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On Slightly Omega Continuous Multifunctions

C. Carpintero
Departamento de Matemáticas
Universidad de Oriente
Cumaná, Venezuela
Facultad de Ciencias Básicas
Universidad del Atlántico
Barranquilla, Colombia
Email: carpintero.carlos@gmail.com

N. Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Tamilnadu, India
Email: nrajesh_topology@yahoo.co.in

E. Rosas
Departamento de Matemáticas
Universidad de Oriente
Cumaná, Venezuela
Facultad de Ciencias Básicas
Universidad del Atlántico
Barranquilla, Colombia
Email: ennisrafael@gmail.com

S. Saranyasri
Department of Mathematics
M. R. K. Institute of Technology
Tamilnadu, India
Email: srisaranya_2010@yahoo.com

Abstract. The purpose of this paper is to introduce and study a new generalization of ω-continuous multifunction called slightly ω-continuous multifunctions in topological spaces.

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1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions.

This implies that both, functions and multifunctions are important tools for studying other properties of spaces and constructing new spaces from previously existing ones. Recently, Zorlutuna introduced the concept of ω-continuous multifunctions [12], ω-continuity which is a weaker form of continuity in ordinary was extended to multifunctions. In 2006 and 2009, Noiri et. al. [10] (respectively [8]) introduced and studied slightly m-continuous multifunctions (respectively slightly ω-continuous functions). In this paper, introduce a new generalization of ω-continuous multifunction called slightly ω-continuous multifunctions in topological spaces.

2. PRELIMINARIES

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [7]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [7] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X,τ), denoted by $\omega O(X,\tau)$, forms a topology on X finer than τ. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X,x) = \{A : A \in \omega O(X)$ and $x \in A\}$. A subset U of X is called a ω-neighborhood of a point $x \in X$ if there exists $V \in \omega O(X,x)$ such that $V \subset U$. By a multifunction $F : (X,\tau) \to (Y,\sigma)$, following [3], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\}$$

and

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$$

In particular, $F^-(Y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \cup_{x\in A}F(x)$. Then F is said to be surjection if $F(X) = Y$. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be surjective if $F(X) = Y$. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be lower ω-continuous [12] (resp. upper ω-continuous) multifunction if $F^-(V) \in \omega O(X)$ (resp. $F^+(V) \in \omega O(X)$) for every $V \in \sigma$.

Definition 1. A topological space (X,τ) is said to be ω-T_2 [2] if for each pair of distinct points x and y in X, there exist disjoint ω-open sets U and V in X such that $x \in U$ and $y \in V$.

Definition 2. A multifunction $F : X \to Y$ is said to be [12]:

(1) upper ω-continuous if for each point $x \in X$ and each open set V containing $F(x)$, there exists $U \in \omega O(X,x)$ such that $F(U) \subset V$;
(2) lower ω-continuous if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X,x)$ such that $U \subset F^-(V)$.

3. Slightly ω-continuous multifunctions

Definition 3. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

(1) upper slightly ω-continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \omega O(X)$ containing x such that $F(U) \subset V$;
(2) lower slightly ω-continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \omega O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;
(3) upper (lower) slightly ω-continuous if it has this property at each point of X.

Remark 4. It is clear that every upper ω-continuous multifunction is upper slightly ω-continuous. But the converse is not true in general, as the following example shows.

Example 5. *Let $X = \mathrm{Re}$ with the topology $\tau = \{\emptyset, \mathrm{Re}, \mathrm{Re} - Q\}$. Define a multifunction $F : (\mathrm{Re}, \tau) \to (\mathrm{Re}, \tau)$ as follows:*

$$F(x) = \begin{cases} Q & \text{if } x \in \mathrm{Re} - Q \\ \mathrm{Re} - Q & \text{if } x \in Q. \end{cases}$$

Then F is upper slightly ω-continuous but is not upper ω-continuous.

