Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 46(1) (2014) pp. 19-24

A note on a Soft Topological Space

Sanjay Roy
Department of Mathematics
South Bantra Ramkrishna Institution
Howrah, West Bengal, India
Email: sanjaypuremath@gmail.com

T.K. Samanta
Department of Mathematics
Uluberia College,
Uluberia, Howrah, West Bengal, India
Email: mumpu_tapas5@yahoo.co.in

Abstract. The aim of this paper is to construct a topology on a soft set. Also the concepts of soft base, soft subbase are introduced and some important theorems are established.

AMS (MOS) Subject Classification Codes: 11B05, 06D72
Key Words: soft set, soft topology, soft base, soft subbase.

1. Introduction and Preliminaries

D. A. Molodtsov[3] introduced the notion of soft set in 1999 as a mathematical tool to deal with uncertainties. He also defined some important operations on soft set such as soft union, soft intersection etc. Later on these definitions have been modified in the paper [2] to define a topology on a soft set and depending upon these modified definitions, in this paper we have established a few theorems relative to base and subbase.
Throughout the work, U refers to an initial universe, E is the set of parameters, $P(U)$ is the power set of U and $A \subseteq E$.

Definition 1. [2] A soft set F_A on the universe U is defined by the set of ordered pairs $F_A = \{(x, F_A(x)) : x \in E, F_A(x) \in P(U)\}$ where $F_A : E \to P(U)$ such that $F_A(x) = \phi$ if x is not an element of A.
The set of all soft sets over U is denoted by $S(U)$.

Definition 2. [2] Let $F_A \in S(U)$. If $F_A(x) = \phi$, for all $x \in E$, then F_A is called a empty soft set, denoted by Φ. $F_A(x) = \phi$ means that there is no element in U related to the parameter $x \in E$.

Definition 3. [2] Let $F_A, G_B \in S(U)$. We say that F_A is a soft subsets of G_B and we write $F_A \sqsubseteq G_B$ if and only if
(i) $A \subseteq B$
(ii) $F_A(x) \subseteq G_B(x)$ for all $x \in E$.

Definition 4. [2] Let $F_A, G_B \in S(U)$. Then F_A and G_B are said to be soft equal, denoted by $F_A = G_B$ if $F_A(x) = G_B(x)$ for all $x \in E$.

Definition 5. [2] Let $F_A, G_B \in S(U)$. Then the soft union of F_A and G_B is also a soft set $F_A \sqcup G_B = H_{A\cup B} \in S(U)$, defined by
$H_{A\cup B}(x) = (F_A \sqcup G_B)(x) = F_A(x) \cup G_B(x)$ for all $x \in E$.

Definition 6. [2] Let $F_A, G_B \in S(U)$. Then the soft intersection of F_A and G_B is also a soft set $F_A \sqcap G_B = H_{A\cap B} \in S(U)$, defined by
$H_{A\cap B}(x) = (F_A \sqcap G_B)(x) = F_A(x) \cap G_B(x)$ for all $x \in E$.

In 2011, N. Cagman has defined a topology on soft set as follows:

Definition 7. [1] Let $F_A \in S(U)$. A soft topology on F_A, denoted by τ, is a collection of soft subsets of F_A having the following properties:
(i) $\Phi, F_A \in \tau$
(ii) $\{F_{A_i} \sqsubseteq F_A : i \in I \subseteq \mathbf{N}\} \subseteq \tau \Rightarrow \sqcup_{i\in I} F_{A_i} \in \tau$
(iii) $\{F_{A_i} \sqsubseteq F_A : 1 \leq i \leq n, n \in \mathbf{N}\} \subseteq \tau \Rightarrow \sqcap_{i=1}^{n} F_{A_i} \in \tau$

