \quad \psi_{2j,\,k}(z)=\frac{f_{2j,\,k}(z)}{z^{p-1}}.$$

Now $Re\,h(z)>1-\frac{1-\gamma}{p}$ yields

$$Re\frac{z\psi'_{2j,\,k}(z)}{\psi_{2j,\,k}(z)}=Re\frac{zf'_{2j,\,k}(z)}{f_{2j,\,k}(z)}-(p-1)>\gamma. \tag{2.7}$$

Inequality (2.7) shows that the function $\psi_{2j,\,k}(z)$ is starlike of order γ, which we denote by $\mathcal{S}^*(\gamma)$. A simple computation shows that

$$\frac{z\,(\phi*f)'\,(z)}{p(\phi*f)_{2j,\,k}(z)}=\frac{\left(\phi*\left(p^{-1}zf'\right)\right)(z)}{(\phi*f_{2j,\,k})\,(z)}=\frac{(\phi*(H\,f_{2j,\,k}))\,(z)}{(\phi*f_{2j,\,k})\,(z)}.$$

Since $\phi/z^{p-1}\in\mathcal{R}_\gamma$ and $\psi_{2j,\,k}\in\mathcal{S}^*(\gamma)$, Lemma 5 yields

$$\frac{(\phi*(H\,f_{2j,\,k}))\,(z)}{(\phi*f_{2j,\,k})\,(z)})\in\overline{co}(H(\mathcal{U})).$$

The subordination $H\prec h$ implies

$$\frac{z\,(\phi*f)'\,(z)}{p(\phi*f)_{2j,\,k}(z)}\prec h(z).$$

Thus $\phi*f\in\mathcal{S}_p^{j,\,k}(h)$. That is

$$f\in\mathcal{S}_p^{j,\,k}(h) \quad\Longrightarrow\quad f*g\in\mathcal{S}_p^{j,\,k}(h) \quad\Longrightarrow\quad \phi*f*g\in\mathcal{S}_p^{j,\,k}(h),$$

or equivalently $\phi*f\in\mathcal{S}_p^{j,\,k}(g,\,h)$. □

Remark 7. Using the condition (1.7) together with the result $f_{2j,\,k}(z) \in \mathcal{S}_p^{j,\,k}(h)$ shows that the functions in $\mathcal{S}_p^{j,\,k}(h)$ are contained in $\mathcal{K}_p^{j,\,k}(h)$. In general, $\mathcal{S}_p^{j,\,k}(g,\,h) \subset \mathcal{K}_p^{j,\,k}(g,\,h)$.

Theorem 8. *Let h be a convex univalent function satisfying*

$$Reh(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \le \gamma < 1),$$

and $\phi \in \mathcal{A}_p$, with $\phi/z^{p-1} \in \mathcal{R}_\gamma$. If $f \in \mathcal{C}_p^{j,\,k}(g,\,h)$ for a fixed function g in $\mathcal{A}_p$, then $\phi * f \in \mathcal{C}_p^{j,\,k}(g,\,h)$.

Proof. From the identity

$$\frac{(z(g*f)'(z))'}{p(g*f)'_{2j,\,k}(z)} = \frac{z\left(g*p^{-1}zf'\right)'(z)}{p\,\left(g*p^{-1}zf'\right)_{2j\,k}(z)},$$

we have $f \in \mathcal{C}_p^{j,\,k}(g,\,h)$ if and only if $\frac{zf'}{p} \in \mathcal{S}_p^{j,\,k}(g,\,h)$ and by Theorem 6 it follows that $\phi * \left(\frac{zf'}{p}\right) = \frac{z}{p}(\phi * f)'(z) \in \mathcal{S}_p^{j,\,k}(g,\,h)$. Hence $\phi * f \in \mathcal{C}_p^{j,\,k}(g,\,h)$. □

Remark 9. Analogous to the result in Theorem 6, it can be proved that $f_{2j,\,k}(z) \in \mathcal{C}_p^{j,\,k}(h)$. Using this result together with condition (1.7) shows that the functions in $\mathcal{C}_p^{j,\,k}(h)$ are contained in $\mathcal{QC}_p^{j,\,k}(h)$. In general, $\mathcal{C}_p^{j,\,k}(g,\,h) \subset \mathcal{QC}_p^{j,\,k}(g,\,h)$.