Theorem 6. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *F is upper slightly ω-continuous;*
(2) *For each $x \in X$ and for each clopen set V such that $x \in F^+(V)$, there exists an ω-open set U containing x such that $U \subset F^+(V)$;*
(3) *For each $x \in X$ and for each clopen set V such that $x \in F^+(Y \backslash V)$, there exists an ω-closed set H such that $x \in X \backslash H$ and $F^-(V) \subset H$;*
(4) *$F^+(V)$ is an ω-open set for any clopen set V of Y;*
(5) *$F^-(V)$ is an ω-closed set for any clopen set V of Y;*
(6) *$F^-(Y \backslash V)$ is an ω-closed set for any clopen set V of Y;*
(7) *$F^+(Y \backslash V)$ is an ω-open set for any clopen set V of Y.*

Proof. (1)⇔(2): Follows from Theorem 3.9 and Definition 3.4 of [10].
(2)⇔(3): Let $x \in X$ and V be a clopen set of Y such that $x \in F^+(Y \backslash V)$. By (ii), there exists an ω-open set U containing x such that $U \subset F^+(Y \backslash V)$. Then $F^-(V) \subset X \backslash U$. Take $H = X \backslash U$. We have $x \in X \backslash H$ and H is ω-open. The converse is similar.
(1)⇔(4): Let $x \in F^+(V)$ and V be a clopen set of Y. By (1), there exists an ω-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$. Since any union of ω-open sets is ω-open, $F^+(V)$ is ω-open. The converse can be shown similarly.
(4)⇔(5): Follows from Theorem 3.9 and Definition 3.4 of [10].
(5)⇔(6)⇔(7) : Clear. □

Definition 7. [8] A function $f : X \to Y$ is called:

(1) slightly ω-continuous at $x \in X$ if for each clopen set V in Y containing $f(x)$, there exists an ω-open set U in X containing x such that $f(U) \subset V$;
(2) slightly ω-continuous if it has this property at each point of X.

Example 8. *Let Re be the real numbers, take three topologies on Re as τ_U, τ_D and τ_I, where τ_U is the usual topology, τ_D the discrete topology and τ_I the indiscrete topology. Let $f : (\mathrm{Re}, \tau_U) \to (\mathrm{Re}, \tau_D)$ and $g : (\mathrm{Re}, \tau_U) \to (\mathrm{Re}, \tau_I)$ be the identity functions. g is slightly ω-continuous but f is not slightly ω-continuous.*

Corollary 9. [8] *Let (X, τ) and (Y, σ) be topological spaces. The following statements are equivalent for a function $f : X \to Y$:*

(1) *f is slightly ω-continuous;*
(2) *for every clopen set $V \subset Y$, $f^{-1}(V)$ is ω-open;*
(3) *for every clopen set $V \subset Y$, $f^{-1}(V)$ is ω-closed.*

Theorem 10. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent :*

(1) *F is lower slightly ω-continuous;*
(2) *For each $x \in X$ and for each clopen set V such that $X \in F^-(V)$, there exists an ω-open set U containing x such that $U \subset F^-(V)$;*
(3) *For each $x \in X$ and for each clopen set V such that $x \in F^-(Y \backslash V)$, there exists an ω-closed set H such that $x \in X \backslash H$ and $F^+(V) \subset H$;*
(4) *$F^-(V)$ is an ω-open set for any clopen set V of Y;*
(5) *$F^+(V)$ is an ω-closed set for any clopen set V of Y;*
(6) *$F^+(Y \backslash V)$ is an ω-closed set for any clopen set V of Y;*
(7) *$F^-(Y \backslash V)$ is an ω-open set for any clopen set V of Y.*

Proof. The proof is similar to that of Theorem 6. □

Lemma 11. [1] *Let A and B be subsets of a topological space (X, τ). If $A \in \omega O(X)$ and $B \in \tau$, then $A \cap B \in \omega O(B)$;*

Theorem 12. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and U be an open subset of X. If F is a lower (upper) slightly ω-continuous multifunction, then multifunction $F_{|_U} : U \to Y$ is a lower (upper) slightly ω-continuous multifunction.*