In the definition 7, N. Cagman has considered that τ is closed under countable union. But in the definition of a topology τ, it is usually considered that τ is closed under arbitrary union, which has not been considered in the definition 7. In fact, the symbol, used in the definition 7 to represent the members of τ, does not bear the meaning properly. It is clarified in the following example:
Let $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_3\}$ and $U = \{u_1, u_2, u_3, u_4, u_5\}$
Let $F_A \in S(U)$ such that $F_A(x) = U$ for all $x \in A$. Then the collection
$\tau = \{\ \Phi,\ F_A,$
$\{\ (e_1, \{u_1\}),\ \ (e_2, \{u_2\})\ \},$
$\{\ (e_1, \{u_1, u_3\}),\ \ (e_2, \{u_2, u_3\})\ \},$
$\{\ (e_1, \{u_1, u_2, u_4, u_5\}),\ \ (e_2, \{u_1, u_2, u_4, u_5\}),\ \ (e_3, U),\ \ (e_4, U)\ \}\ \ \}$
of some soft subsets of F_A is a soft topology on F_A. Now if we like to denote the members of τ according to the above definition, then it is not possible to represent the third and fourth members of τ simultaneously.

To avoid this difficulties, we first redefine the soft topology in section 2 and thereafter we establish the soft base, soft subbase and a few important theorems related to these concepts.

2. Soft topological space

In this section we introduce some basic definitions and theorems of soft topological spaces.

Definition 8. A soft topology τ on soft set F_A is a family of soft subsets of F_A satisfying the following properties
(i) $\Phi, F_A \in \tau$
(ii) If $G_B, H_C \in \tau$, then $G_B \sqcap H_C \in \tau$
(iii) If $F^{\alpha}_{A_\alpha} \in \tau$ for all $\alpha \in \Lambda$, an index set, then $\sqcup_{\alpha\in\Lambda} F^{\alpha}_{A_\alpha} \in \tau$
If τ is a soft topology on a soft set F_A, the pair (F_A, τ) is called the soft topological space.

Example 9. *Let* $E = \{e_1, e_2, e_3, e_4, e_5\}$, $A = \{e_1, e_2, e_3, e_4\}$ *and* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Let $F_A \in S(U)$ *where* $F_A(e_1) = \{1, 5, 8\}$, $F_A(e_2) = \{2, 6, 9\}$, $F_A(e_3) = \{3, 7, 9\}$, $F_A(e_4) = \{4, 7, 10\}$. *Now let us consider the collection*

τ={ $\Phi,\ F_A,$
$\{\ (e_2,\{2\})\ \},$
$\{\ (e_4,\{4\})\ \},$
$\{\ (e_1,\{1\}),\ (e_3,\{3\})\ \},$
$\{\ (e_2,\{2\}),\ (e_4,\{4\})\ \},$
$\{\ (e_2,\{2,9\}),\ (e_4,\{4,7\})\ \},$
$\{\ (e_1,\{1\}),\ (e_2,\{2\}),\ (e_3,\{3\})\ \},$
$\{\ (e_1,\{1\}),\ (e_3,\{3\}),\ (e_4,\{4\})\ \},$
$\{\ (e_1,\{1,5\}),\ (e_2,\{2,6\}),\ (e_3,\{3,7\})\ \},$
$\{\ (e_1,\{1,8\}),\ (e_3,\{3,9\}),\ (e_4,\{4,10\})\ \},$
$\{\ (e_1,\{1\}),\ (e_2,\{2\}),\ (e_3,\{3\}),\ (e_4,\{4\})\ \},$
$\{\ (e_1,\{1\}),\ (e_2,\{2,9\}),\ (e_3,\{3\}),\ (e_4,\{4,7\})\ \},$
$\{\ (e_1,\{1,8\}),\ (e_2,\{2\}),\ (e_3,\{3,9\}),\ (e_4,\{4,10\})\ \},$
$\{\ (e_1,\{1,5\}),\ (e_2,\{2,6\}),\ (e_3,\{3,7\}),\ (e_4,\{4\})\ \},$
$\{\ (e_1,\{1,8\}),\ (e_2,\{2,9\}),\ (e_3,\{3,9\}),\ (e_4,\{4,7,10\})\ \},$
$\{\ (e_1,\{1,5\}),\ (e_2,\{2,6,9\}),\ (e_3,\{3,7\}),\ (e_4,\{4,7\})\ \},$
$\{\ (e_1,\{1,5,8\}),\ (e_2,\{2,6\}),\ (e_3,\{3,7,9\}),\ (e_4,\{4,10\})\ \}\ \}$
of some soft subsets of F_A. Then obviously, τ forms a soft topology on a soft set F_A.