Using the arguments similar to those detailed in Theorem 6 and Theorem 8, we can prove the following two Theorems. We therefore, choose to omit the details involved.

Theorem 10. *Let h be a convex univalent function satisfying*

$$Re\,h(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \le \gamma < 1),$$

and $\phi \in \mathcal{A}_p$ with $\phi(z)/z^{p-1} \in \mathcal{R}_\gamma$. If $f \in \mathcal{K}_p^{j,\,k}(g,\,h)$, then $\phi * f \in \mathcal{K}_p^{j,\,k}(g,\,h)$.

Theorem 11. *Let h be a convex univalent function satisfying*

$$Re\,h(z) > 1 - \frac{1-\gamma}{p}, \quad (0 \le \gamma < 1),$$

and $\phi \in \mathcal{A}_p$ with $\phi(z)/z^{p-1} \in \mathcal{R}_\gamma$. If $f \in \mathcal{QC}_p^{j,\,k}(g,\,h)$, then $\phi * f \in \mathcal{QC}_p^{j,\,k}(g,\,h)$.

3. Integral Representation

Theorem 12. *Let $f \in \mathcal{S}_p^{j,\,k}(g,\,h)$, then we have*

$$s_{2j,\,k}(z) = z^p\,\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^\nu\zeta)\right) + \overline{\phi\left(w(\varepsilon^\nu\overline{\zeta})\right)} - 2\right]\,d\zeta\right\}, \tag{3.1}$$

where $s_{2j,\,k}(z) = (f*g)_{j,\,k}(z)$, and $w(z)$ is analytic in $\mathcal{U}$ with $w(0) = 0$, $\mid w(z) \mid < 1$.

Proof. From the definition of $\mathcal{S}_p^{j,\,k}(g,\,h)$, we have

$$\frac{z\,(f*g)'\,(z)}{p\,s_{2j,\,k}(z)} = \phi\,(w(z))\,, \tag{3.2}$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $\mid w(z) \mid < 1$. Substituting z by $\varepsilon^\nu z$ in the equality (3.2) respectively ($\nu = 0,\,1,\,2,\,\ldots,\,k-1,\,\varepsilon^k = 1$), we have

$$\frac{\varepsilon^\nu z\,(f*g)'\,(\varepsilon^\nu z)}{p\,s_{2j,\,k}(\varepsilon^\nu z)} = \phi\,(w(\varepsilon^\nu z)) \tag{3.3}$$

On simple computation, we get

$$\frac{\overline{\varepsilon^{\nu}\overline{z}}\,\overline{(f*g)'(\varepsilon^{\nu}\overline{z})}}{p\,\overline{s_{2j,\,k}(\varepsilon^{\nu}\overline{z})}} = \overline{\phi\left(w(\varepsilon^{\nu}\overline{z})\right)}. \tag{3.4}$$

Proceeding as in Theorem 6, we have

$$\frac{z s'_{2j,\,k}(z)}{p\,s_{2j,\,k}(z)} = \frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\phi\left(w(\varepsilon^{\nu}z)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{z})\right)}\right],$$

which can be rewritten as

$$\frac{s'_{2j,\,k}(z)}{s_{2j,\,k}(z)} - \frac{p}{z} = \frac{p}{2k}\sum_{\nu=0}^{k-1}\frac{1}{z}\left[\phi\left(w(\varepsilon^{\nu}z)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{z})\right)} - 2\right].$$

Integrating this equality, we get

$$\log\left\{\frac{s_{2j,\,k}(z)}{z^p}\right\} = \frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)} - 2\right]\,d\zeta,$$

or equivalently,

$$s_{2j,\,k}(z) = z^p\,\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)} - 2\right]\,d\zeta\right\}.$$

This completes the proof of Theorem 12. □

Theorem 13. *Let* $f \in \mathcal{S}_p^{j,\,k}(g,\,h)$*, then we have*

$$s(z) = \int_0^{\eta} p\,z^{p-1}\,\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)} - 2\right]\,d\zeta\right\}\cdot\phi\left(w(z)\right)dz$$

where $s(z) = (f*g)(z)$ *and* $w(z)$ *is analytic in* $\mathcal{U}$ *with* $w(0) = 0$, $|\,w(z)\,| < 1$.

