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Coupled Coincidence Point Theorems for (ψ, α, β)-Weak Contractions in Partially Ordered Partial Metric Spaces

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Abstract. In this paper, we obtain coupled coincidence and coupled fixed points of mappings satisfying a non linear contractive condition in partially ordered partial metric spaces. Our results generalize the theorems of H.Aydi [4] and W. Shatanawi et.al. [20]. We also provide an example to support our results.

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1. Introduction

In 1994 the partial metric space was introduced by Matthews [16] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation.

The notion of coupled fixed point of a mapping $F : X \times X \to X$ was introduced by Bhaskar and Lakshmikantham in [17] and investigated some coupled fixed point theorems in partially ordered sets. Later on, many authors investigated many coupled fixed point results in differerent spaces such as usual metric spaces, fuzzy metric spaces, partial metric spaces and partially ordered metric spaces(see [2 - 21]).

In this paper, we prove results of coupled coincidence points of mappings satisfying a nonlinear contractive condition in partially ordered partial metric spaces. Our results generalize the results of H.Aydi [4] and W. Shatanawi et.al.[20].
Consistent with [12, 16, 17, 18], the following definitions and results will be needed in sequel.
A set with a partial order $\preceq$ is called partially ordered set.

Definition 1. Any two elements x and y of a set X, which is partially ordered by a binary relation $\preceq$, are either comparable or incomparable. Specifically, the elements x and y are comparable if and only if $x \preceq y$ or $y \preceq x$. Otherwise, x and y are incomparable.

Definition 2 ([16])**.** A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

(p_1): $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2): $p(x, x) \leq p(x, y), p(y, y) \leq p(x, y)$,
(p_3): $p(x, y) = p(y, x)$,
(p_4): $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

(X, p) is called a partial metric space.

It is clear that $p(x, y) = 0$ implies $x = y$ from (p_1) and (p_2).
But if $x = y$, $p(x, y)$ may not be zero. A basic example of a partial metric space is the pair (R^+, p), where $p(x, y) = max\{x, y\}$ for all $x, y \in R^+$.

Each partial metric p on X generates τ_0 topology τ_p on X which has a base the family of open p - balls $\{B_p(x, \varepsilon)|x \in X, \varepsilon > 0\}$ for all $x \in X$ and $\varepsilon > 0$, where $B_p(x, \varepsilon) = \{y \in X|\ p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $d_p : X \times X \to R^+$ given by
$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

Definition 3 ([16])**.** Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is said to converge to a point $x \in X$ if and only if $p(x, x) = \lim\limits_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in (X, p) is said to be Cauchy sequence if $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and is finite .
(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, w. r. to τ_p, to a point $x \in X$ such that $p(x, x) = \lim\limits_{n,m\to\infty} p(x_n, x_m)$.

Lemma 4 ([16])**.** *Let (X, p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).*
(b) *(X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore,*
$\lim\limits_{n\to\infty} d_p(x_n, x) = 0$ *if and only if* $p(x, x) = \lim\limits_{n\to\infty} p(x_n, x) = \lim\limits_{n,m\to\infty} p(x_n, x_m)$.

Lemma 5 ([3])**.** *Let (X, p) be a partial metric space and $x_n \to z$ as $n \to \infty$ in (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Definition 6 ([17])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 7 ([12])**.** An element $(x, y) \in X \times X$ is called

(g_1) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.
(g_2) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Definition 8 ([12])**.** The mappings $F : X \times X \to X$ and $f : X \to X$ are called w - compatible if $f(F(x, y)) = F(fx, fy)$ and $f(F(y, x)) = F(fy, fx)$ whenever $fx = F(x, y)$ and $fy = F(y, x)$.

Definition 9 ([17]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed monotone property if $F(x, y)$ is monotone non - decreasing in x and is monotone non - increasing in y; that is, for any $x, y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X$$

and

$$y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

Definition 10 ([18]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed g- monotone property if $F(x, y)$ is monotone g - non - decreasing in x and is monotone g- non - increasing in y; that is, for any $x, y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

In recent year 2011, Aydi [4], proved the following coupled fixed point theorems in partial metric spaces.

Theorem 11. *Let (X, p) be a complete partial metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$*

$$p(F(x, y), F(u, v)) \leq kp(x, u) + lp(y, v),$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Theorem 12. *Let (X, p) be a complete partial metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$*

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), x) + lp(F(u, v), u),$$

where k, l are non - negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

The main theorems of Shatanawi et. al. [20], are

Theorem 13 (Theorem 3, [20]). . *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that for $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, we have*

$$\psi(p(F(x, y), F(u, v))) \leq \psi(\max\{p(x, u), p(y, v)\}) - \phi(\max\{p(x, u), p(y, v)\}),$$

where ψ and ϕ are altering distance functions. If there exists $(x_0, y_0) \in X \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Theorem 14 (Theorem 4, [20]). . *Let $(X, \preceq)$ be a partially ordered set and (X, p) be a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property. Assume there are two altering distance functions such that*

$$(14.1)\psi(p(F(x, y), F(u, v))) \leq \psi(\max\{p(x, u), p(y, v)\}) - \phi(\max\{p(x, u), p(y, v)\}),$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $v \succeq y$. Also, suppose that X has the following properties:

(i) *if $\{x_n\}$ is a non - decreasing sequence and $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$, then $x_n \preceq x$ for all n,*

(ii) *if* $\{x_n\}$ *is a non - increasing sequence and* $x \in X$ *with* $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$, *then* $x \preceq x_n$ *for all* n.

