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Magnetohydrodynamic Flow due to a Stretching Surface in Rotating Fluid

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Abstract. In this study, we investigate analytical treatment of three dimensional MHD viscous flow in a rotating frame. The flow is generated due to uniform stretching of a sheet in x direction. Analytic solution is obtained through peturbation method after transforming the governing partial differential equation to ordinary differential equation using suitable similarity transformations. Graphs are plotted to observe the effects of involved parameters on velocity profiles.

Key Words: Steady flow, MHD, rotating flow, stretching sheet, analytic solution.

1. Introduction

The increasing number of technical applications using megnetohydrodynamics (MHD) effects has made it desirable to extend many available hydrodynamic solutions to include the effects of magnetic fields in electrically conducting viscous fluids. For example, liquid-metal MHD flows take their roots in conventional hydrodynamics of incompressible media, which becomes important in metallurgical industry, nuclear reactor, sodium cooling system, energy storage and power generation [4, 7, 9]. The basic equations of incompressible MHD are nonlinear and there are many interesting cases where these equations become linear in terms of unknown quantities and may be solved without any difficulty. This discussion may be made interesting work in [2, 3, 5, 8, 10]. Hayat et al.[6] obtained the solution for MHD pipe flow for fourth grade fluid.

Since it is admitted that the viscous flow started due to a stretching sheet is essentially multidimensional and hence the governing equations are always non linear Navier-Stokes equations. In this way, the study of such flows possess a great challenge to the mathematicians while seeking analytic solution of these problems. The pioneering study in this regard was performed by Crane [10] who obtained a closed form solution to his problem. Crane [1] considered two dimensional problem by considering stretching of sheet in one direction. Later in 1984, Wang [11] investigated three dimensional flow due to a stretching sheet being stretched in two lateral directions, Further in 1988, Wang [12] studied three dimensional flow of viscous fluid over a stretching surface for the case when fluid and sheet rotate like a rigid body. In this case, Wang considered the wall stretching only in one direction but the problem was three dimensional due to the rotating frame assumption. Wang obtained perturbation solution to this problem by perturbing the parameter λ which is ratio of rotating rate to stretching rate. In this study, we execute the analysis for the case of electrically conducting fluids by imposing a uniform magnetic field perpendicular to the sheet. Analytic solution is obtained by following the same procedure as performed by Wang [12]. Effects of different involved parameters are observed through graphs. The paper is organized into five sections. Section 2 contains mathematical formulation of the problem; perturbation solution is given in section 3; discussion part is numbered as section 4 and finally the conclusion is given in section 5.

2. Development of the problem

We consider a stretching surface in rigid body rotation with fluid. The motion of the fluid is three dimensional due to Coriolis force. Let (u, v, w) be the velocity components in the direction of cartesian axes respectively with the axis rotating at an angular velocity Ω in the z-direction. The fluid is electrically conducting and magnetic field of strength B_0 is applied perpendicular to the plate.

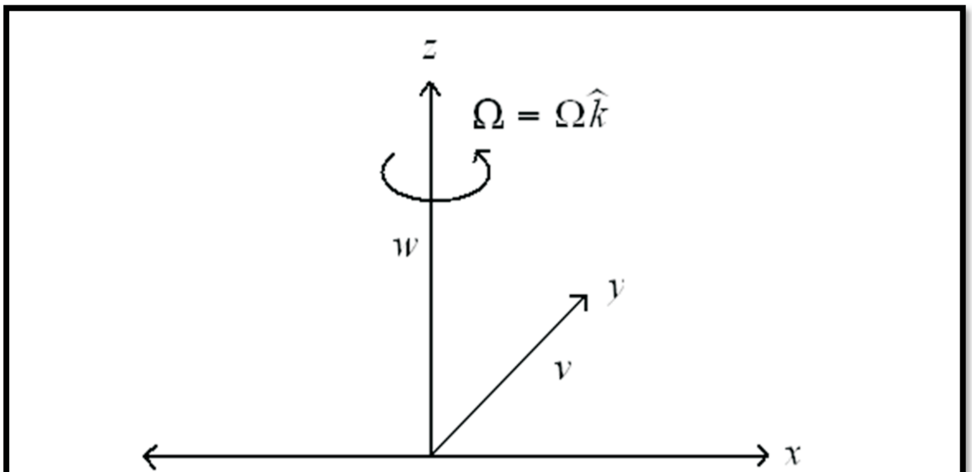

FIGURE 1. Geometry of problem

In this way, the velocity vector is chosen of the form:

