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Exact analytical solutions for a longitudinal flow of a fractional Maxwell fluid between two coaxial cylinders

A. U. Awan
Department of Mathematics
University of the Punjab
Lahore, Pakistan
Email: auawan2003@yahoo.com

M. Imran
Department of Mathematics
Government College University
Faisalabad, Pakistan
Email:drmimrachaudhry@gmail.com

M. Athar
Division of Science and Technology
University of Education
Township Campus, Lahore, Pakistan
Email:athar_sms@yahoo.com

M. Kamran
COMSATS Institute of Information Technology
Wah Cantt, Pakistan
Email:getkamran@gmail.com

Abstract. In this paper the velocity field and the adequate shear stress corresponding to the longitudinal flow of a fractional Maxwell fluid, between two infinite coaxial circular cylinders, are determined by applying the Laplace and finite Hankel transforms. Initially both cylinders are at rest and at time $t = 0^+$ both cylinders begin to translate along their common axis with different constant accelerations. The solutions that have been obtained are presented in terms of generalized G functions. The expressions for the velocity field and the shear stress are in the most simplified form, and the point worth mentioning is that these expressions are free from integral of the generalized G functions, in contrast with [20], in which the expression for the velocity field involves integral of the generalized G functions. Moreover, these solutions satisfy both the governing differential equation and all imposed initial and boundary conditions. The corresponding solutions for ordinary Maxwell and Newtonian fluids are obtained as limiting case of general solutions. Furthermore, the solutions for the motion between the cylinders, when one of them is at rest, can also be obtained from our results.

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Key Words: Fractional Maxwell fluid, Coaxial cylinders, Velocity field, Shear stress, Laplace and finite Hankel transforms.

1. INTRODUCTION

The study of viscoelastic fluids has many applications in industrial processes. They include the extrusion of polymer fluids, cooling of the metallic plate in a bath, food stuffs, exotic lubricants, colloidal and suspension solutions. The classical Navier-Stokes theory is inadequate to describe the flows of such kinds of fluids. The non-Newtonian characteristics include stress relaxation, the normal stress difference, shear thinning or shear-thickening and many other. Due to complexity of non-Newtonian fluids, various models for viscoelastic fluids have been proposed. The first exact solutions for unsteady motions of such fluids seem to be obtained by Srivastava [27]. However, the exact analytic solutions for non-Newtonian fluid flows is not an easy task and therefore in literature such exact solutions are rare. In spite of several challenges, many investigations regarding the exact analytic solutions for flows of non-Newtonian fluids have been performed [1, 5, 6, 7, 9, 21, 34, 36, 37, 38].

There is a great interest of theoretical and applied scientists to study the fluid flows in the neighborhood of translating or oscillating bodies. As early as Casarella and Laura [2] obtained an exact solution for the motion of the linearly viscous fluid due to both longitudinal and torsional oscillations of the rod. Later, Rajagopal [25] found two simple but elegant solutions for the flow of a second grade fluid induced by the longitudinal and torsional oscillations of an infinite rod. These solutions have been already extended to Oldroyd-B fluids by Rajagopal and Bhatnagar [26]. Others interesting results have been recently obtained by Hayat *at al* [10], Rajagopal [24], Fetecau and Corina Fetecau [4], Tong and Shan [33], Corina Fetecau *at al* [8], and Jamil and Fectecau [14].

During last decade, fractional calculus has been successfully applied in the constitutive modeling of non-Newtonian fluids. The main objective for this approach is that a fractional model could describe simply and elegantly the complex behavior of a viscoelastic material. For instance, the exponential relaxation moduli of the existing ordinary models (e.g. the Maxwell model) [11] can not be described the characteristic of the relaxation processes of many materials exhibit an algebraic decay. The relaxation process of some material even jump from one power law state to another [12, 28]. However, experimental evidence show that these characteristic can be easily described and correlated by fractional models [12, 16]. These fractional models with compact expressions can be physically visualized as an infinite hierarchical arrangement of springs and dashpots [16, 35]; thus they are a natural generalization of the ordinary models. A theoretical advantage of these fractional models with few parameters that they could describe a large number of different equivalent and complicated combinations of the conventional springs and dashpots, which could well be impractical to realize with the traditional springs and dashpots arrangements. These fractional models with only one or two extra parameters involved, can exhibit a large variety of dynamical behaviors, and thus they are the appropriate tools for describing the complex characteristic of viscoelastic materials.

Recently one of the viscoelastic fluid namely Maxwell fluid has received special attention. In a simple shear flow of a real fluid, Maxwell model predicts a linear relation between shear rate and shear stress. Furthermore, for the Maxwell model it was not possible to achieve satisfactory fit of experimental data over the entire range of frequencies [17]. A very good fit of experimental data was achieved when the ordinary Maxwell model has been replaced by the Maxwell model with fractional calculus [18]. The starting point of the fractional derivative models of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional differential operator. This generalization allows us to define precisely non-integer order integrals or derivatives [22]. There is a vast literature dealing with such fluids, but we shall recall here only a few of the most recent papers [13, 19, 23, 29, 30, 32] in cylindrical domains.