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq y_0$ *then there exist* $x, y \in X$ *such that* $x = F(x, y)$ *and* $y = F(y, x)$.

The aim of the paper is to study necessary conditions for the existence of coupled coincidence points and coupled common fixed points of (ψ, α, β)-weak contractions in partially ordered partial metric spaces. For more details on (ψ, α, β)-weakly contractive mappings we refer the reader to [1].

Now we prove our main result.

2. Main Results

Definition 15. The function $\varphi : R^+ \to R^+$ is called an altering distance function if the following properties are satisfied:

(i) φ is continuous and non - decreasing.

(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 16. . *Let* $(X, \preceq)$ *be partially order set and* p *be a partial metric such that* (X, p) *be partial metric space. Let* $F : X \times X \to X$, $g : X \to X$ *be such that*

(16.1) $\psi\left(p\left(F(x, y), F(u, v)\right)\right) \le \alpha\left(M\left(x, y, u, v\right)\right) - \beta\left(M\left(x, y, u, v\right)\right)$, $\forall x, y, u, v \in X$, $gx \preceq gu$ *and* $gy \succeq gv$, *where* $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$*are such that* ψ *is an altering distance function,* α *is continuous,* β *is lower semi continuous,* $\alpha(0) = \beta(0) = 0$ *and* $\psi(t) - \alpha(t) + \beta(t) > 0$, *for all* $t > 0$ *and*

$$M\left(x, y, u, v\right) = \max\left\{\begin{array}{c} p(gx, gu), p(gy, gv), p(gx, F(x, y)), \\ p(gy, F(y, x)), p(gu, F(u, v)), p(gv, F(v, u)), \\ \frac{1}{2}\left[p(gx, F(u, v)) + p(gu, F(x, y))\right] \end{array}\right\},$$

(16.2) $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X,

(16.3) F *has a mixed* g *- monotone property,*

(16.4) (a) *If a non - decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n,
(b) *If a non - increasing sequence* $\{y_n\} \to y$, *then* $y \preceq y_n$ *for all* n.

If there exist $x_0, y_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$, *then* F *and* g *have coupled coincidence point in* $X \times X$.

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$.
Since $F(X \times X) \subseteq g(X)$, we choose $x_1, y_1 \in X$ such that

$$\begin{aligned} gx_0 &\preceq F(x_0, y_0) = gx_1 \text{ and} \\ gy_o &\succeq F(y_0, x_0) = gy_1 \end{aligned}$$

and choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$.
Since F has the mixed g- monotone property, we obtain
$gx_0 \preceq gx_1 \preceq gx_2$ and $gy_o \succeq gy_1 \succeq gy_2$. Continuing this process, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} a_n &= gx_n = F(x_{n-1}, y_{n-1}) \text{ and} \\ b_n &= gy_n = F(y_{n-1}, x_{n-1}), n = 0, 1, 2, \cdots \end{aligned}$$

with

$$\left.\begin{array}{c} gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \text{ and} \\ gy_0 \succeq gy_1 \succeq gy_2 \succeq \cdots \end{array}\right\} - - - - - - - - (I)$$

Choose $R_n = \max\{p(a_n, a_{n+1}), p(b_n, b_{n+1})\}$.

Case (a): if $R_n = 0$ then $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$. Thus (x_n, y_n) is a coupled coincidence point of g and F in $X \times X$.

Case (b): Let $R_n \neq 0$.

From (I) we have $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$. Now using (16.1) we obtain

$$\begin{aligned}\psi(p(a_n, a_{n+1})) &= \psi(p(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\ &\leq \alpha(M(x_{n-1}, y_{n-1}, x_n, y_n)) - \beta(M(x_{n-1}, y_{n-1}, x_n, y_n)).\end{aligned}$$

$$M(x_{n-1}, y_{n-1}, x_n, y_n) = \max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), \\ p(gy_n, gy_{n+1}), \frac{1}{2}\left[p(gx_{n-1}, gx_{n+1}) + \right. \\ \left. p(gx_n, gx_n)\right] \end{array}\right.$$

But

$$\begin{aligned}&\tfrac{1}{2}\left[p(gx_{n-1}, gx_{n+1}) + p(gx_n, gx_n)\right] \\ &\qquad \leq \tfrac{1}{2}\left[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})\right] \\ &\qquad \leq \max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}.\end{aligned}$$

Therefore $M(x_{n-1}, y_{n-1}, x_n, y_n) = \max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array}\right\}$.