$$V = [u(x, y, z), v(x, y, z), w(x, y, z)],$$

and the Navier-Stokes equation in rotating frame is given by (in vector notation):

$$\rho[\frac{dV}{dt} + 2\Omega \times V + \Omega \times (\Omega \times r)] = -\nabla p + \mu \nabla^2 V + J \times B,$$

and in component form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v = -\frac{1}{\rho}\hat{p_x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) - \frac{\sigma B_0^2 u}{\rho}, \quad (2.1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u = -\frac{1}{\rho}\hat{p_y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) - \frac{\sigma B_0^2 v}{\rho}, \quad (2.2)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\hat{p_z} + \nu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}), \quad (2.3)$$

where $\hat{p} = p - (\frac{1}{2})\rho\Omega^2 r^2$ is the modified pressure and the continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (2.4)$$

Let the surface (at z=0)be stretched in x-direction such that the velocities at boundary are defined as

$$u = ax, v = w = 0, \quad (2.5)$$

where 'a' has the dimensions $[T^{-1}]$ representing the stretching rate. We introduce the dimensionless quantities defined by

$$u = axf'(\eta), v = axh(\eta), w = -\sqrt{a\nu}f(\eta), \eta = z\sqrt{\frac{a}{\nu}}. \quad (2.6)$$

Due to Eq. (2.6), Eq. (2.4) is satisfied identically and Eqs. (2.1) - (2.5) reduce to ordinary differential equations

$$(f')^2 - ff'' - 2\lambda h = f''' - Kf', \quad (2.7)$$

$$f'h - fh' + 2\lambda f' = h'' - Kh, \quad (2.8)$$

subject to the boundary conditions

$$f'(0) = 1, h(0) = 0, f(0) = 0, \quad (2.9)$$

$$h(\infty) = 0, f'(\infty) = 0. \quad (2.10)$$

where $\lambda = (\Omega/a)$ is an important non dimensional parameter signifying the relative importance of rotation rate to stretching rate and $K = \frac{\sigma B_0^2}{a\rho}$

3. SOLUTION OF THE PROBLEM

We use the perturbation technique to solve our problem. Perturbation solutions are obtained by assuming the parameter λ as small or large parameter.

Perturbation for small λ

The problem given by Eqs. (2.7) - (2.10) constitutes a nonlinear problem with no exact analytic solution. For small λ, one may perturb the solution as follows:

$$f = f_0 + O(\lambda^2), h = \lambda h_1 + O(\lambda^3). \quad (3.1)$$

Substituting Eq.(3.1) into Eqs.(2.7) - (2.10), we get zeroth order and first order problems for which the solutions are given below:

Zeroth order solution

$$f_0 = \frac{1 - e^{-(\sqrt{1+K})\eta}}{\sqrt{1+K}} \tag{3.2}$$

First order solution

$$h_1 = -\frac{2}{2+K}\eta e^{-(1-\frac{k}{2})\eta}, \tag{3.3}$$

In this way, Eq. (3.1) becomes

$$f = \frac{1 - e^{-(\sqrt{1+K})\eta}}{\sqrt{1+K}} + O(\lambda^2), \tag{3.4}$$

$$h = \lambda[-\frac{2}{2+K}\eta e^{-(1-\frac{k}{2})\eta}] + O(\lambda^3). \tag{3.5}$$

Perturbation for large λ

The problem becomes singular when λ is large. We set $\lambda = (1/(\epsilon^2))$ and stretch the boundary layer by introducing new scallings:

$$H = \epsilon\zeta. \tag{3.6}$$

$$f = \epsilon F_0(\zeta) + \epsilon^3 F_1(\zeta) + O(\epsilon^5), \tag{3.7}$$

$$h = H_0(\zeta) + \epsilon^2 H_1(\zeta) + O(\epsilon^4), \tag{3.8}$$

Using Eqs.(3.6)-(3.8) in Eqs. (2.7)-(2.10) and solving the resulting equations, we find that

$$F_0 = \frac{1}{2} + \frac{1}{2}(sin\zeta - cos\zeta)e^{-\zeta}, \tag{3.9}$$

and

$$H_1 = -\frac{1}{2}e^{-\zeta}sin\zeta. \tag{3.10}$$

First order system

The higher order problems are more complicated, so we let

$$\Phi = H_1 + F_1^{'} \tag{3.11}$$

$$\Phi^{''} + 2i\Phi = F_0^{'}(H_0 + iF_0^{'}) - F_0(H_0^{'} + iF_0^{''}) + K(H_0 + F_0^{'}). \tag{3.12}$$