In this paper, we are interested into the longitudinal motion of a fracional Maxwell fluid between two infinite coaxial circular cylinders, both of them translate along their common axis with given constant accelerations. The velocity field and associated tangential shear stress are determined by means of Laplace and finite Hankel transforms, and are presented in terms of Bessel and generalized G functions. It is worthy to point out that the solutions that have been obtained satisfy both the governing differential equations as well as all imposed initial and boundary conditions. The solutions corresponding to the ordinary Maxwell and Newtonian fluids, performing the same motion, are also obtained as limiting cases of general solutions. Furthermore, the respective solutions for the longitudinal motion between the cylinders, when one of them is at rest, are obtained from general solutions.

2. THE DIFFERENTIAL EQUATIONS GOVERNING THE FLOW

For the problem under consideration, we shall assume a velocity field **v** and an extra-stress tensor **S** of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\mathbf{e}_z; \quad \mathbf{S} = \mathbf{S}(r,t), \tag{2.1}$$

where $\mathbf{e}_z$ is the unit vector in the z-direction of a cylindrical coordinate system r, θ, z. For such flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{v}(r,0) = \mathbf{0}; \qquad \mathbf{S}(r,0) = \mathbf{0}, \tag{2.2}$$

The balance of the linear momentum, in the absence of a pressure gradient in the axial direction ($\partial_\theta p = 0$ due to the rotational symmetry [26]), and the constitutive equation corresponding to Maxwell fluid lead to the relevant partial differential equation [6]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu\frac{\partial v(r,t)}{\partial r}. \tag{2.3}$$

where $\tau(r,t) = S_{rz}(r,t)$ is the shear stress which is different of zero, and μ is the dynamic viscosity of the fluid, and λ is the relaxation time. The equations of motion, in the absence of body forces, reduce to (see for instance Rajagopal and Bhatnagar[26]):

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \text{ and } -\frac{\partial p}{\partial z} + \frac{\partial \tau}{\partial r} + \frac{\tau}{r} = \rho\frac{\partial v}{\partial t}, \tag{2.4}$$

eliminating τ between Eqs. (2.3) and (2.4), we attain to the governing equation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial t}=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t). \tag{2.5}$$

The governing equations corresponding to an incompressible fractional Maxwell fluid, performing the same motion, are given by [5, 6, 8, 23, 30, 32, 33]

$$(1+\lambda D_t^{\beta})\frac{\partial v(r,t)}{\partial t}=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t); \quad (1+\lambda D_t^{\beta})\tau(r,t)=\mu\frac{\partial v(r,t)}{\partial r}, \tag{2.6}$$

where the fractional differential operator D_t^{β} is defined by [22, 31]

$$D_t^{\beta}f(t)=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t\frac{f(\tau)}{(t-\tau)^{\beta}}d\tau; \quad 0\leq\beta<1, \tag{2.7}$$

and $\Gamma(\cdot)$ is the Gamma function. Of course, the new material constant λ, although we keep the same notation, has the dimension of t^{β}. For $\beta\to 1$, it tends to the relaxation time. In the following the system of fractional partial differential equations (2.6), with appropriate initial and boundary conditions, will be solved by means of Laplace and finite Hankel transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used [5, 6, 7, 8, 13, 19, 23, 29, 30, 32].

3. Statement of the problem and the exact analytic solutions

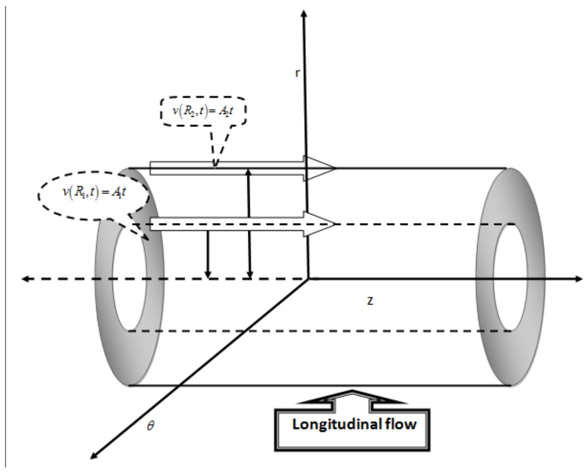

FIGURE 1. Geometry of the problem.

Let us consider an incompressible fractional Maxwell fluid at rest in an annular region between two straight circular cylinders of radii R_1 and $R_2(>R_1)$ as shown in Fig, 1. At time $t=0^+$, both cylinders with radii R_1 and R_2 begin to slide along their common axis with constant accelerations A_1 and A_2 respectively. Owing to the shear, the fluid is gradually moved, its velocity being of the form $(2.1)_1$.

The governing equations are (2.6), while the appropriate initial and boundary conditions are

$$v(r,0)=\frac{\partial v(r,0)}{\partial t}=0,\quad r\in[R_1,R_2]\,, \tag{3.1}$$

$$v(R_1,t)=A_1t,\quad v(R_2,t)=A_2t\ \ \text{for}\ \ t\geq 0\,, \tag{3.2}$$

where A_1 and A_2 have the unit m/s^2.