Hence

$$\begin{aligned}\psi(p(a_n, a_{n+1})) &\leq \alpha\left(\max\left\{\begin{array}{c} p(a_{n-1}, a_n), p(b_{n-1}, b_n), \\ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \end{array}\right\}\right) \\ &\quad -\beta\left(\max\left\{\begin{array}{c} p(a_{n-1}, a_n), p(b_{n-1}, b_n), \\ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \end{array}\right\}\right).\end{aligned}$$

Similarly

$$\begin{aligned}\psi(p(b_n, b_{n+1})) &\leq \alpha\left(\max\left\{\begin{array}{c} p(a_{n-1}, a_n), p(b_{n-1}, b_n), \\ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \end{array}\right\}\right) \\ &\quad -\beta\left(\max\left\{\begin{array}{c} p(a_{n-1}, a_n), p(b_{n-1}, b_n), \\ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \end{array}\right\}\right).\end{aligned}$$

Now,

$$\begin{aligned}\psi\left(\max p(a_n, a_{n+1}), p(b_n, b_{n+1})\right) &= \max\{\psi(p(a_n, a_{n+1})), \psi(p(b_n, b_{n+1}))\} \\ &\leq \alpha\left(\max\left\{\begin{array}{c} p(a_{n-1}, a_n), p(b_{n-1}, b_n), \\ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \end{array}\right\}\right) \\ &\quad -\beta\left(\max\left\{\begin{array}{c} p(a_{n-1}, a_n), p(b_{n-1}, b_n), \\ p(a_n, a_{n+1}), p(b_n, b_{n+1}) \end{array}\right\}\right) \\ &= \alpha\left(\max\left\{R_{n-1}, R_n\right\}\right) - \beta\left(\max\left\{R_{n-1}, R_n\right\}\right).\end{aligned}$$

So,

$$\psi\left(R_n\right) \leq \alpha\left(\max\left\{R_{n-1}, R_n\right\}\right) - \beta\left(\max\left\{R_{n-1}, R_n\right\}\right).$$

If R_n is maximum, we get $\psi(R_n) \leq \alpha(R_n) - \beta(R_n)$.

Thus $\psi(R_n) - \alpha(R_n) + \beta(R_n) \leq 0$, by our assumptions we have $R_n = 0$ which is a contradiction. Hence

$$\psi(R_n) \leq \alpha(R_{n-1}) - \beta(R_{n-1}) \tag{2.1}$$

Since $R_{n-1} > 0$, from our assumptions we have

$$\alpha(R_{n-1}) - \beta(R_{n-1}) < \psi(R_{n-1}).$$

Thus from (2.1) we get, $\psi(R_n) \leq \psi(R_{n-1})$.

Since ψ is non - decreasing, we obtain

$$R_n \leq R_{n-1}.$$

Thus $\{R_n\}$ is non - increasing sequence of non - negative real numbers and must converges to a real number $r \geq 0$ say.
Suppose $r > 0$.
Letting $n \to \infty$ in (2.1), we get

$$\psi(r) \leq \alpha(r) - \beta(r),$$

which implies that r = 0, a contradiction.
Hence $r = 0$. Thus

$$\lim_{n\to\infty} \max\{p(a_n, a_{n+1}), p(b_n, b_{n+1})\} = 0.$$

$$\lim_{n\to\infty} p(a_n, a_{n+1}) = 0 = \lim_{n\to\infty} p(b_n, b_{n+1}). \tag{2.2}$$

Hence from (p_2),

$$\lim_{n\to\infty} p(a_n, a_n) = 0 = \lim_{n\to\infty} p(b_n, b_n). \tag{2.3}$$

From (2.2) and (2.3) and by definition of d_p, we get

$$\lim_{n\to\infty} d_p(a_n, a_{n+1}) = 0 = \lim_{n\to\infty} d_p(b_n, b_{n+1}). \tag{2.4}$$

Now we prove that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences.
We shall show that for every $\epsilon > 0$, there exists $k \in N$ such that if $m, n \geq k$,

$$\max\ \{d_p(a_n, a_m), d_p(b_n, b_m)\} < \epsilon.$$

Suppose the above statement is false.
Then there exist an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$,

$$\max\{d_p(a_{m_k}, a_{n_k}), d_p(b_{m_k}, b_{n_k})\} \geq \epsilon. \tag{2.5}$$

and

$$\max\{d_p(a_{m_k}, a_{n_k-1}), d_p(b_{m_k}, b_{n_k-1})\} < \epsilon. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}
\epsilon &\leq \max\{d_p(a_{m_k}, a_{n_k}), d_p(b_{m_k}, b_{n_k})\} \\
&\leq \max\{d_p(a_{m_k}, a_{n_k-1}), d_p(b_{m_k}, b_{n_k-1})\} \\
&\quad + \max\{d_p(a_{n_k-1}, a_{n_k}), d_p(b_{n_k-1}, b_{n_k})\} \\
&< \epsilon + \max\{d_p(a_{n_k-1}, a_{n_k}), d_p(b_{n_k-1}, b_{n_k})\}.
\end{aligned}$$