Using the values of F_0 , H_0 and their derivatives in above equation, we get

$$\Phi^{''} + 2i\Phi = \frac{i-1}{2}e^{-2\zeta} + (\frac{i+1}{2} + iK)e^{-(1-i)\zeta}. \tag{3.13}$$

with the boundary conditions

$$\Phi(0) = 0, \Phi(\infty) = 0. \tag{3.14}$$

Solving Eqs. (3.13) and (3.14), we find

$$\Phi = \frac{3i-1}{20}(e^{-2\zeta} - e^{-(1-i)\zeta} - \frac{i}{4}e^{-(1-i)\zeta}\zeta - \frac{i-1}{2}Ke^{-(1-i)\zeta}\zeta). \tag{3.15}$$

Since

$$\Phi = H_1 + iF_1^{'} \tag{3.16}$$

so

$$F_1^{'} = Im(\Phi), \tag{3.17}$$

$$F_1 = Im\int_0^{\zeta}\Phi d\zeta, \tag{3.18}$$

In this way, the above equation becomes

$$F_1 = Im\int_0^{\zeta}(\frac{3i-1}{20}(e^{-2\zeta} - e^{-(1-i)\zeta} - \frac{i}{4}e^{-(1-i)\zeta}\zeta - \frac{i-1}{2}Ke^{-(1-i)\zeta}\zeta))d\zeta, \quad (3.19)$$

$$F_1 = Im[\frac{3i-1}{40}e^{-2\zeta} + (1+i)e^{-(1-i)\zeta} + i + (\frac{i}{8} + \frac{i-1}{4}K)$$
$$(1+i)\zeta e^{-(1-i)\zeta} + ie^{-(1-i)\zeta} - i]. \quad (3.20)$$

Thus for large λ, we have

$$f = \epsilon F_0(\zeta) + \epsilon^3 F_1(\zeta) + O(\epsilon^5), \quad (3.21)$$

$$f^{'} = F_0^{'}(\zeta) + \epsilon^2 F_1^{'}(\zeta), \quad (3.22)$$

$$f^{''} = (F_0^{''}(\zeta)/\epsilon) + \epsilon F_1^{''}(\zeta). \quad (3.23)$$

so

$$f_0^{''}(0) = \frac{F_0^{''}}{\epsilon} + F_1^{''}(0), \quad (3.24)$$

$$F_0^{''}(0) = -1. \quad (3.25)$$

$$F_1 = Im[\frac{3i-1}{40}-e^{-2\zeta} + (1+i)e^{-(1-i)\zeta} + i(\frac{i}{8} + \frac{i-1}{4}k)$$
$$(1+i)\zeta e^{-(1-i)\zeta} + ie^{-(1-i)\zeta} - i], \quad (3.26)$$

$$F_1^{'} = Im[\frac{1-3i}{40}-2e^{-2\zeta} - (1-i^2)e^{-(1-i)\zeta} + +(\frac{i}{8} + \frac{i-1}{4}K)$$
$$(1+i)\zeta e^{-(1-i)\zeta} - (1-i^2)e^{-(1-i)\zeta}] \quad (3.27)$$

$$F_1^{''} = Im[\frac{1-3i}{40}4e^{-2\zeta} + 2(1-i)e^{-(1-i)\zeta} + (\frac{i}{8} + \frac{i-1}{4}K)$$
$$-4e^{-(1-i)\zeta} + 2(1-i)\zeta e^{-(1-i)\zeta} + 2e^{-(1-i)\zeta}], \quad (3.28)$$

$$F_1^{''}(0) = Im[\frac{8-14i-20iK+20K}{40}] = -\frac{7+10K}{20}. \quad (3.29)$$

Using Eqs. (3.25) and (3.29) in Eq. (3.24), we get

$$f^{''}(0) = \frac{-1}{\epsilon} - \frac{7+10K}{20}\epsilon + O(\epsilon^3) \quad (3.30)$$

$$= -\sqrt{\lambda} - \frac{7+10}{20\sqrt{\lambda}} + O(\lambda^{-\frac{3}{2}}) \quad (3.31)$$