3.1. **Calculation of the velocity field.** Applying the Laplace transform to Eq. $(2.6)_1$ and (3.2), and using initial conditions (3.1), we get

$$\big(q+\lambda q^{\beta+1}\big)\overline{v}(r,q)=\nu\bigg(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\bigg)\overline{v}(r,q),\quad r\in(R_1,R_2), \tag{3.3}$$

$$\overline{v}(R_1,q)=\frac{A_1}{q^2},\ \ \overline{v}(R_2,q)=\frac{A_2}{q^2}\,, \tag{3.4}$$

where the image function $\overline{v}(r,q)$ is the Laplace transform of the functions $v(r,t)$.

In the following, let us denote by [3]

$$\overline{v}_H(r_n,q)=\int_{R_1}^{R_2} r\overline{v}(r,q)B(r,r_n)dr\,,\quad n=1,2,3,... \tag{3.5}$$

the finite Hankel transform of the function $\overline{v}(r,q)$, and the inverse Hankel transform of the function $\overline{v}_H(r_n,q)$ is given by

$$\overline{v}(r,q)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_0^2(R_1r_n)B(r,r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\overline{v}_H(r_n,q)\,. \tag{3.6}$$

where

$$B(r,r_n)=J_0(rr_n)Y_0(R_2r_n)-J_0(R_2r_n)Y_0(rr_n)\,, \tag{3.7}$$

while r_n are the positive roots of the transcendental equation $B(R_1,r)=0$, and $J_p(\cdot)$ and $Y_p(\cdot)$ are Bessel functions of the first and second kind of order p.

Multiplying now both sides of (3.3) by $rB(r,r_n)$, then integrating it with respect to r from R_1 to R_2, and taking into account the Eqs. (3.4) along with the following relations

$$\frac{d}{dr}B(r,r_n)=-r_n\big[J_1(rr_n)Y_0(R_2r_n)-J_0(R_2r_n)Y_1(rr_n)\big]\,, \tag{3.8}$$

and

$$J_0(z)Y_1(z)-J_1(z)Y_0(z)=-\frac{2}{\pi z}\,, \tag{3.9}$$

and the result which we can easily prove

$$\int_{R_1}^{R_2} r\bigg(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\bigg)\overline{v}(r,q)B(r,r_n)dr=-\frac{2J_0(R_2r_n)\overline{v}(R_1,q)}{\pi J_0(R_1r_n)}-r_n^2\overline{v}_H(r_n,q),$$

we find that

$$\overline{v}_H(r_n,q)=\frac{2\nu\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{\pi J_0(R_1r_n)}\,\frac{1}{q^2(q+\lambda q^{\beta+1}+\nu r_n^2)}\,. \tag{3.10}$$

Eq. (3.10) can be written in the equivalent form as

$$\begin{aligned}\overline{v}_H(r_n,q) &= \frac{2\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{\pi r_n^2J_0(R_1r_n)}\frac{1}{q^2}- \\ &- \frac{2\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{\pi r_n^2J_0(R_1r_n)}\frac{1+\lambda q^\beta}{q(q+\lambda q^{\beta+1}+\nu r_n^2)},\end{aligned} \tag{3.11}$$

Applying the inverse Hankel transform to Eq. (3.11), and using identities

$$\frac{\ln(R_2/r)}{\ln(R_2/R_1)} = -\pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)J_0(R_2r_n)B(r,r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}, \tag{3.12}$$

and

$$\frac{\ln(r/R_1)}{\ln(R_2/R_1)} = \pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B(r,r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}, \tag{3.13}$$

we get

$$\begin{aligned}\overline{v}(r,q) &= \frac{1}{\ln(R_2/R_1)}\left[A_1\ln\left(\frac{R_2}{r}\right)+A_2\ln\left(\frac{r}{R_1}\right)\right]\frac{1}{q^2}- \\ &- \pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)B(r,r_n)\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\frac{1+\lambda q^\beta}{q(q+\lambda q^{\beta+1}+\nu r_n^2)}.\end{aligned} \tag{3.14}$$

Apply the discrete inverse Laplace transform to Eq. (3.14), then using the expansion

$$\begin{aligned}\frac{1+\lambda q^\beta}{q(q+\lambda q^{\beta+1}+\nu r_n^2)} &= \frac{1+\lambda q^\beta}{q}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{1}{\lambda q^{k+1}\left(q^\beta+\lambda^{-1}\right)^{k+1}} = \\ &= \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{q^{-k-2}}{\left(q^\beta+\lambda^{-1}\right)^{k+1}}+\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{q^{\beta-k-2}}{\left(q^\beta+\lambda^{-1}\right)^{k+1}},\end{aligned} \tag{3.15}$$

and the formulae [13]

$$\begin{aligned}L^{-1}\left\{\frac{1}{q^a}\right\}=\frac{t^{a-1}}{\Gamma(a)},\ \ a>0,\ \ L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} &= G_{a,b,c}(d,t), \\ \mathrm{Re}(ac-b)>0,\quad \left|\frac{d}{q^a}\right|<1,&\end{aligned} \tag{3.16}$$

where $G_{a,b,c}(d,t)$ are the generalized G functions defined as [38]

