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Some Generalizations of Hermite-Hadamard Type Integral Inequalities and their Applications

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Abstract. In this paper, we establish various inequalities for some differentiable mappings that are linked with the illustrious Hermite- Hadamard integral inequality for mappings whose derivatives are $(h{-}(\alpha, m))$-convex.The generalized integral inequalities contribute some better estimates than some already presented. The inequalities are then applied to numerical integration and some special means.

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1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function defined on the interval I of real numbers. Then f is called convex if

$$f(t\,x + (1-t)\,y) \leq \; t\,f(x) \; + \; (1-t)\,f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR. There are many results associated with convex functions in the area of inequalities, but one of those is the classical Hermite Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

for $a, b \in I$, with $a < b$.
In [5], H. Hudzik and L. Maligranda considered, among others, the class of functions which are $s-$convex in the first and second sense. This class is defined as follows:

Definition 1. A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s-$convex or f belongs to the class K_s^i if

$$f(\mu\,x + \nu\,y) \leq \; \mu^s\,f(x) \; + \; \nu^s\,f(y) \tag{1.2}$$

holds for all $x, y \in [0, \infty)$, $\mu, \nu \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Note that, if $\mu^s + \nu^s = 1$, the above class of convex functions is called s-convex functions in first sense and represented by K_s^1 and if $\mu + \nu = 1$ the above class is called s-convex in second sense and represented by K_s^2.
It may be noted that every 1-convex function is convex. In the same paper [5] H. Hudzik and L. Maligranda discussed a few results connecting with $s-$convex functions in second sense and some new results about Hadamard's inequality for $s-$convex functions are discussed in [4], while on the other hand there are many important inequalities connecting with 1-convex (convex) functions [4], but one of these is (1.1).
In [11], V.G. Mihesan presented the class of (α, m)-convex functions as reproduced below:

Definition 2. The function $f : [0, b] \to \mathbb{R}$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1 - t^\alpha) f(y)$$

Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one receives the following classes of functions respectively: increasing, α-starshaped, starshaped, m-convex, convex and α-convex.
Denote by $K_m^\alpha(b)$ the set of all (α, m)-convex functions on $[0, b]$ with $f(0) \leq 0$. For recent results and generalizations referring m-convex and (α, m)-convex functions see [1], [2] and [16].
M. Muddassar et. al., define a new class of convex functions in [15] named as s-(α, m)-convex functions as reproduced below

Definition 3. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-(α, m)-convex function in first sense or f belongs to the class $K_{m,1}^{\alpha,s}$, if for all $x, y \in [0, \infty)$ and $\mu \in [0, 1]$, the following inequality holds:

$$f(\mu x + (1-\mu)y) \leq (\mu^{\alpha s}) f(x) + m (1 - \mu^{\alpha s}) f\left(\frac{y}{m}\right)$$

where $(\alpha, m) \in [0, 1]^2$ and for some fixed $s \in (0, 1]$.

Definition 4. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-(α, m)-convex function in second sense or f belongs to the class $K_{m,2}^{\alpha,s}$, if for all $x, y \in [0, \infty)$ and $\mu, \nu \in [0, 1]$, the following inequality holds:

$$f(\mu x + (1-\mu)y) \leq (\mu^{\alpha})^s f(x) + m (1 - \mu^{\alpha})^s f\left(\frac{y}{m}\right)$$

where $(\alpha, m) \in [0, 1]^2$ and for some fixed $s \in (0, 1]$.

Note that for $s = 1$, we get $K_m^\alpha(I)$ class of convex functions and for $\alpha = 1$ and $m = 1$, we get $K_s^1(I)$ and $K_s^2(I)$ class of convex functions.
In [19], S. Varošanec define the following class of convex functions as reproduced below:

Definition 5. Let $h : \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an h-convex function (or that f belongs to the class $SX(h, I)$) if f is non-negative and for all $x, y \in I$, $\mu, \nu \in (0, 1)$ and $\mu + \nu = 1$, we have

$$f(\mu x + \nu y) \leq h(\mu) f(x) + h(\nu) f(y)$$

if the above inequality reversed, then f is said to be h-concave (or $f \in SV(h, I)$).