Proof. Let V be any clopen subset of Y, $x \in U$ and $x \in F^-_{|_U}(V)$. Since F is lower slightly ω-continuous multifunction, it follows that there exists an ω-open set G containing x such that $G \subset F^-(V)$. By Lemma 11, we have $x \in G \cap U \in \omega O(U)$ and $G \cap U \subset F^-_{|_U}(V)$. This shows that the restriction multifunction $F_{|_U}$ is a lower slightly ω-continuous. The proof of the upper slightly ω-continuity of $F_{|_U}$ can be done by a similar manner. □

Corollary 13. *If $f : (X, \tau) \to (Y, \sigma)$ is slightly ω-continuous and A is an open subset of X, then the restriction $f_{|_A}$: $A \to Y$ is slightly ω-continuous.*

Lemma 14. [9] *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following holds:*

(1) *$G_F^+(A \times B) = A \cap F^+(B)$;*
(2) *$G_F^-(A \times B) = A \cap F^-(B)$*

for any subset A of X and B of Y.

Theorem 15. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction. If the graph multifunction of F is an upper slightly ω-continuous, then F is an upper slightly ω-continuous.*

Proof. Suppose that $x \in X$ and V any clopen subset of Y such that $x \in F^+(V)$. We obtain that $x \in G_F^+(X \times V)$ and that $X \times V$ is a clopen set. Since the graph multifunction G_F is upper slightly ω-continuous, it follows that there exists an ω-open set U of X containing x such that $U \subset G_F^+(X \times V)$. Since $U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$. We obtain that $U \subset F^+(V)$. Thus, F is upper slightly ω-continuous. □

Theorem 16. *If $G_F : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is lower slightly ω-continuous, then $F : (X, \tau) \to (Y, \sigma)$ is lower slightly ω-continuous multifunction.*

Proof. Suppose that G_F is lower slightly ω-continuous. Let $x \in X$ and V be any clopen set of Y such that $x \in F^-(V)$. Then $X \times V$ is clopen in $X \times Y$ and $G_F(x) \cap (X \times V)$ = $(\{x\} \times F(x)) \cap (X \times V)$ = $\{x\} \times (F(x) \cap V) \neq \varnothing$. Since G_F is lower slightly ω-continuous, there exists an ω-open U containing x such that $U \subset G_F^-(X \times V)$; hence $U \subset F^-(V)$. This shows that F is lower slightly ω-continuous. □

Theorem 17. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha))$ = $\{x_\alpha\}$. If F is an upper (lower) slightly ω-continuous multifunction, then $P_\alpha \circ F$ is an upper (lower) slightly ω-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a clopen set in $(X_{\alpha 0}, \tau_{\alpha 0})$. Then

$$(P_{\alpha 0} \circ F)^+(V_{\alpha 0}) = F^+(P_{\alpha 0}^+(V_{\alpha 0})) = F^+(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$$

(resp. $(P_{\alpha 0} \circ F)^-(V_{\alpha 0})$ = $F^-(P_{\alpha 0}^-(V_{\alpha 0}))$ = $F^-(V_{\alpha 0} \times \prod_{\alpha \neq \alpha 0} X_\alpha))$. Since F is an upper (lower) slightly ω-continuous multifunction and since $V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a clopen set, it follows that $F^+(V_{\alpha 0} \times \prod_{\alpha \neq 0} X_\alpha)$ (resp. $F^-(V_{\alpha 0} \times \prod_{\alpha \neq \alpha 0} X_\alpha)$) is an ω-open set in (X, τ). This shows that $P_{\alpha 0} \circ F$ is an upper (lower) slightly ω-continuous multifunction. Hence, we obtain that $P_\alpha \circ F$ is an upper (lower) slightly ω-continuous multifunction for each $\alpha \in J$. □

Theorem 18. *Let $(X, \tau), (Y, \sigma), (Z, \eta)$ be a topological spaces and multifunctions F_1: $X \to Y$, F_2: $X \to Z$. Let $F_1 \times F_2$: $X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x)$ = $F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) slightly ω-continuous multifunction, then F_1 and F_2 are upper (lower) slightly ω-continuous multifunctions.*

Proof. Let $x \in X$, $K \subset Y$ and $H \subset Z$ be clopen sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and thus, $F_1(x) \times F_2(x)$ = $(F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper slightly ω-continuous multifunction, it follows that there exists an ω-open set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, F_1 and F_2 are upper slightly ω-continuous multifunction. The proof of the lower slightly ω-continuity of F_1 and F_2 is similar to the above. □

Definition 19. [5] Let (X, τ) be a topological space. X is said to be a strongly normal space if for every disjoint closed subsets K and F of X, there exist two clopen sets U and V such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

Recall that a multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be punctually closed if for each $x \in X$, $F(x)$ is closed.