Letting $k \to \infty$ and using (2.4), we get

$$\lim_{k\to\infty} \max\{d_p(a_{m_k}, a_{n_k}), d_p(b_{m_k}, b_{n_k})\} = \epsilon. \tag{2.7}$$

By definition of d_p and using (2.3) we get

$$\lim_{k\to\infty} \max\left\{\ p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{n_k})\ \right\} = \frac{\epsilon}{2}. \tag{2.8}$$

Since

$$d_p(a_{n_k+1}, a_{m_{k+1}}) \leq d_p(a_{n_{k+1}}, a_{n_k}) + d_p(a_{n_k}, a_{m_k}) + d_p(a_{m_k}, a_{m_{k+1}}) \tag{2.9}$$

and

$$d_p(b_{n_{k+1}}, b_{m_{k+1}}) \leq d_p(b_{n_{k+1}}, b_{n_k}) + d_p(b_{n_k}, b_{m_k}) + d_p(b_{m_k}, b_{m_{k+1}}) \tag{2.10}$$

we have

$$\max\left\{\begin{array}{c} d_p(a_{n_k+1}, a_{m_{k+1}}), \\ d_p(b_{n_{k+1}}, b_{m_{k+1}}) \end{array}\right\} \leq \max\{d_p(a_{n_{k+1}}, a_{n_k}), d_p(b_{n_{k+1}}, b_{n_k})\} + \max\{d_p(a_{n_k}, a_{m_k}), d_p(b_{n_k}, b_{m_k})\} + \max\{d_p(a_{m_k}, a_{m_{k+1}}), d_p(b_{m_k}, b_{m_{k+1}})\}.$$

Letting $k \to \infty$, using (2.4) and (2.7), we get

$$\lim_{k\to\infty} \max\{d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}})\} \leq \epsilon. \tag{2.11}$$

Again since

$$d_p(a_{n_k}, a_{m_k}) \leq d_p(a_{n_k}, a_{n_{k+1}}) + d_p(a_{n_{k+1}}, a_{m_{k+1}}) + d_p(a_{m_{k+1}}, a_{m_k}) \tag{2.12}$$

and

$$d_p(b_{n_k}, b_{m_k}) \leq d_p(b_{n_k}, b_{n_{k+1}}) + d_p(b_{n_{k+1}}, b_{m_{k+1}}) + d_p(b_{m_{k+1}}, b_{m_k}) \tag{2.13}$$

we have

$$\max\left\{\begin{array}{c} d_p(a_{n_k}, a_{m_k}), \\ d_p(b_{n_k}, b_{m_k}) \end{array}\right\} \leq \max\{d_p(a_{n_k}, a_{n_{k+1}}), d_p(b_{n_k}, b_{n_{k+1}})\} + \max\{d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}})\} + \max\{d_p(a_{m_{k+1}}, a_{m_k}), d_p(b_{m_{k+1}}, b_{m_k})\}.$$

Letting $k \to \infty$ and using (2.4), we have

$$\epsilon \leq \lim_{k\to\infty} \max\{d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}})\}. \tag{2.14}$$

Now, from (2.11) and (2.14), we get

$$\lim_{k\to\infty} \max\{d_p(a_{n_{k+1}}, a_{m_{k+1}}), d_p(b_{n_{k+1}}, b_{m_{k+1}})\} = \epsilon. \tag{2.15}$$

By definition of d_p and using (2.3) we get

$$\lim_{k\to\infty} \max\left\{\ p(a_{m_{k+1}}, a_{n_{k+1}}), p(b_{m_{k+1}}, b_{n_{k+1}})\ \right\} = \frac{\epsilon}{2}. \tag{2.16}$$

As $n_k > m_k$, we have $gx_{m_k} \preceq gx_{n_k}$ and $gy_{m_k} \succeq gy_{n_k}$. Putting $x = x_{m_k}$, $y = y_{m_k}$, $u = x_{n_k}$ and $v = y_{n_k}$ in (16.1), we have

$$\begin{aligned}
\psi(p(a_{m_{k+1}}, a_{n_{k+1}}) &= \psi\left(p(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k}))\right) \\
&\leq \alpha\left(M(x_{m_k}, y_{m_k}, x_{n_k}, y_{n_k})\right) - \beta\left(M(x_{m_k}, y_{m_k}, x_{n_k}, y_{n_k})\right). \\
&= \alpha\left(\max\left\{\begin{array}{c} p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{m_{k+1}}), \\ p(b_{m_k}, b_{m_{k+1}}), p(a_{n_k}, a_{n_{k+1}}), p(b_{n_k}, b_{n_{k+1}}), \\ \frac{1}{2}[p(a_{m_k}, a_{n_{k+1}}) + p(a_{n_k}, a_{m_{k+1}})] \end{array}\right\}\right) \\
&\quad -\beta\left(\max\left\{\begin{array}{c} p(a_{m_k}, a_{n_k}), p(b_{m_k}, b_{n_k}), p(a_{m_k}, a_{m_{k+1}}), \\ p(b_{m_k}, b_{m_{k+1}}), p(a_{n_k}, a_{n_{k+1}}), p(b_{n_k}, b_{n_{k+1}}), \\ \frac{1}{2}[p(a_{m_k}, a_{n_{k+1}}) + p(a_{n_k}, a_{m_{k+1}})] \end{array}\right\}\right),
\end{aligned}$$