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{d^j\,\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{3.17}$$

we find that

$$
\begin{aligned}
v(r,t) &= \frac{1}{\ln(R_2/R_1)}\left[A_1\ln\left(\frac{R_2}{r}\right)+A_2\ln\left(\frac{r}{R_1}\right)\right]t- \\
&-\pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)B(r,r_n)\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times \\
&\times\ \Big[\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right)+ \\
&+\ \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,\beta-k-2,k+1}\left(-\lambda^{-1},t\right)\Big].
\end{aligned}
\tag{3.18}
$$

3.2. **Calculation of the shear stress.** Applying the Laplace transform to Eq. $(2.6)_2$, and having in mind initial condition $(2.2)_1$, we get

$$
(1+\lambda q^\beta)\overline{\tau}(r,q)=\mu\frac{\partial\overline{v}(r,q)}{\partial r}\,. \tag{3.19}
$$

Using Eq. (3.14) and the relation

$$
\frac{1}{q(q+\lambda q^{\beta+1}+\nu r_n^2)}=\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{1}{q^{k+2}\left(q^\beta+\lambda^{-1}\right)^{k+1}}\,,
$$

we find that

$$
\begin{aligned}
\overline{\tau}(r,q) &= \frac{\mu(A_2-A_1)}{\lambda r\ln\big(R_2/R_1\big)}\,\frac{1}{q^2\left(q^\beta+\lambda^{-1}\right)}+ \\
&+\ \frac{\pi\mu}{\lambda}\sum_{n=1}^{\infty}\frac{r_nJ_0(R_1r_n)B_1(r,r_n)\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times \\
&\times\ \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{1}{q^{k+2}\left(q^\beta+\lambda^{-1}\right)^{k+1}}\,,
\end{aligned}
\tag{3.20}
$$

where

$$
B_1(r,r_n)=J_1(rr_n)Y_0(R_2r_n)-J_0(R_2r_n)Y_1(rr_n)\,. \tag{3.21}
$$

Now, applying the inverse Laplace transform to Eq. (3.20) and using Eq. $(3.16)_2$, we get the shear stress under the form

$$
\begin{aligned}
\tau(r,t) &= \frac{\mu(A_2-A_1)}{\lambda r\ln\big(R_2/R_1\big)}R_{\beta,-2}\left(-\lambda^{-1},t\right)+ \\
&+\ \frac{\pi\mu}{\lambda}\sum_{n=1}^{\infty}\frac{r_nJ_0(R_1r_n)B_1(r,r_n)\big[A_2J_0(R_1r_n)-A_1J_0(R_2r_n)\big]}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times \\
&\times\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right)\,.
\end{aligned}
\tag{3.22}
$$

4. LIMITING CASES

1. Making $\beta \to 1$ into Eqs. (3.18) and (3.22), we find the velocity field and the shear stress

$$\begin{aligned} v_{M}(r,t) &= \frac{1}{\ln(R_2/R_1)}\left[A_1 \ln\left(\frac{R_2}{r}\right) + A_2 \ln\left(\frac{r}{R_1}\right)\right] t - \\ &- \pi \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) B(r, r_n)\left[A_2 J_0(R_1 r_n) - A_1 J_0(R_2 r_n)\right]}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)} \times \\ &\times \Big[\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\lambda^{-1}, t\right) + \\ &+ \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\lambda^{-1}, t\right)\Big], \end{aligned} \tag{4.1}$$

$$\begin{aligned} \tau_{M}(r,t) &= \frac{\mu(A_2 - A_1)}{\lambda r \ln\left(R_2/R_1\right)} R_{1,-2}\left(-\lambda^{-1}, t\right) + \\ &+ \frac{\pi\mu}{\lambda} \sum_{n=1}^{\infty} \frac{r_n J_0(R_1 r_n) B_1(r, r_n)\Big[A_2 J_0(R_1 r_n) - A_1 J_0(R_2 r_n)\Big]}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)} \times \\ &\times \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\lambda^{-1}, t\right), \end{aligned} \tag{4.2}$$

corresponding to an ordinary Maxwell fluid performing the same motion.