Definition 10. If τ is a soft topology on F_A, then the member of τ is called soft open sets in (F_A,τ).

Definition 11. A collection β of some soft subsets of F_A is called a soft open base or simply a base for some soft topology on F_A if the following conditions hold:
(i) $\Phi\in\beta$.
(ii) $\sqcup\beta= F_A$ i.e., for each $e\in A$ and $x\in F_A(e)$, there exists $G_B\in\beta$ such that $x\in G_B(e)$, where $B\subseteq A$.
(iii) If $G_B,H_C\in\beta$ then for each $e\in B\cap C$ and $x\in(G_B\sqcap H_C)(e)=G_B(e)\cap H_C(e)$ there exists $I_D\in\beta$ such that $I_D\sqsubseteq G_B\sqcap H_C$ and $x\in I_D(e)$, where $D\subseteq B\cap C$.

Example 12. *Let us consider the previous example 9 and we take*
β={ $\Phi,$
$\{\ (e_2,\{2\})\ \},$
$\{\ (e_4,\{4\})\ \},$
$\{\ (e_1,\{1\}),\ (e_3,\{3\})\ \},$
$\{\ (e_2,\{2,9\}),\ (e_4,\{4,7\})\ \},$
$\{\ (e_1,\{1,5\}),\ (e_2,\{2,6\}),\ (e_3,\{3,7\})\ \},$
$\{\ (e_1,\{1,8\}),\ (e_3,\{3,9\}),\ (e_4,\{4,10\})\ \}\ \}$
Then obviously, β forms a soft base for the topology τ on F_A.

Theorem 13. *Let β be a soft base for a soft topology on F_A. Suppose τ_β consists of those soft subset G_B of F_A for which corresponding to each $e\in B$ and $x\in G_B(e)$, there exists $H_C\in\beta$ such that $H_C\sqsubseteq G_B$ and $x\in H_C(e)$, where $C\subseteq B$. Then τ_β is a soft topology on F_A.*

Proof. We have $\Phi\in\tau_\beta$ by default.
Again by definition of soft base, we have for each $e\in A$ and $x\in F_A(e)$, there exists $G_B\in\beta$ such that $x\in G_B(e), B\subseteq A$. So $F_A\in\tau_\beta$.
Now let $G_B,H_C\in\tau_\beta$. Then for each $e\in B\cap C$ and $x\in G_B(e)\cap H_C(e)$, there exists $G^{'}_{B'},H^{'}_{C'}\in\beta$, where $B^{'}\subseteq B$ and $C^{'}\subseteq C$ such that $G^{'}_{B'}\sqsubseteq G_B$, $H^{'}_{C'}\sqsubseteq H_C$ and also $x\in G^{'}_{B'}(e), x\in H^{'}_{C'}(e)$.

Let $I_D = G'_{B'} \sqcap H'_{C'} \in \beta$, where $D = B' \cap C'$. Then obviously $x \in I_D(e)$.
Thus for $e \in B \cap C$ and $x \in (G_B \sqcap H_C)(e)$, there exists $I_D \in \beta$ such that $I_D \sqsubseteq G_B \sqcap H_C$ and $x \in I_D(e)$. So $G_B \sqcap H_C \in \tau_\beta$.
Again let $F^{\alpha}_{A_\alpha} \in \tau_\beta$, for all $\alpha \in \Lambda$, an index set.
Let $G_B = \sqcup_{\alpha \in \Lambda} F^{\alpha}_{A_\alpha}, e \in B$ and $x \in G_B(e)$, where $B = \cup_{\alpha \in \Lambda} A_\alpha$.
Then there exists $\alpha \in \Lambda$ such that $x \in F^{\alpha}_{A_\alpha}(e)$.
Since $F^{\alpha}_{A_\alpha} \in \tau_\beta$ and $x \in F^{\alpha}_{A_\alpha}(e)$, there exists $H_C \in \beta$ such that $H_C \sqsubseteq F^{\alpha}_{A_\alpha}$ and $x \in H_C(e)$ i.e., $H_C \sqsubseteq G_B$ and $x \in H_C(e)$.
Therefore $G_B \in \tau_\beta$ i.e., $\sqcup_{\alpha \in \Lambda} F^{\alpha}_{A_\alpha} \in \tau_\beta$
Hence τ_β is a soft topology on F_A. □

Definition 14. Suppose β is a soft base for a soft topology on F_A. Then τ_β, described in above theorem, is called the soft topology generated by β and β is called the soft base for τ_β.