Evidently, if $h(\mu) = \mu$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$; if $h(\mu) = \frac{1}{\mu}$, then $SX(h, I) = Q(I)$; if $h(\mu) = 1$, then $P(I) \subseteq SX(h, I)$; and if $h(\mu) = \mu^s$, where $s \in (0, 1]$, then $K_s^2 \subseteq SX(h, I)$. In [16], M. E. Özdemir et. al., define a new class of convex functions as below:

Definition 6. Let $h : \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an $(h - (\alpha, m))$-convex function (or that f belongs to the class $SX((h_(\alpha, m)), I))$ if f is non-negative and for all $x, y \in I$ and $\lambda \in (0, 1)$ for $(\alpha, m) \in [0, 1]^2$, we have

$$f(\lambda x + m(1 - \lambda)y) \leq h^{\alpha}(\lambda)f(x) + m(1 - h^{\alpha}(\lambda))f(y)$$

if the above inequality is reversed, then f is said to be $(h - (\alpha, m), I)$-concave, *i.e.*, $f \in SV(h - (\alpha, m), I)$.

Evidently, if $h(\lambda) = \lambda$, then all non-negative convex functions belong to $K_m^{\alpha}(I)$. In [4] S. S. Dragomir et al. discoursed inequalities for differentiable and twice differentiable functions associating with the H-H Inequality on the footing of the following Lemmas.

Lemma 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I° (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1([a, b])$, then we have*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)}{4}\int_0^1 (1-t)\left[f'\left(ta + (1-t)\frac{a+b}{2}\right) - f'\left(tb + (1-t)\frac{a+b}{2}\right)\right]dt \qquad (1.3)$$

In [3], Dragomir and Agarwal constituted the following results linked with the right part of (1.3) as well as to apply them for some primary inequalities for real numbers and numerical integration.

Lemma 8. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I°, $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$, then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta + (1-t)b)dt \qquad (1.4)$$

Here We feed definition of Beta function of Euler type which will be useful in our next discussion, which is for $x, y > 0$ defined as

$$\beta(x, y) = \frac{\Gamma(x).\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

This paper is in the direction of the results discussed in [6] but here we use $(h - (\alpha, m))$-convex functions instead of s-convex function. After this introduction, in section 2 we found some new integral inequalities of the type of Hermite Hadamard's for generalized convex functions. In section 3 we give some new applications of the results from section 2 for some special means. The inequalities are then applied to numerical integration in section 4.

2. Main Results

The following theorems were obtained by using the $(h - (\alpha, m))$-convex function.

Theorem 9. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$. If the mapping $|f'|$ is $(h - (\alpha, m))$-convex on $[a, b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\left[\left\{|f'(a)|+|f'(b)|+2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right\}\right.$$
$$\left.\int_0^1 (1-t)h^\alpha(t)dt + \frac{m}{2}\left|f'\left(\frac{a+b}{2m}\right)\right|\right] \tag{2.1}$$

Proof. Taking modulus on both sides of lemma 7, we get

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{4}\int_0^1 |1-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right.\\ &\qquad \left.-f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\\ &= \frac{b-a}{4}\left\{\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt\right.\\ &\qquad \left.+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right\}\end{aligned} \tag{2.2}$$

Since the mapping $|f'|$ is $(h - (\alpha, m))$ convex on $[a, b]$, then

$$|f'(tx+(1-t)y)| \leq h^\alpha(t)\,|f'(x)| + m(1-h^\alpha(t))\left|f'(\frac{y}{m})\right|$$