2. Now making $\lambda \to 0$ into Eqs. (4.1) and (4.2) and using

$$\lim_{\lambda \to 0} \frac{1}{\lambda^k} G_{1,b,k}(-\lambda^{-1}, t) = \frac{t^{-b-1}}{\Gamma(-b)}, \quad b < 0,$$

we find the velocity field and the shear stress

$$\begin{aligned} v_{N}(r,t) &= \frac{1}{\ln(R_2/R_1)}\left[A_1 \ln\left(\frac{R_2}{r}\right) + A_2 \ln\left(\frac{r}{R_1}\right)\right] t - \qquad (4.3) \\ &- \frac{\pi}{\nu} \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) B(r, r_n)\left[A_2 J_0(R_1 r_n) - A_1 J_0(R_2 r_n)\right]}{r_n^2\left[J_0^2(R_1 r_n) - J_0^2(R_2 r_n)\right]}\left(1 - e^{-\nu r_n^2 t}\right), \end{aligned}$$

$$\begin{aligned} \tau_{N}(r,t) &= \frac{\mu(A_2 - A_1)}{r \ln\left(R_2/R_1\right)} t + \qquad (4.4) \\ &+ \pi\rho \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) B_1(r, r_n)\left[A_2 J_0(R_1 r_n) - A_1 J_0(R_2 r_n)\right]}{r_n\left[J_0^2(R_1 r_n) - J_0^2(R_2 r_n)\right]}\left(1 - e^{-\nu r_n^2 t}\right). \end{aligned}$$

corresponding to a Newtonian fluid performing the same motion.

5. CONCLUSIONS AND NUMERICAL RESULTS

In this paper, the velocity field and the adequate tangential shear stress, corresponding to the flow of a fractional Maxwell fluid between two infinite circular cylinders which slide along common axis, are determined by using Laplace and finite Hankel transforms. At time $t = 0^+$ both cylinders begin to move along their common axis with constant acceleration A_1 and A_2. The solutions are presented in terms of Bessel ($J_0(\cdot)$, $Y_0(\cdot)$, $J_1(\cdot)$

and $Y_1(\cdot)$) and generalized G functions, satisfy the corresponding governing equations as well as all imposed initial and boundary conditions.

Making $A_1 = 0$ and $A_2 = A$ into Eqs. (3.18) and (3.22), for instance, we obtain the velocity field and the shear stress

$$\begin{aligned} v(r,t) = {} & \frac{A\ln(r/R_1)}{\ln(R_2/R_1)}t - A\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B(r,r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times \\ & \times\Big[\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right)+ \\ & + \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,\beta-k-2,k+1}\left(-\lambda^{-1},t\right)\Big], \end{aligned} \tag{5.1}$$

$$\begin{aligned} \tau(r,t) \;=\; & \frac{\mu A}{\lambda r\ln\left(R_2/R_1\right)}R_{\beta,-2}\left(-\lambda^{-1},t\right)+\frac{\pi\mu A}{\lambda}\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)B_1(r,r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times \\ \times & \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right) \end{aligned} \tag{5.2}$$

corresponding to the flow between cylinders when the inner cylinder is at rest.

Fig. 2 shows the profile of the velocity field and shear stress corresponding to the Eqs. (5.1) and (5.2) for different values of time, when the inner cylinder is at rest. It shows that both velocity and shear stress are increasing functions with regards to t, it also shows that velocity is an increasing function of r on the whole flow domain.

Similarly, making $A_1 = A$ and $A_2 = 0$ into Eqs. (3.18) and (3.22), we obtain the velocity field and the shear stress

$$\begin{aligned} v(r,t) \;=\; & \frac{A\ln(R_2/r)}{\ln(R_2/R_1)}t + A\pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)J_0(R_2r_n)B(r,r_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times \\ & \times\Big[\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right)+ \\ & + \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,\beta-k-2,k+1}\left(-\lambda^{-1},t\right)\Big], \end{aligned} \tag{5.3}$$

$$\begin{aligned} \tau(r,t) = {} & \frac{\mu A}{\lambda r\ln\left(R_2/R_1\right)}R_{\beta,-2}\left(-\lambda^{-1},t\right)-\frac{A\pi\mu}{\lambda}\sum_{n=1}^{\infty}\frac{r_nJ_0(R_1r_n)J_0(R_2r_n)B_1(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)} \\ \times & \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-2,k+1}\left(-\lambda^{-1},t\right) \end{aligned} \tag{5.4}$$

corresponding to the flow between cylinders when the outer cylinder is at rest.

Fig. 3 show the profile of the velocity field and shear stress corresponding to the Eqs. (5.3) and (5.4) for different values of time, when the outer cylinder is at rest. It shows that velocity as well as shear stress (in absolute value) is an increasing function with regards to t, as like in Fig. 2. While velocity is a decreasing function of r on the whole flow domain, in contrast to Fig 2.

In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity $v(r,t)$ and the shear stress $\tau(r,t)$ given by Eqs. (3.18) and (3.22), have been drawn against *r* for different values of the time *t* and of the material parameters. Figs. 4 shows the influence of time on the fluid motion. It is clearly seen that the velocity (in absolute value), as well as the shear stress is an increasing function of *t*. In Figs. 5, it is shown the influence of the kinematic viscosity ν on the fluid motion. It is clearly seen that the velocity is a decreasing function of ν. The influence of the relaxation time λ on the fluid motion is shown in Figs. 6, as expected both the velocity and the shear stress are decreasing functions with respect to λ. In Fig. 7, it is shown the influence of the fractional parameter β on the fluid motion. It is clearly seen that the velocity is a decreasing function with respect to β, while the shear stress is an increasing one with regards to β.