Further, since

$$H_0 = -sin\zeta\ e^{-\zeta}, \quad (3.32)$$

therefore

$$H_0^{'}(0) = -1 \quad (3.33)$$

and

$$H_1(\zeta) = Re[\frac{3i-1}{20}(e^{-2\zeta} - e^{-(1-i)\zeta} - \frac{i}{4}e^{-(1-i)\zeta}\zeta - \frac{i-1}{2}Ke^{-(1-i)\zeta}\zeta)], \quad (3.34)$$

therefore,

$$H_1^1(\zeta) = Re[\frac{3i-1}{20} - 2e^{-2\zeta} + (1-i)e^{-(1-i)\zeta} - (\frac{i}{4} + \frac{i-1}{2}K)$$
$$e^{-(1-i)\zeta} - (1-i)e^{-(1-i)\zeta}\zeta], \quad (3.35)$$

And consequently,

$$H_1^{'}(0) = \frac{4+10K}{20}. \quad (3.36)$$

From Eq. (3.7), we have

$$f(\infty) = \epsilon F_0(\infty) + \epsilon^3 F_1(\infty) + O(\epsilon^5), \quad (3.37)$$

and due to Eqs.(3.9) and (3.26), we respectively have

$$F_0(\infty) = \frac{1}{2}. \quad (3.38)$$

and

$$F_1(\infty) = \frac{1+10K}{40}. \quad (3.39)$$

Using Eqs. (3.38) and (3.39) in Eq. (3.37), we get

$$f(\infty) = \frac{1}{2}\epsilon + \frac{1+10K}{40}\epsilon^3 + O(\epsilon^5), \quad (3.40)$$

$$= \frac{1}{2\sqrt{\lambda}} + \frac{1+10K}{40\lambda\sqrt{\lambda}} + O(\lambda^{-\frac{5}{2}}). \quad (3.41)$$

From Eq. (3.8), we have

$$h = H_0(\zeta) + \epsilon^2 H_1(\zeta) + O(\epsilon^3), \quad (3.42)$$

$$h^{'} = \frac{H_0^{'}(\zeta)}{\epsilon} + \epsilon H_1^{'}(\zeta) + O(\epsilon^3), \quad (3.43)$$

$$h^{'}(0) = \frac{H_0^{'}(0)}{\epsilon} + \epsilon H_1^{'}(\zeta) + O(\epsilon^3), \quad (3.44)$$

Using Eqs. (3.33) and (3.36) in Eq. (3.44), we get

$$h^{'}(0) = \frac{-1}{\epsilon} + \frac{4+10K}{20}\epsilon + O(\epsilon^3), \quad (3.45)$$

$$= -\sqrt{\lambda} - \frac{4+10K}{20\sqrt{\lambda}} + O(\lambda^{-\frac{3}{2}}). \quad (3.46)$$

From Eq. (3.20), we have

$$F_1 = \frac{1}{40}[(-9sin\zeta + 2cos\zeta)e^{-\zeta} - 3e^{-2\zeta} - 5\zeta e^{-\zeta}(sin\zeta - \cos\zeta)$$
$$+20K(-\zeta e^{-\zeta}sin\zeta - \frac{e^{-5}}{2}(sin\zeta + \cos\zeta) + 1 + 10K)] \quad (3.47)$$

and from Eq. (3.15), we have

$$H_1 = Re\Phi = \frac{1}{20}[(\cos\zeta + 3sin\zeta)e^{-\zeta} - e^{-2\zeta} + 5e^{-\zeta}\zeta sin\zeta + 10Ke^{-\zeta}(sin\zeta + \cos\zeta)]. \quad (3.48)$$

Using Eqs.(3.9) and (3.47) in Eq.(3.7), we reach the final expressions:

$f = \epsilon[\frac{1}{2} + \frac{1}{2}(\sin\zeta - \cos\zeta)e^{-\zeta}] + \frac{\epsilon^3}{40}[(-9\sin\zeta + 2\cos\zeta)e^{-\zeta} - 3e^{-2\zeta} - 5\zeta e^{-\zeta}(\sin\zeta - \cos\zeta) + 20K(-\zeta e^{-\zeta}\sin\zeta - \frac{e^{-\zeta}}{2}(\sin\zeta + \cos\zeta)) + 1 + 10K] + O(\epsilon^5), f \frac{1}{\sqrt{\lambda}}[\frac{1}{2} + \frac{1}{2}(\sin\zeta -$