Inequation (2.2) becomes

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\int_0^1 (1-t)\left\{|f'(a)|\,h^\alpha(t)+m\left|f'\left(\frac{a+b}{2m}\right)\right|\right.\right.$$
$$\left.(1-h^\alpha(t))\right\}dt + \int_0^1 (1-t)\left\{|f'(b)|\,h^\alpha(t)+m\left|f'\left(\frac{a+b}{2m}\right)\right|(1-h^\alpha(t))\right\}dt\Bigg] \tag{2.3}$$

which completes the proof. □

Theorem 10. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$. If the mapping $|f'|^q$ is $(h - (\alpha, m))$-convex on $[a, b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left\{\left(|f'(a)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\times\right.\right.$$
$$\left.\int_0^1 h^\alpha(t)dt + m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right\}^{\frac{1}{q}} + \left\{\left(|f'(b)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\times\right.$$
$$\left.\left.\int_0^1 h^\alpha(t)dt + m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right\}^{\frac{1}{q}}\right] \tag{2.4}$$

Proof. By utilizing the Hölder's Integral Inequality on the first integral in the right of (2.2), we get

$$\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt \le \left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \tag{2.5}$$

Here

$$\int_0^1 (1-t)^p dt = \frac{1}{p+1} \tag{2.6}$$

and

$$\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt = |f'(a)|^q \int_0^1 h^\alpha(t)dt + m\left|f'\left(\frac{a+b}{2m}\right)\right|^q \int_0^1 \left(1-h^\alpha(t)\right)dt \tag{2.7}$$

Using the inequalities (2.6) and (2.7), the inequality (2.5) turns to

$$\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left(|f'(a)|^q \int_0^1 h^\alpha(t)dt + m\left|f'\left(\frac{a+b}{2m}\right)\right|^q \int_0^1 \left(1-h^\alpha(t)\right)dt\right)^{\frac{1}{q}} \tag{2.8}$$

similarly

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left(|f'(b)|^q \int_0^1 h^\alpha(t)dt + m\left|f'\left(\frac{a+b}{2m}\right)\right|^q \int_0^1 \left(1-h^\alpha(t)\right)dt\right)^{\frac{1}{q}} \tag{2.9}$$

which completes the proof. □

Corollary 11. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f'|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left\{|f'(a)|+|f'(b)| - 2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right\} \int_0^1 h^{\frac{\alpha}{q}}(t)dt + 2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right] \tag{2.10}$$

Proof. Proof is very similar to the above theorem but at the end we use the following fact: $\sum_{m=1}^{n-1}\left(\Phi_m+\Psi_m\right)^r \le \sum_{m=1}^{n-1}(\Phi_m)^r + \sum_{m=1}^{n-1}(\Psi_m)^r$ for $(0 < r < 1)$ and for each m both $\Phi_m, \Psi_m \ge 0$ □

Theorem 12. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f'|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$,* *then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2^{\frac{2p+1}{p}}}\left[\left(\left\{|f'(a)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right\}\right.\right.$$

$$\left.\int_0^1 (1-t)h^{\alpha}(t)dt + \frac{m}{2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left\{|f'(b)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right\}\right.$$

$$\left.\left.\int_0^1 (1-t)h^{\alpha}(t)dt + \frac{m}{2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right] \tag{2.11}$$

Proof. By utilizing the Hölder's Integral Inequality on the first integral in the right of (2.2), we get

$$\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt \leq \left(\int_0^1 (1-t)dt\right)^{\frac{1}{p}}$$

$$\left(\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left[\left(|f'(a)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)h^{\alpha}(t) + m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\left\{|f'(a)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right\}\int_0^1 (1-t)h^{\alpha}(t)dt + \frac{m}{2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}} \tag{2.12}$$

similarly

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt \leq \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\left\{|f'(b)|^q - m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right\}\right.$$

$$\left.\int_0^1 (1-t)h^{\alpha}(t)dt + \frac{m}{2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}} \tag{2.13}$$

which completes the proof. □

Versions of these results for twice differentiable functions are given underneath. These can be proved in a like way based on Lemma 8.