Finally, for comparison, the diagrams of $v(r,t)$ and $\tau(r,t)$ corresponding to the three models (fractional Maxwell, ordinary Maxwell and Newtonian) are together depicted in Fig. 8 for the same values of the common material constants and time *t*. In all cases the velocity of the fluid is an increasing function with respect to *r*. The Newtonian fluid is the swiftest, while the Maxwell fluid is the slowest on the whole flow domain. One thing is of worth mentioning that units of the material constants are SI units in all figures, and the roots r_n have been approximated by $n\pi/(R_2 - R_1)$.

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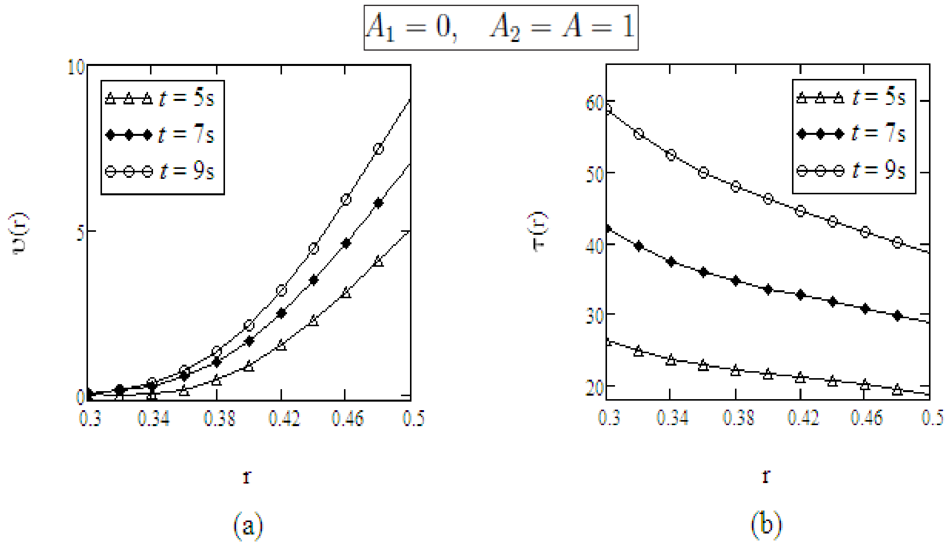

FIGURE 2. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (5.1) and (5.2) for $R_1 = 0.3,\ R_2 = 0.5,\ A_1 = 0,\ A_2 = A = 1,\ \nu = 0.004,\ \mu = 2.916,\ \lambda = 4,\ \beta = 0.5$, and different values of t.

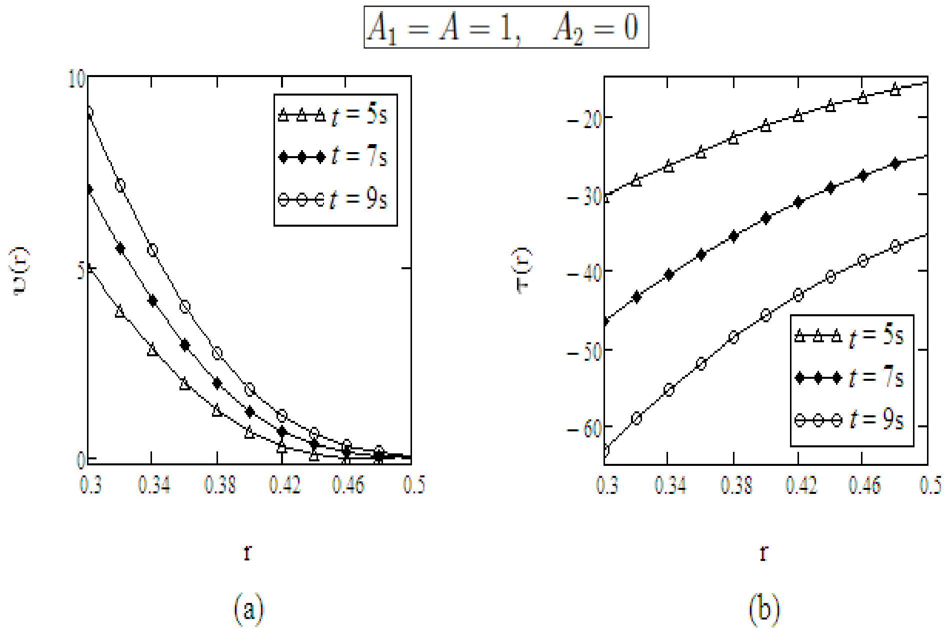

FIGURE 3. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (5.3) and (5.4) for $R_1 = 0.3,\ R_2 = 0.5,\ A_1 = A = 1,\ A_2 = 0,\ \nu = 0.004,\ \mu = 2.916,\ \lambda = 4,\ \beta = 0.5$, and different values of t.