$\cos\zeta)e^{-\zeta}]+\frac{\epsilon^3}{40}[(-9\sin\zeta+2\cos\zeta)e^{-\zeta}-3e^{-2\zeta}-5\zeta e^{-\zeta}(\sin\zeta-\cos\zeta)+20K(-\zeta e^{-\zeta}\sin\zeta-\frac{e^{-\zeta}}{2}(\sin\zeta+\cos\zeta))+1+10K]+O(\epsilon^5)$, $f^{'}=\cos\zeta e^{-\zeta}+\frac{1}{20\lambda}[(\sin\zeta-\cos\zeta)e^{-\zeta}+3e^{-2\zeta}$

$$-5\zeta e^{-\zeta}\cos\zeta+10Ke^{-\zeta}(\sin\zeta-\cos\zeta)]+O(\lambda^2). \quad (3.49)$$

Using Eqs. (3.32) and (3.48) in Eq. (3.8), we get

$$h=-\frac{1}{2}e^{-\zeta}\sin\zeta+\epsilon^2\frac{1}{20}[(\cos\zeta+3\sin\zeta)e^{-\zeta}-e^{-2\zeta}+5e^{-\zeta}\zeta\sin\zeta$$
$$+10Ke^{-\zeta}(\sin\zeta+\cos\zeta)O(\epsilon)], \quad (3.50)$$
$$h^{'}=\sqrt{\lambda}\frac{1}{2}e^{-\zeta}(\sin\zeta-\cos\zeta)+\frac{1}{20\lambda}[(\sin\zeta-\cos\zeta)e^{-\zeta}+3e^{-2\zeta}-5\zeta e^{-\zeta}\cos\zeta$$
$$+10Ke^{-\zeta}(\sin\zeta-\cos\zeta)]+O(\epsilon^{-\frac{3}{2}}). \quad (3.51)$$

4. DISCUSSION

In order to investigate the effects of magnetic field on the velocity profiles, graphs are plotted in Figs. 2-9. It is noticed that at $K = 0$ (in the absence of magnetic field), the results of Wang [12] are recovered. In Figs 2-4, the velocity $f(\eta)$ is plotted against η for varying values of λ and finding K at different values. From all these figures, it is observed that by increasing the values of λ, velocity f decreases and by increasing the value of the magnetic parameter K, the velocity $f(\eta)$ increases at the boundary and oscillates in increasing direction of η. The x component of velocity, namely,$f^{'}(\eta)$ is plotted against η for different values of λ and K in Figs. 5 and 6. From here, it is observed that by increasing the values of λ the velocity decreases and as a consequence, the layer thickness also decreases. In Figs 7-9 the velocity function $h(\eta)$ is plotted for different values of λ and finding K at different values. Clearly, by increasing values of λ, boundary layer thickness decreases. The effect of magnetic field is observed to control the reverse flow and for large values of the parameter K, the velocity increases at the stretching surface.

5. CONCLUSION

In this study, the perturbation solution is obtained for steady flow of incompressible viscous fluid over a stretching in a rotating frame of reference. Solutions of small and large values of λ are computed separately. Effect of magnetic field on velocity profiles is studied through graphs. It is observed that by increasing magnetic field, the boundary layer thickness decreases. It is also noted that the strong magnetic field depreciates the reverse flow.

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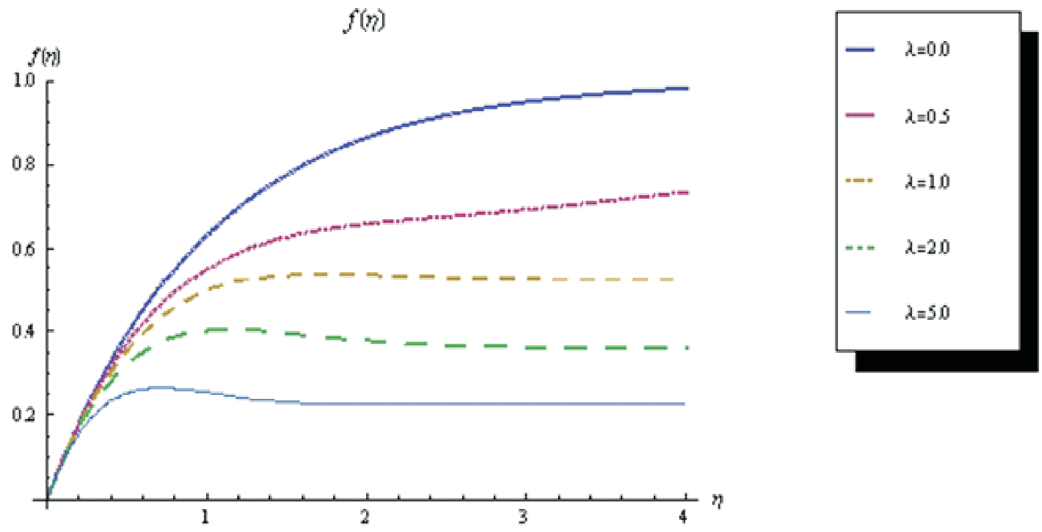