Corollary 13. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f'|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2^{\frac{2p+1}{p}}}\left[\left\{|f'(a)|+|f'(b)|-2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right\}\right.$$

$$\left.\int_0^1 \left(1-\frac{t}{q}\right)h^{\frac{\alpha}{q}}(t)dt + m\left|f'\left(\frac{a+b}{2m}\right)\right|\right] \tag{2.14}$$

Theorem 14. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f''|$ is $(h-(\alpha,m))$-convex on $[a,b]$,* *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2}\left[\left\{|f''(a)|-m\left|f''\left(\frac{b}{m}\right)\right|\right\}\right.$$
$$\left.\int_0^1 t(1-t)h^{\alpha}(t)dt+\frac{m}{6}\left|f''\left(\frac{b}{m}\right)\right|\right] \quad (2.15)$$

Theorem 15. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f''|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left[\left\{|f''(a)|^q-m\times\right.\right.$$
$$\left.\left.\left|f''\left(\frac{b}{m}\right)\right|^q\right\}\int_0^1 h^{\alpha}(t)dt+m\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \quad (2.16)$$

Corollary 16. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f''|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left[\{|f''(a)|-m\times\right.$$
$$\left.\left|f''\left(\frac{b}{m}\right)\right|\right\}\int_0^1 h^{\frac{\alpha}{q}}(t)dt+m\left|f''\left(\frac{b}{m}\right)\right|\right] \quad (2.17)$$

Theorem 17. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f''|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2.6^{\frac{1}{p}}}\left[\left\{|f''(a)|^q-m\left|f''\left(\frac{b}{m}\right)\right|^q\right\}\right.$$
$$\left.\int_0^1 t(1-t)h^{\alpha}(t)dt+\frac{m}{6}\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \quad (2.18)$$

Corollary 18. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$. If the mapping $|f''|^q$ is $(h-(\alpha,m))$-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2.6^{\frac{1}{p}}}\left[\left\{|f''(a)|-m\left|f''\left(\frac{b}{m}\right)\right|\right\}\right.$$
$$\left.\int_0^1 t^{\frac{1}{q}}\left(1-\frac{t}{q}\right)h^{\frac{\alpha}{q}}(t)dt+\frac{m}{6}\left|f''\left(\frac{b}{m}\right)\right|\right] \quad (2.19)$$

3. Application to some special means

Let us recall the following means for any two positive numbers a and b.

(1) *The Arithmetic mean*

$$A \equiv A(a,b) = \frac{a+b}{2}$$

(2) *The Harmonic mean*

$$H \equiv H(a,b) = \frac{2ab}{a+b}$$

(3) *The $p-$Logarithmic mean*

$$L_p \equiv L_p(a,b) = \begin{cases} a, & \text{if } a = b; \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b. \end{cases}$$

(4) The $Identric\ mean$

$$I \equiv I(a,b) = \begin{cases} a, & \text{if } a = b; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b. \end{cases}$$

(5) *The Logarithmic mean*

$$L \equiv L(a,b) = \begin{cases} a, & \text{if } a = b; \\ \frac{b-a}{\ln b\, -\, \ln a}, & \text{if } a \neq b. \end{cases}$$

The following inequality is well known in the literature in [11]:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 19. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$. Then one has the inequality.*

$$|\mathrm{G}(a,b)\ -\ \mathrm{L}(a,b)| \leq \frac{\ln b - \ln a}{4(p+1)^{\frac{1}{p}}}\left[\mathrm{A}\left(|a|,|b|\right)\ +\ \mathrm{G}\left(|a|,|b|\right)\right]. \tag{3.1}$$

Proof. By Corollary 11 applied for the mapping $f(x) = e^x$ setting $h(t) = t$, $\alpha = 1$, $m = 1$ and $q = 1$ we have the above inequality (3.1). □

Proposition 20. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$, then*

$$\left|\frac{\mathrm{A}(a,b)}{\mathrm{I}(a,b)}\right| \leq \exp\left\{\frac{b-a}{3.2^{\frac{2p+1}{p}}}\left(\mathrm{H}^{-1}\left(|a|,|b|\right) + 2\mathrm{A}^{-1}\left(|a|,|b|\right)\right)\right\}$$