Proof. Let $f \in \mathcal{S}_p^{j,\,k}(g,\,h)$. Then from the definition, we have

$$\begin{aligned} s'(z) &= \frac{p\,s_{2j,\,k}(z)}{z}\cdot\phi\left(w(z)\right) \\ &= p\,z^{p-1}\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)} - 2\right]\,d\zeta\right\}\cdot\phi\left(w(z)\right). \end{aligned}$$

Integrating the above equality will prove the assertions of the theorem. □

Theorem 14. *Let* $f \in \mathcal{C}_p^{j,\,k}(g,\,h)$*, then we have*

$$s_{2j,\,k}(z) = \int_0^{\eta} z^{p-1}\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)} - 2\right]\,d\zeta\right\}dz,$$

where $s_{2j,\,k}(z) = (f*g)_{2j,\,k}(z)$, *and* $w(z)$ *is analytic in* $\mathcal{U}$ *with* $w(0) = 0$, $|\,w(z)\,| < 1$.

Theorem 15. *Let* $f \in \mathcal{C}_p^{j,\,k}(g,\,h)$*, then we have*

$$s(z) = \int_0^{\xi}\frac{p}{\eta}\int_0^{\eta} z^{p-1}\exp\left\{\frac{p}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right) + \overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)} - 2\right]\,d\zeta\right\}dz\,d\eta,$$

where $s(z) = (f*g)(z)$ *and* $w(z)$ *is analytic in* $\mathcal{U}$ *with* $w(0) = 0$, $|\,w(z)\,| < 1$.

References

[1] R. M. Ali, A. O. Badghaish and V. Ravichandran, Multivalent functions with respect to n-ply points and symmetric conjugate points, Comput. Math. Appl. **60** (2010), no. 11, 2926–2935.

[2] R. Chandrashekar et al., Convolutions of meromorphic multivalent functions with respect to n-ply points and symmetric conjugate points, Appl. Math. Comput. (2011),

[3] R. Md. El-Ashwah and D. K. Thomas, Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc. **2** (1987), no. 1, 85–100.

[4] A. W. Goodman, *Univalent functions. Vol. I*, Mariner, Tampa, FL, 1983.

[5] I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Dekker, New York, 2003.

[6] P. Liczberski and J. Połubiński, On (j,k)-symmetrical functions, Math. Bohem. **120** (1995), no. 1, 13–28.

[7] M. Haji Mohd et al., Subclasses of meromorphic functions associated with convolution, J. Inequal. Appl. **2009**, Art. ID 190291, 9 pp.

[8] K. R. Karthikeyan, Some classes of analytic functions with respect to symmetric conjugate points, Submitted.

[9] S. Ruscheweyh, *Convolutions in geometric function theory*, Séminaire de Mathématiques Supérieures, 83, Presses Univ. Montréal, Montreal, QC, 1982.

[10] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan **11** (1959), 72–75.

[11] T. V. Sudharsan, P. Balasubrahmanyam and K. G. Subramanian, On functions starlike with respect to symmetric and conjugate points, Taiwanese J. Math. **2** (1998), no. 1, 57–68.

[12] Z.-G. Wang, C.-Y. Gao and S.-M. Yuan, On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points, J. Math. Anal. Appl. **322** (2006), no. 1, 97–106.

[13] Z.-G. Wang, Y.-P. Jiang and H. M. Srivastava, Some subclasses of multivalent analytic functions involving the Dziok-Srivastava operator, Integral Transforms Spec. Funct. **19** (2008), no. 1-2, 129–146.

[14] Z.-G. Wang and C.-Y. Gao, On starlike and convex functions with respect to $2k$-symmetric conjugate points, Tamsui Oxf. J. Math. Sci. **24** (2008), no. 3, 277–287.

[15] D.-G. Yang and J.-L. Liu, On Sakaguchi functions, Int. J. Math. Math. Sci. **2003**, no. 30, 1923–1931.

[16] S.-M. Yuan and Z.-M. Liu, Some properties of α-convex and α-quasiconvex functions with respect to n-symmetric points, Appl. Math. Comput. **188** (2007), no. 2, 1142–1150.