By using (p_4) we get

$$\psi(p(a_{m_{k+1}},a_{n_{k+1}}) \le \alpha\left(\max\left\{\begin{array}{c} p(a_{m_k},a_{n_k}),p(b_{m_k},b_{n_k}),p(a_{m_k},a_{m_{k+1}}),\\ p(b_{m_k},b_{m_{k+1}}),p(a_{n_k},a_{n_{k+1}}),p(b_{n_k},b_{n_{k+1}}),\\ \frac{1}{2}[p(a_{m_k},a_{n_k})+p(a_{n_k},a_{n_{k+1}})+\\ p(a_{n_k},a_{m_k})+p(a_{m_k},a_{m_{k+1}})] \end{array}\right\}\right)$$

$$-\beta\left(\max\left\{\begin{array}{c} p(a_{m_k},a_{n_k}),p(b_{m_k},b_{n_k}),p(a_{m_k},a_{m_{k+1}}),\\ p(b_{m_k},b_{m_{k+1}}),p(a_{n_k},a_{n_{k+1}}),p(b_{n_k},b_{n_{k+1}}),\\ \frac{1}{2}[p(a_{m_k},a_{n_k})+p(a_{n_k},a_{n_{k+1}})+\\ p(a_{n_k},a_{m_k})+p(a_{m_k},a_{m_{k+1}})] \end{array}\right\}\right).$$

Similarly we have

$$\psi(p(b_{m_{k+1}},b_{n_{k+1}}) \le \alpha\left(\max\left\{\begin{array}{c} p(b_{m_k},b_{n_k}),p(a_{m_k},a_{n_k}),p(b_{m_k},b_{m_{k+1}}),\\ p(a_{m_k},a_{m_{k+1}}),p(b_{n_k},b_{n_{k+1}}),p(a_{n_k},a_{n_{k+1}}),\\ \frac{1}{2}[p(b_{m_k},b_{n_k})+p(b_{n_k},b_{n_{k+1}})+\\ p(b_{n_k},b_{m_k})+p(b_{m_k},b_{m_{k+1}})] \end{array}\right\}\right)$$

$$-\beta\left(\max\left\{\begin{array}{c} p(b_{m_k},b_{n_k}),p(a_{m_k},a_{n_k}),p(b_{m_k},b_{m_{k+1}}),\\ p(a_{m_k},a_{m_{k+1}}),p(b_{n_k},b_{n_{k+1}}),p(a_{n_k},a_{n_{k+1}}),\\ \frac{1}{2}[p(b_{m_k},b_{n_k})+p(b_{n_k},b_{n_{k+1}})+\\ p(b_{n_k},b_{m_k})+p(b_{m_k},b_{m_{k+1}})] \end{array}\right\}\right),$$

Now we have

$$\psi(\max\ \{p(a_{m_{k+1}},a_{n_{k+1}}),p(b_{m_{k+1}},b_{n_{k+1}})\})$$

$$\le \alpha\left(\max\left\{\begin{array}{c} p(b_{m_k},b_{n_k}),p(a_{m_k},a_{n_k}),p(b_{m_k},b_{m_{k+1}}),\\ p(a_{m_k},a_{m_{k+1}}),p(b_{n_k},b_{n_{k+1}}),p(a_{n_k},a_{n_{k+1}}), \end{array}\right\}\right)$$

$$-\beta\left(\max\left\{\begin{array}{c} p(b_{m_k},b_{n_k}),p(a_{m_k},a_{n_k}),p(b_{m_k},b_{m_{k+1}}),\\ p(a_{m_k},a_{m_{k+1}}),p(b_{n_k},b_{n_{k+1}}),p(a_{n_k},a_{n_{k+1}}), \end{array}\right\}\right),$$

Letting $k\to\infty$ and using (2.4),(2.8),(2.16), we have

$$\psi(\frac{\epsilon}{2}) \le \alpha(\frac{\epsilon}{2}) - \beta(\frac{\epsilon}{2}).$$