Theorem 20. *Let $F : (X, \tau) \to (Y, \sigma)$ be a punctually closed from a topological space X to a strongly normal space Y an upper slightly ω-continuous multifunction and let $F(x) \cap F(y)$ = $\varnothing$ for each pair of distinct points x and y of X. Then X is an ω-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y)$ = $\varnothing$. Since Y is strongly normal, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint ω-open sets containing x and y, respectively and hence (X, τ) is ω-T_2. □

Definition 21. [11] [2] A topological space (X, τ) is said to be mildly compact (resp. ω-compact) if every clopen (resp. ω-open) cover of X has a finite subcover.

Theorem 22. *Let $F : (X, \tau) \to (Y, \sigma)$ be a surjective an upper slightly ω-continuous multifunction such that $F(x)$ is mildly compact for each $x \in X$. If X is ω-compact space, then Y is mildly compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a clopen cover of Y. Since $F(x)$ is mildly compact for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \cup\{V_\alpha\colon \alpha \in \Lambda(x)\}$. Put $V(x) = \cup\{V_\alpha\colon \alpha \in \Lambda(x)\}$. Since F is an upper slightly ω-continuous, there exists an ω-open set $U(x)$ of X containing x such that $F(U(x)) \subset V(x)$. Then $\{U(x) : x \in X\}$ is an ω-open cover of X and since X is ω-compact, there exists a finite number of points, say, x_1, x_2, x_3,... x_n in X such that

$$X = \cup\{U(x_i) : i = 1, 2, ..., n\}$$

Hence we have

$$Y = F(X) = F(\overset{n}{\underset{i=1}{\cup}} U(x_i)) = \overset{n}{\underset{i=1}{\cup}} F(U(x_i)) \subset \overset{n}{\underset{i=1}{\cup}} V(x_i) = \overset{n}{\underset{i=1}{\cup}} \underset{\alpha\in\Lambda(x_i)}{\cup} V_\alpha.$$

This shows that Y is mildly compact. □

Definition 23. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction. The multigraph $G(F)$ is said to be ω-co-closed if for each $(x, y) \notin G(F)$, there exist ω-open set U and clopen set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \varnothing$.

Definition 24. [6] A topological space (X, τ) is said to be clopen T_2 (clopen Hausdorff) if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 25. *If a multifunction $F : (X, \tau) \to (Y, \sigma)$ is an upper slightly ω-continuous such that $F(x)$ is mildly compact relative to Y for each $x \in X$ and Y is a clopen Hausdorff space, then the multigraph $G(F)$ of F is ω-co-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. That is $y \notin F(x)$. Since Y is clopen Hausdorff, for each $z \in F(x)$, there exist disjoint clopen sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(z)$. Then $\{U(z) : z \in F(x)\}$ is a clopen cover of $F(x)$ and since $F(x)$ is mildly compact, there exists a finite number of points, say, z_1, z_2, ... , z_n in $F(x)$ such that $F(x) \subset \cup\{U(z_i) : i$ = 1, 2, ..., $n\}$. Put $U = \cup\{U(z_i)$: i = 1, 2, ..., $n\}$ and $V = \cap\ \{V(y_i)$: i = 1, 2, ..., $n\}$. Then U and V are clopen sets in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \varnothing$. Since F is upper slightly ω-continuous multifunction, there exists an ω-open set W of X containing x such that $F(W) \subset U$. We have $(x, y) \in W \times V \subset (X \times Y) \setminus G(F)$. We obtain that $(W \times V) \cap G(F) = \varnothing$; hence $G(F)$ is ω-co-closed in $X \times Y$. □

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