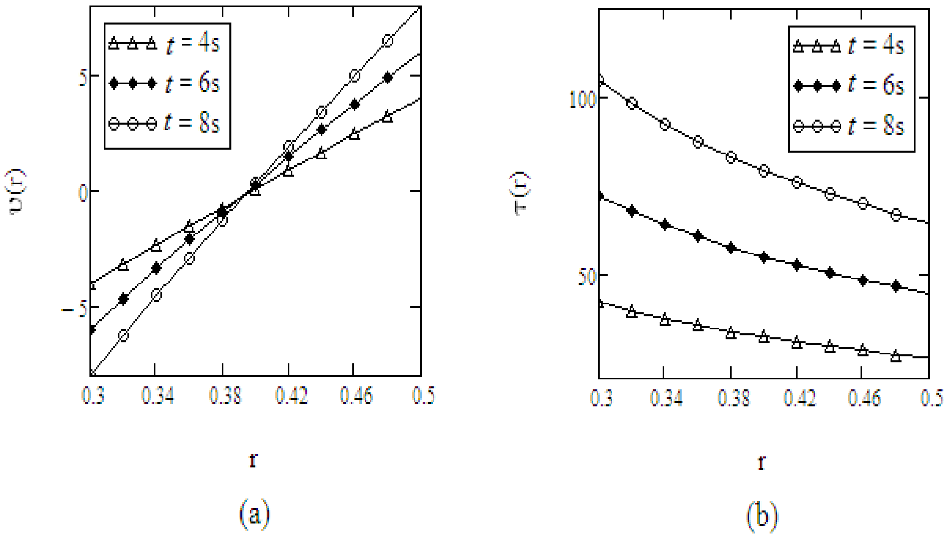

FIGURE 4. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (3.18) and (3.22) for $R_1 = 0.3$, $R_2 = 0.5$, $A_1 = -1$, $A_2 = 1$, $\nu = 0.004$, $\mu = 2.916$, $\lambda = 4$, $\beta = 0.5$, and different values of t.

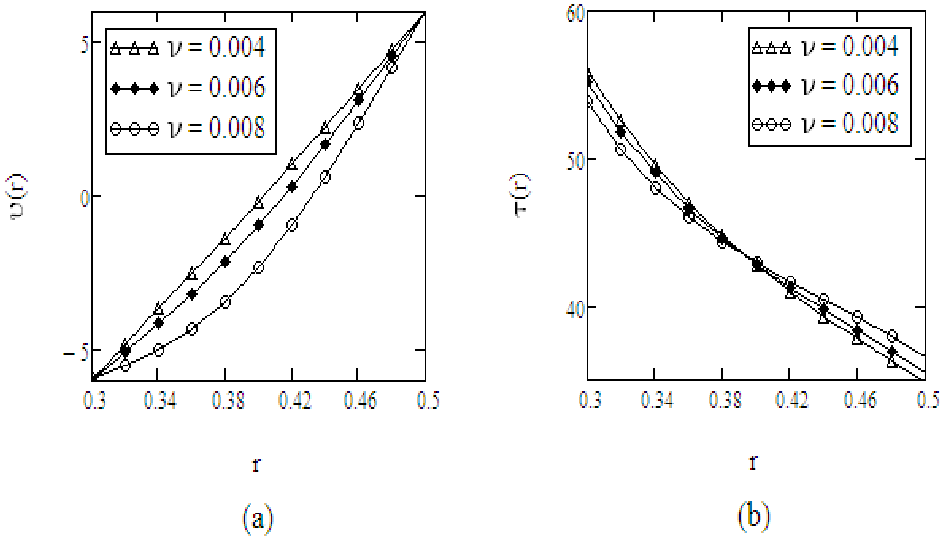

FIGURE 5. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (3.18) and (3.22) for $R_1 = 0.3$, $R_2 = 0.5$, $A_1 = -1$, $A_2 = 1$, $t = 6s$, $\mu = 30$, $\lambda = 2.5$, $\beta = 0.1$, and different values of ν.