Thus $\psi(\frac{\epsilon}{2}) - \alpha(\frac{\epsilon}{2}) + \beta(\frac{\epsilon}{2}) \le 0$ and hence $\epsilon = 0$, which is a contradiction. Consequently, $\{a_n\}$ and $\{b_n\}$ are Cauchy.
Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the metric space (X, d_p).
Hence we have $\lim\limits_{n\to\infty} d_p(gx_n, gx_m) = 0 = \lim\limits_{n\to\infty} d_p(gy_n, gy_m)$.
Now from definition of d_p and from (2.3) we have

$$\lim_{n\to\infty} p(gx_n, gx_m) = 0 = \lim_{n\to\infty} p(gy_n, gy_m). \tag{2.17}$$

Suppose $g(X)$ is a complete sub space of X. Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in complete metric space $(g(X), d_p)$, it follows that $\{gx_n\}$ and $\{gy_n\}$ are converge to some r and s in $g(X)$.
Thus

$$\lim_{n\to\infty} d_p(gx_n, r) = 0$$

and

$$\lim_{n\to\infty} d_p(gy_n, s) = 0$$

for some r and s in $g(X)$.
Since $r, s \in g(X)$, there exist $x, y \in X$ such that $r = gx$ and $s = gy$.
Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences and $gx_n \to r$ and $gy_n \to s$, it follows that $gx_{n+1} \to r$ and $gy_{n+1} \to s$.
From Lemma 4(b) and (2.17), we obtain

$$p(r, r) = \lim_{n\to\infty} p(gx_n, r) = p(s, s) = \lim_{n\to\infty} p(gy_n, s) = 0. \tag{2.18}$$

Now from Lemma 5 and (2.18), we have

$$\lim_{n\to\infty} p(F(x, y), gx_n) = p(F(x, y), r) \text{ and}$$

$$\lim_{n\to\infty} p(F(y, x), gy_n) = p(F(y, x), s).$$

From (p_4), we have that

$$\begin{aligned} p(r, F(x, y)) &\le p(r, gx_{n+1}) + p(gx_{n+1}, F(x, y)) - p(gx_{n+1}, gx_{n+1}) \\ &\le p(r, gx_{n+1}) + p(gx_{n+1}, F(x, y)). \end{aligned}$$

Letting $n \to \infty$, we have

$$p(r, F(x, y)) \le 0 + \lim_{n\to\infty} p(F(x_n, y_n), F(x, y)).$$

Also from (16.4), we get $gx_n \preceq gx$ and $gy_n \succeq gy$. Since ψ is continuous and non - decreasing function, we get

$$\begin{aligned} \psi(p(r, F(x, y)) &\le \lim_{n\to\infty} \psi(p(F(x_n, y_n), F(x, y))) \\ &\le \lim_{n\to\infty} \left[\alpha(M(x_n, y_n, x, y)) - \beta(M(x_n, y_n, x, y))\right]. \end{aligned}$$

$$\begin{aligned} M(x_n, y_n, x, y) &= \max \left\{ \begin{array}{c} p(gx_n, r), p(gy_n, s), p(gx_n, gx_{n+1}), \\ p(gy_n, gy_{n+1}), p(r, F(x, y)), p(s, F(y, x)), \\ \frac{1}{2}\left[p(gx_n, F(x, y)) + p(r, gx_{n+1})\right] \end{array} \right\} \\ &\to \max\{p(r, F(x, y)), p(s, F(y, x))\} \text{ as } n \to \infty. \end{aligned}$$

Therefore

$$\psi(p(r, F(x, y))) \le \alpha \left(\max \left\{ \begin{array}{c} p(r, F(x, y)), \\ p(s, F(y, x)) \end{array} \right\} \right) - \beta \left(\max \left\{ \begin{array}{c} p(r, F(x, y)), \\ p(s, F(y, x)) \end{array} \right\} \right).$$

Similarly,

$$\psi(p(s, F(y, x))) \le \alpha \left(\max \left\{ \begin{array}{c} p(r, F(x, y)), \\ p(s, F(y, x)) \end{array} \right\} \right) - \beta \left(\max \left\{ \begin{array}{c} p(r, F(x, y)), \\ p(s, F(y, x)) \end{array} \right\} \right).$$

Hence

$$\begin{aligned} &\psi(\max\{p(r, F(x, y)), p(s, F(y, x))\}) \\ &= \max\{\psi(p(r, F(x, y))), \psi(p(s, F(y, x))\} \\ &\le \alpha \left(\max \left\{ \begin{array}{c} p(r, F(x, y)), \\ p(s, F(y, x)) \end{array} \right\} \right) - \beta \left(\max \left\{ \begin{array}{c} p(r, F(x, y)), \\ p(s, F(y, x)) \end{array} \right\} \right). \end{aligned}$$

Hence from our assumptions, we have

$$\max\left\{ \ p(r, F(x, y)), p(s, F(y, x)) \ \right\} = 0$$

so that $F(x,y)=r$ and $F(y,x)=s$.
Hence $F(x,y)=gx=r$ and $F(y,x)=gy=s$.
Hence F and g have coupled coincidence point (x,y) in $X\times X$.