FIGURE 2. *The variation of* λ *and* K *on* $f(\eta)$, $K = 0.0$

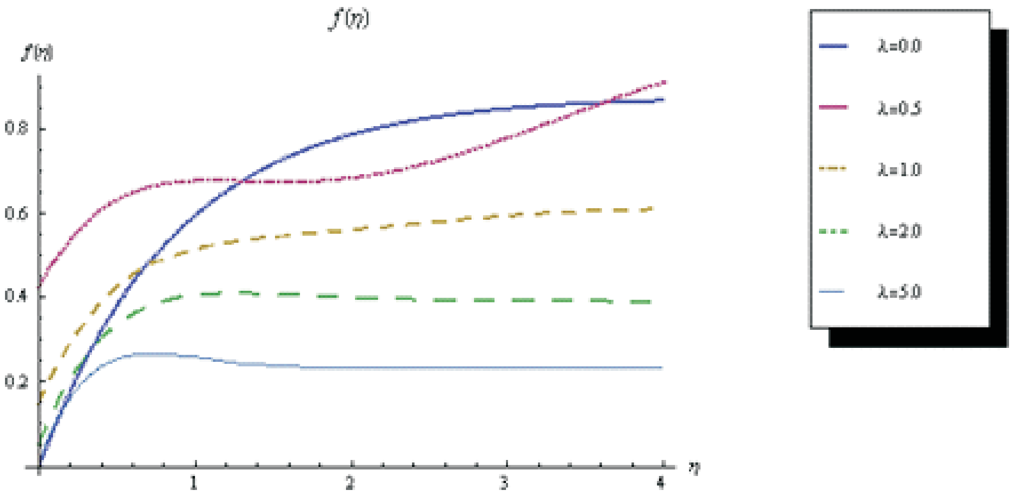

FIGURE 3. *The variation of* λ *and* K *on* $f(\eta)$, $K = 0.1$

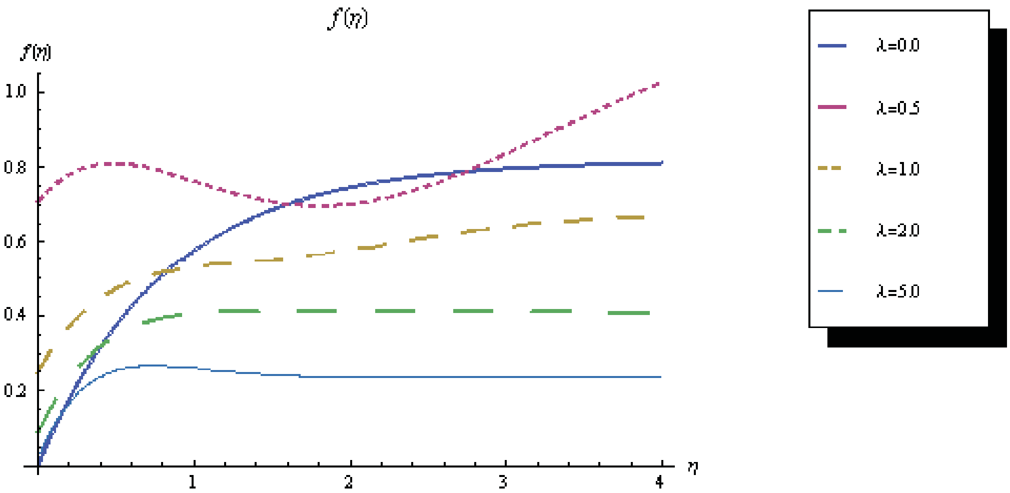

FIGURE 4. *The variation of* λ *and* K *on* $f(\eta)$, $K = 0.3$