Example 15. *Let $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_3\}$ and $U = \{1, 2, 3, 4\}$.*
Let $F_A \in S(U)$ where $F_A(e_i) = U$ for $i = 1, 2, 3$. Now let us consider the collection
β={ Φ, { $(e_1, \{1, 2\})$, $(e_2, \{2, 3\})$ }, {$(e_1, \{3, 4\})$, $(e_2, \{1\})$}, {$(e_2, \{4\})$, (e_3, U) } }
of some soft subsets of F_A. Then Obviously, β is a soft base for some soft topology on F_A. Then the soft topology generated by β is
τ_β={ Φ, F_A,
{ $(e_1, \{1, 2\})$, $(e_2, \{2, 3\})$ },
{$(e_1, \{3, 4\})$, $(e_2, \{1\})$},
{$(e_2, \{4\})$, (e_3, U) },
{ (e_1, U), $(e_2, \{1, 2, 3\})$ },
{ $(e_1, \{1, 2\})$, $(e_2, \{2, 3, 4\})$, (e_3, U) },
{ $(e_1, \{3, 4\})$, $(e_2, \{1, 4\})$, (e_3, U) } }

Theorem 16. *Let β be a soft base for a soft topology on F_A. Then $G_B \in \tau_\beta$ if and only if $G_B = \sqcup_{\alpha \in \Lambda} G^{\alpha}_{B_\alpha}$, where $G^{\alpha}_{B_\alpha} \in \beta$ for each $\alpha \in \Lambda$, Λ an index set.*

Proof. Since every member of β is also a member of τ_β, we have any union of members of β is a member of τ_β.
Conversely, let $G_B \in \tau_\beta$.
If $G_B = \Phi$, the proof of the theorem is obvious.
If G_B is not equal to Φ, for each $e \in B$ and $x \in G_B(e)$, there exists $X^e_{B^x_e} \in \beta$ with $B^x_e \subseteq B$ Such that $X^e_{B^x_e} \sqsubseteq G_B$ and $x \in X^e_{B^x_e}(e)$ [here we shall choose X corresponding to x].
Since each $B^x_e \subseteq B$, $\cup_{e \in B}\{\cup_{x \in G_B(e)} B^x_e\} \subseteq B$.
Again for each $e \in B$, there exists B^x_e such that $e \in B^x_e$. So $B \subseteq \cup_{e \in B}\{\cup_{x \in G_B(e)} B^x_e\}$.
Therefore $B = \cup_{e \in B}\{\cup_{x \in G_B(e)} B^x_e\}$.
Let $H_B = \sqcup_{e \in B}\{\sqcup_{x \in G_B(e)} X^e_{B^x_e}\}$
Now it is enough to show that $H_B = G_B$.
Obviously, $H_B \sqsubseteq G_B$ as each $X^e_{B^x_e} \sqsubseteq G_B$.
Let $a \in B$ and $y \in G_B(a)$. Then we have $Y^a_{B^y_a} \in \beta$ such that $Y^a_{B^y_a} \sqsubseteq G_B$ and $y \in Y^a_{B^y_a}(a)$.
Again since $Y^a_{B^y_a} \sqsubseteq H_B$. Therefore $y \in H_B(a)$
Thus $G_B(a) \subseteq H_B(a)$ for all $a \in B$. So, $G_B \sqsubseteq H_B$
Hence $H_B = G_B$. This completes the proof. □

Theorem 17. *Let (F_A, τ) be a soft topological space and β be a sub collection of τ such that every member of τ is a union of some members of β. Then β is a soft base for the soft topology τ on F_A.*