□

Theorem 17. *In addition to the hypothesis of Theorem 16, suppose that for every $(x,y),(x^1,y^1)\in X\times X$ there exists $(u,v)\in X\times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^1,y^1),F(y^1,x^1))$. If (x,y) and (x^1,y^1) are coupled coincidence points of F and g, then*

$$\begin{aligned} F(x,y) &= gx = gx^1 = F(x^1,y^1) \text{ and} \\ T(y,x) &= gy = gy^1 = F(y^1,x^1). \end{aligned}$$

Moreover if (F,g) is w - compatible, then F and g have a unique common coupled fixed point in $X\times X$.

Proof. From Theorem 16, there exists $(x,y)\in X\times X$ such that

$$F(x,y)=gx=r \text{ and } F(y,x)=gy=s.$$

Thus the existence of coupled coincidence point of F and g is conformed.
Now if (x^1,y^1) be another coupled coincidence point of F and g , we will prove that

$$gx=gx^1 \text{ and } gy=gy^1 \tag{2.19}$$

By additional assumption, there is $(u,v)\in X\times X$ such that
$(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^1,y^1),F(y^1,x^1))$.
Let $u_0=u,\ v_0=v,\ x_0=x,\ y_0=y,\ x^1_0=x^1$ and $y^1_0=y^1$.
Since $F(X\times X)\subseteq g(X)$, we can construct the sequences $\{gu_n\},\{gv_n\},\{gx_n\},\{gy_n\}$, $\{gx^1_n\}$ and $\{gy^1_n\}$ such that

$$\begin{aligned} gu_{n+1} &= F(u_n,v_n), gv_{n+1}=F(v_n,u_n), \\ gx_{n+1} &= F(x_n,y_n), gy_{n+1}=F(y_n,x_n), \\ gx^1_{n+1} &= F(x^1_n,y^1_n) \text{ and } gy^1_{n+1}=F(y^1_n,x^1_n), n=0,1,2,\cdots \end{aligned}$$

Since $(gx,gy)=(F(x,y),F(y,x))$= (gx_1,gy_1) and
$(F(u,v),F(v,u))=(gu_1,gv_1)$ are comparable, then $gx\preceq gu_1$ and $gy\succeq gv_1$.
One can show that $gx\preceq gu_n$ and $gy\succeq gv_n$ for all n.
As in Theorem 16. we conclude that $\{gu_{n+1}\}\to gx$ and $\{gv_{n+1}\}\to gy$.
Analogously, we can show that $\{gu_{n+1}\}\to gx^1$ and $\{gv_{n+1}\}\to gy^1$ in
$(g(X),d_p)$. Since $g(X)$ is complete and $\{gu_{n+1}\}$ converges to gx and gx^1, we get
$gx=gx^1$. Similarly $gy=gy^1$. Hence (2.19) is proved .
Since (F,g) is w - compatible, then

$$\begin{aligned} gr &= g(gx)=g(F(x,y))=F(gx,gy)=F(r,s) \text{ and} \\ gs &= g(gy)=g(F(y,x))=F(gy,gx)=F(s,r). \end{aligned}$$

Hence the pair (r,s) is also coupled coincidence point of F and g. From (2.19), we have

$$gr=gx \text{ and } gs=gy.$$

Therefore

$$r=gr=F(r,s) \text{ and } s=gs=F(s,r).$$

Thus (r,s) is common coupled fixed point of F and g.
The uniqueness of common coupled fixed point of F and g follows easily in the similar lines of the above argument. □

Example 18. *Let $X = [0,1]$ we define a partial order "$\preceq$" on X as $x \preceq y$ if and only if $x \leq y$ for all $x, y \in X$. Let a partial metric space p on X be defined by*
$p(x,y) = \max\{x,y\}$, for all $x, y \in X$, then (X,p) is a complete partial metric space. Define $F : X \times X \to X$ as

$$F(x,y) = \left|\frac{x-y}{2}\right|$$

for all $x, y \in X$ and $g : X \to X$ be defined by

$$gx = x.$$

Let $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = 4t$, $\alpha(t) = 7t$ and $\beta(t) = \frac{7}{2}(t)$. clearly ψ is an altering distance function, α is continuous and β is lower semicontinous, $\alpha(0) = 0$, $\beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) = \frac{t}{2} > 0$ for all $t > 0$.
Now, let $x \preceq u$ and $y \succeq v$. So we have

$$\begin{aligned}
\psi\left(p(F(x,y), F(u,v))\right) &= 4\left(\max\left\{\frac{|x-y|}{2}, \frac{|u-v|}{2}\right\}\right) \\
&= 2\max\left\{|x-y|, |u-v|\right\} \\
&= 2\max\left\{x-y, y-x, u-v, v-u\right\} \\
&\leq 2\max\left\{x, y, u, v\right\} \\
&= 2\max\left\{p(gx,gy), p(gu,gv)\right\} \\
&\leq 2M(x,y,u,v) \\
&\leq 7M(x,y,u,v) - \frac{7}{2}M(x,y,u,v).
\end{aligned}$$