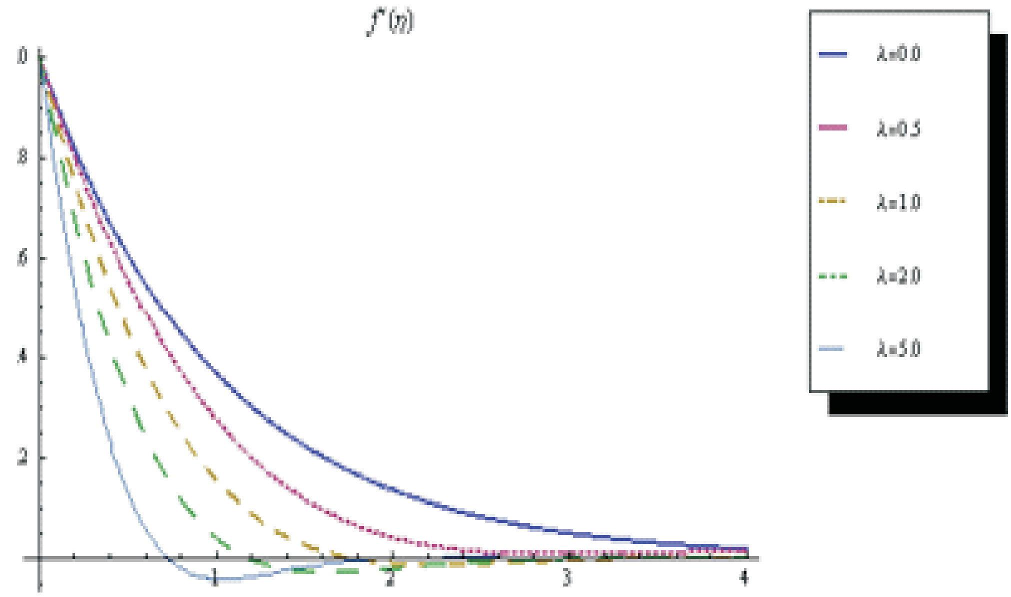

FIGURE 5. *The variation of* λ *and* K *on* $f^{'}(\eta)$, $K = 0.0$

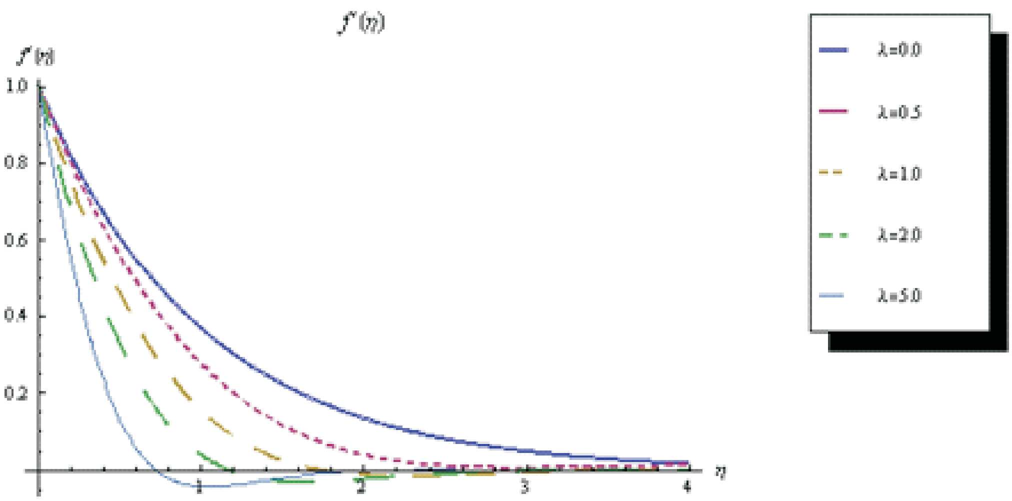

FIGURE 6. *The variation of* λ *and* K *on* $f^{'}(\eta)$, $K = 0.5$

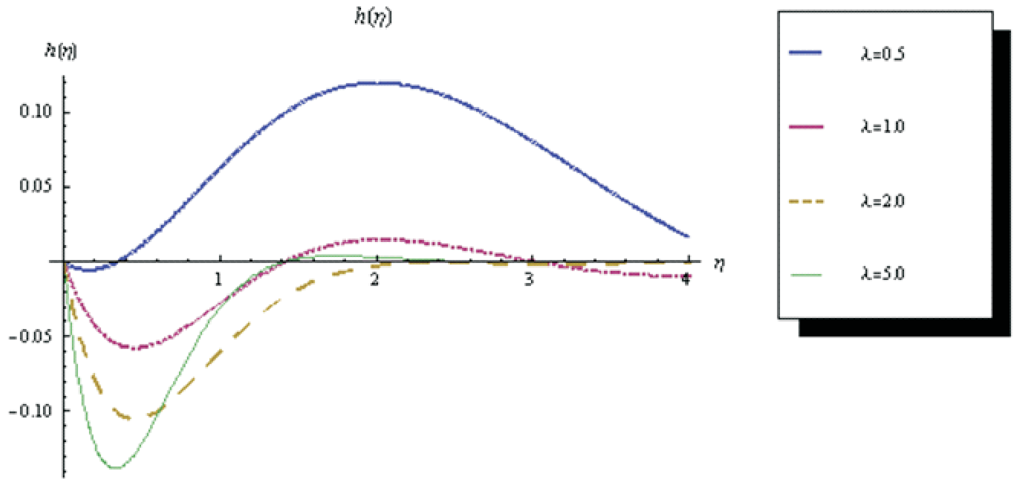