Proof. Since $\Phi \in \tau$, $\Phi \in\beta$.
Again since $F_A \in\tau$ and every member of β is a soft subset of F_A, $F_A =\sqcup\beta$.
Let $G_{A_1}, H_{A_2} \in\beta$. Then $G_{A_1}, H_{A_2} \in\tau\Rightarrow G_{A_1} \sqcap H_{A_2} \in\tau$.
Then there exist $B^{\alpha}_{C_\alpha} \in\beta, \alpha \in \Lambda$ such that $G_{A_1} \sqcap H_{A_2} = \sqcup\{B^{\alpha}_{C_\alpha} : \alpha \in \Lambda\}$
obviously each $B^{\alpha}_{C_\alpha} \sqsubseteq G_{A_1} \sqcap H_{A_2}$.
Now let $e \in A_1 \cap A_2$ and $x \in (G_{A_1} \sqcap H_{A_2})(e)$. Then $x \in \sqcup\{B^{\alpha}_{C_\alpha}(e) : \alpha \in \Lambda\}$.
So there exists $\alpha \in \Lambda$ such that $x \in B^{\alpha}_{C_\alpha}(e)$.
Therefore for each $e \in A_1 \cap A_2$ and $x \in (G_{A_1} \sqcap H_{A_2})(e)$, there exists $B^{\alpha}_{C_\alpha} \in\beta$ such that $B^{\alpha}_{C_\alpha} \sqsubseteq G_{A_1} \sqcap H_{A_2}$ and $x \in B^{\alpha}_{C_\alpha}(e)$.
Hence β is a soft base for the soft topology τ on F_A. □

Definition 18. A collection Ω of members of a soft topology τ is said to be subbase for τ if and only if the collection of all finite intersections of members of Ω is a base for τ.

Example 19. *Let us consider the previous example 12 and we take*
Ω={ { $(e_2, \{2, 9\}),\ (e_4, \{4, 7\})$ },
{ $(e_1, \{1, 5\}),\ (e_2, \{2, 6\}),\ (e_3, \{3, 7\})$ },
{ $(e_1, \{1, 8\}),\ (e_3, \{3, 9\}),\ (e_4, \{4, 10\})$ } }
Then obviously, β can be obtain by the collection of all intersections of members of Ω. So Ω is a subbase for τ.

Theorem 20. *A collection Ω of soft subsets of F_A is a subbase for a suitable soft topology τ on F_A if and only if*
(i) $\Phi \in \Omega$ or Φ is the intersection of a finite number of members of Ω.
(ii) $F_A = \sqcup\Omega$.

Proof. First let Ω is a subbase for τ and β be a base generated by Ω.
Since $\Phi \in\beta$, either $\Phi \in \Omega$ or Φ is expressible as an intersection of finitely many members of Ω.
Let $e \in A$ and $x \in F_A(e)$. Since $\sqcup\beta= F_A$, there exists $G_B \in\beta$ such that $x \in G_B(e)$.
Again since $G_B \in\beta$, there exists $S^i_{B_i} \in \Omega$, $i = 1, 2, \cdots, n$
such that $G_B = \sqcap^n_{i=1} S^i_{B_i}$
Therefore $x \in \cap^n_{i=1} S^i_{B_i}(e) \Rightarrow x \in S^i_{B_i}(e)$, for each $i = 1, 2, \cdots, n$
Hence $F_A = \sqcup\Omega$.
Conversely let Ω be a collection of some soft subsets of F_A satisfying the conditions (i) and (ii). Let β be the collection of all finite intersections of members of Ω. Now it is enough to show that β forms base for a suitable soft topology.
Since β is the collection of all finite intersections of members of Ω, by assumption (i) we get $\Phi \in\beta$ and by (ii) we get $\sqcup\beta= F_A$.
Again let $G_B, H_C \in\beta$
Since $G_B \in\beta$, there exist $G^i_{B_i} \in \Omega$, for $i = 1, 2, \cdots, n$ such that $G_B = \sqcap^n_{i=1} G^i_{B_i}$, where $B = \cap^n_{i=1} B_i$
Again since $H_C \in\beta$, there exists $H^j_{C_j} \in \Omega$, for $j = 1, 2, \cdots, m$ such that
$H_C = \sqcap^m_{j=1} H^j_{C_j}$, where $C = \cap^m_{j=1} C_j$
Therefore $G_B \sqcap H_C = (\sqcap^n_{i=1} G^i_{B_i}) \sqcap (\sqcap^m_{j=1} H^j_{C_j}) \in\beta$.
i.e., $G_B \sqcap H_C \in\beta$. This completes the proof. □

Acknowledgements
The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form.

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