Proof. Follows from Corollary 13 for the mapping $f(x) = -\ln(x)$ setting $h(t) = t$, $\alpha = 1$, $m = 1$ and $q = 1$. □

Another result which is connected with p-Logarithmic mean $L_p(a,b)$ is the following one:

Proposition 21. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$, then*

$$\left|\mathrm{H}^{-1}(a,b)\ -\ \mathrm{L}^{-1}(a,b)\right| \leq (b-a)^2\beta^{\frac{1}{p}}(p+1,p+1)\mathrm{H}^{-1}\left(|a|^3,|b|^3\right)$$

Proof. Follows by Corollary 16, for the mapping $f(x) = \frac{1}{x}$ setting $h(t) = t$, $\alpha = 1$, $m = 1$ and $q = 1$. □

Proposition 22. *Let $p > 1$, $0 < a < b$ and $q = \frac{p}{p-1}$, then*

$$\left|\mathrm{A}(a^n,b^n) - \mathrm{L}_p^p(a,b)\right| \leq |n(n-1)|\frac{(b-a)^2}{2.6^{\frac{p+1}{p}}}\mathrm{A}\left(|a|^{p-2},|b|^{p-2}\right)$$

Proof. Follows by Corollary 18, for the mapping $f(x) = (1-x)^n$ setting $h(t) = t$, $\alpha = 1$, $m = 1$ and $q = 1$. □

4. Error Estimates for Midpoint Formula and Trapezoidal Formula

Let K be the $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = S(f,K) + R(f,K) \tag{4.1}$$

where

$$S(f,K) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$

for the midpoint version and $R(f, K)$ denotes the related approximation error.

$$S(f,K) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

for the trapezoidal version and $R(f, K)$ denotes the related approximation error.

Proposition 23. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L^1[a,b]$, where $a, b \in I$ with $a < b$ and $|f'|$ is convex on $[a, b]$, then*

$$|R(f,K)| \leq \frac{1}{2^{\frac{2p+1}{p}}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{2} \left(|f'(x_i)| + |f'(x_{i+1})|\right). \tag{4.2}$$

Proof. By applying subdivisions $[x_i, x_{i+1}]$ of the division k for $i = 0, 1, 2, ..., n-1$ on Corollary 13 setting $h(t) = t$, $\alpha = 1$, $m = 1$ and $q = 1$ taking into account that $|f'|$ is convex, we have

$$\left|\frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_{i+1} + x_i}{2}\right)\right| \leq \frac{x_{i+1} - x_i}{2^{\frac{3p+1}{p}}}\left(|f'(x_{i+1})| + |f'(x_i)|\right) \tag{4.3}$$

Taking sum over i from 0 to $n-1$, we get

$$\begin{aligned}\left|\int_a^b f(x)dx - S(f,K)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_{i+1} + x_i}{2}\right)(x_{i+1} - x_i)\right\}\right| \\ &\leq \sum_{i=0}^{n-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - (x_{i+1} - x_i) f\left(\frac{x_{i+1} + x_i}{2}\right)\right\}\right| \\ &= \sum_{i=0}^{n-1}(x_{i+1} - x_i)\left|\left\{\frac{1}{(x_{i+1} - x_i)}\int_{x_i}^{x_{i+1}} f(x)dx\right.\right. \\ &\qquad \left.\left. - f\left(\frac{x_{i+1} + x_i}{2}\right)\right\}\right| \end{aligned} \tag{4.4}$$

By combining (4.3) and (4.4), we get (4.2). Which completes the proof. □

Proposition 24. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^o such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$ and $|f''|$ is (α, m)-convex on $[a, b]$, then*

$$|R(f,K)| \leq \frac{\beta(\alpha+2,2)}{(6)^{\frac{1}{p}}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{2}\left(|f''(x_i)| + m\alpha(\alpha+5)\left|f''\left(\frac{x_{i+1}}{m}\right)\right|\right)$$

Proof. Proof is very similar as that of Proposition 23 by using corollary 18 setting $h(t) = t$. □

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