FIGURE 7. *The variation of* λ *and* K *on* $h(\eta)$, $K = 0.0$

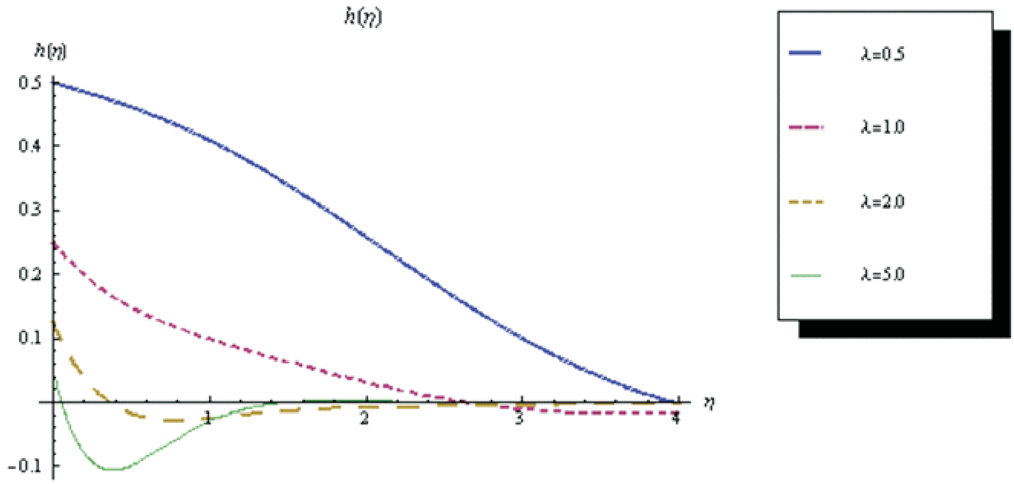

FIGURE 8. *The variation of* λ *and* K *on* $h(\eta)$, $K = 0.1$

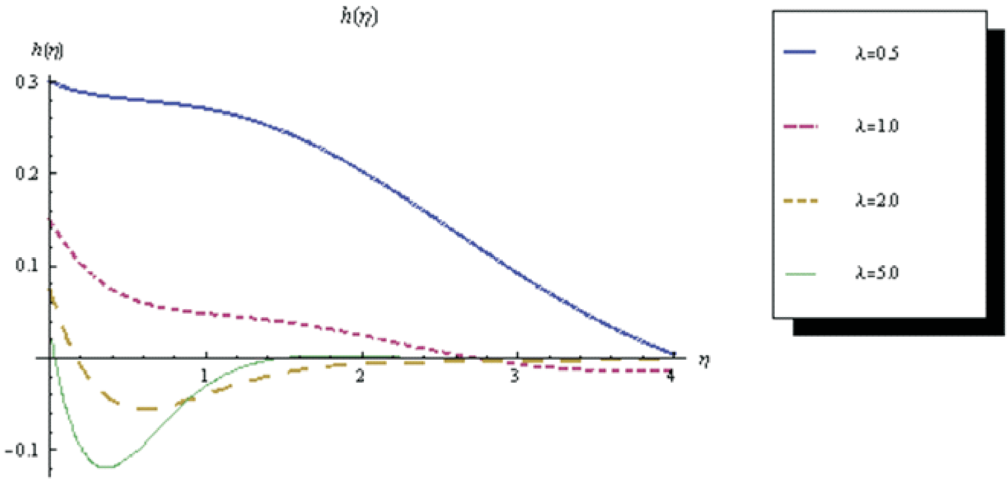

FIGURE 9. *The variation of* λ *and* K *on* $h(\eta)$, $K = 0.3$