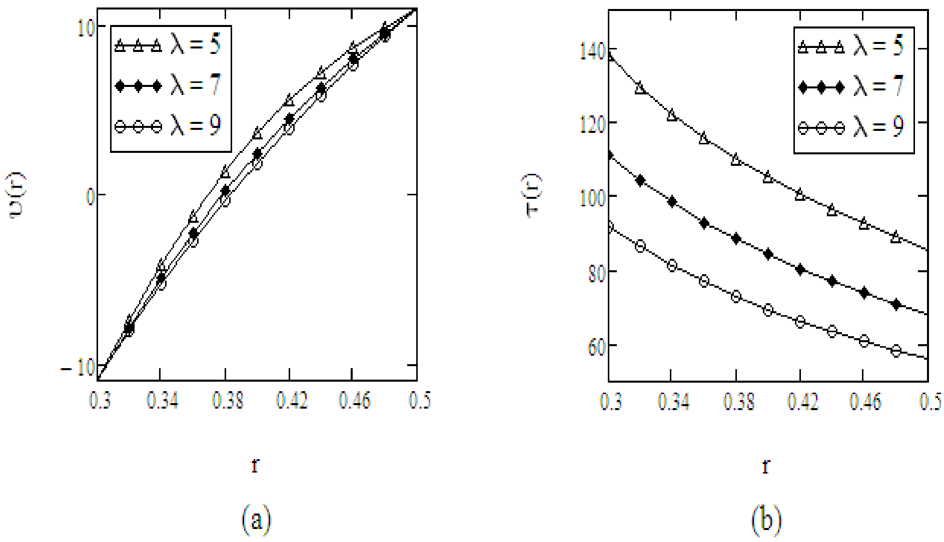

FIGURE 6. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (3.18) and (3.22) for $R_1 = 0.3$, $R_2 = 0.5$, $A_1 = -1$, $A_2 = 1$, $t = 11s$, $\nu = 0.005$, $\mu = 2,916$, $\beta = 0.5$, and different values of λ.

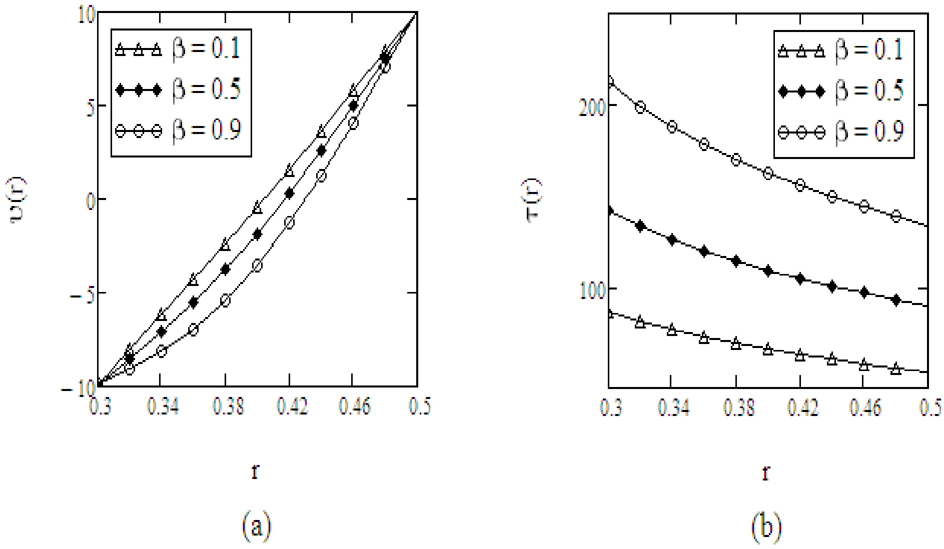

FIGURE 7. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (3.18) and (3.22) for $R_1 = 0.3$, $R_2 = 0.5$, $A_1 = -1$, $A_2 = 1$, $t = 10s$, $\nu = 0.005$, $\mu = 2,916$, $\lambda = 4$ and different values of $\beta = 0.6$.

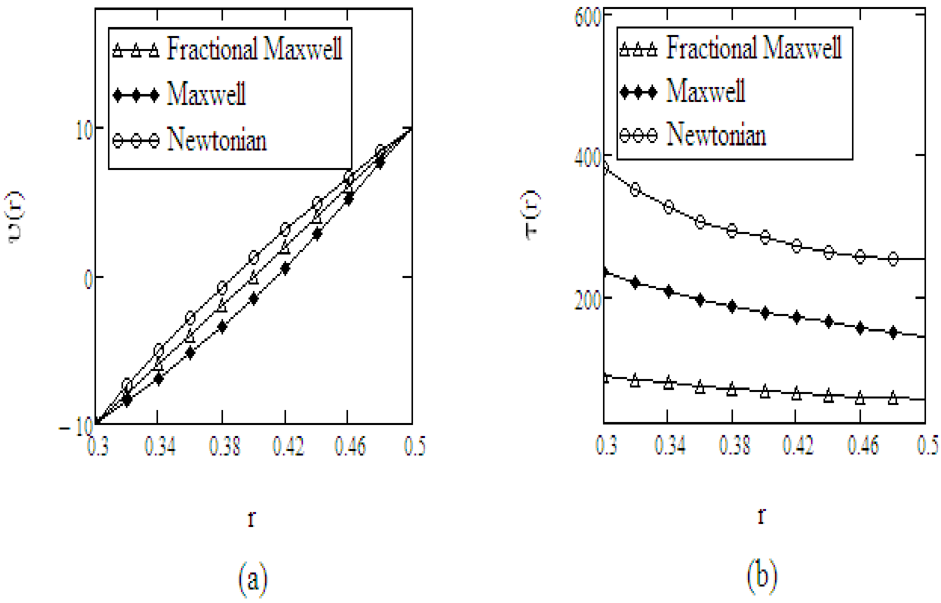

FIGURE 8. Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ corresponding to the Newtonian, Maxwell and fractional Maxwell fluids, for $R_1 = 0.3$, $R_2 = 0.5$, $A_1 = -1$, $A_2 = 1$, $t = 10s$, $\nu = 0.004$, $\mu = 2.916$, $\lambda = 4$ and $\beta = 0.1$