So

$$\psi\left(p(F(x,y), F(u,v))\right) \leq \alpha(M(x,y,u,v)) - \beta(M(x,y,u,v)).$$

Therefore, all of the conditions of Theorem 17 are satisfied. Moreover, $(0,0)$ is the unique coupled coincidence point of F and g.
Now we show that Theorem 14 is not applicable in this example.
For this let (x,y) = (0, 1), (u,v) = (0, 0) and $\phi = \beta$.
Then

$$\psi\left(p(F(x,y), F(u,v))\right) = \psi(p(\frac{1}{2}, 0)) = 4 \times \frac{1}{2} = 2.$$
$$\psi\left(\max\left\{p(x,u), p(y,v)\right\}\right) = \psi\left(\max\left\{0,1\right\}\right) = \psi(1) = 4.$$
$$\phi\left(\max\left\{p(x,u), p(y,v)\right\}\right) = \phi\left(\max\left\{0,1\right\}\right) = \beta(1) = \frac{7}{2}.$$

Thus

$$\psi\left(p(F(x,y), F(u,v))\right) > \psi\left(\max\left\{p(x,u), p(y,v)\right\}\right) - \phi\left(\max\left\{p(x,u), p(y,v)\right\}\right).$$

Hence, the condition (14.1) is not satisfied.

References

[1] Binayak S.Choudhury and Amaresh Kundu, (ψ, α, β)-weak contractions in partially ordered metric spaces, Applied Mathematics Letters, 25(2012), 6 -10.

[2] E. Karapnar, Couple fixed point theorems for nonlinear contractions in cone metric spaces, Computers and Mathematics with Applications, vol. 59, no. 12, pp. 3656 - 3668, 2010.

[3] E. Karapinar, Generalizations of Caristi Kirks theorem on partial metric spaces, Fixed Point Theory and Applications, vol. 2011, article 7, 2011.

[4] H. Aydi, Some coupled fixed point results on partial metric spaces, International Journal of Mathematics and Mathematical Sciences, vol. 2011, Article ID 647091, 11 pages, 2011.

[5] H. Aydi, B. Damjanovic, B. Samet, and W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Mathematical and Computer Modelling, vol. 54, no.9-10, pp. 2443 - 2450, 2011.

[6] H. Aydi, E. Karapinar and W. Shatanawi, Coupled fixed point results for $(\psi - \phi)$-weakly contractive condition in ordered partial metric spaces, Computers and Mathematics with Applications, vol. 62, no. 12, pp. 4449 - 4460, 2011.

[7] H. Aydi, Some fixed point results in ordered partial metric spaces, The Journal of Nonlinear Science and Applications, vol. 4, no. 2, pp. 1 - 12, 2011.

[8] H. Aydi, Fixed point results for weakly contractive mappings in ordered partial metric spaces, Journal of Advanced Mathematical Studies, vol. 4, no. 2, pp. 1 - 12, 2011.

[9] Ishak Altun, Ferhan Sola and Hakam Simsek, Generalized contractions on partial metic spaces, Topology and its Applications, 157(2010), 2778 - 2785.

[10] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, Topology and its Applications, vol. 157, no. 18, pp. 2778 - 2785, 2010.

[11] L. C iric, B. Samet, H. Aydi and C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, Applied Mathematics and Computation, vol. 218, no. 6, pp. 2398 - 2406, 2011.

[12] M. Abbas, M. Ali Khan and S. Radenovic, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Appl. Math. Comput., 217, (2010), 195 - 202.

[13] N. V. Luong and N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Analysis. Theory, Methods and Applications, vol. 74, no. 3, pp. 983 - 992, 2011.

[14] S. Sedghi, I. Altun and N. Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Analysis. Theory, Methods and Applications, vol. 72, no. 3-4, pp. 1298 - 1304, 2010.

[15] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory and Applications, vol. 2010, Article ID 493298, 6 pages, 2010.

[16] S. G. Matthews, Partial metric topology, in Papers on General Topology and Applications, vol. 728, pp.183197, The New York Academy of Sciences, New York, NY, USA, 1994.

[17] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis. Theory, Methods and Applications, vol. 65, no. 7, pp. 1379 - 1393,2006.

[18] V. Lakshmikantham and L. C iric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis. Theory, Methods and Applications, vol. 70, no. 12, pp. 4341 - 4349, 2009.

[19] W. Shatanawi, Coupled fixed point theorems in generalized metric spaces, Hacettepe Journal of Mathematics and Statistics, vol. 40, no. 3, pp. 441 - 447, 2011.

[20] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, Mathematical and Computer Modelling, vol. 55, pp. 680 - 687, 2012.

[21] W. Shatanawi, On w-compatible mappings and common coupled coincidence point in cone metric spaces, Applied Mathematics Letters, vol. 25, no. 6, pp. 925 - 931, 2012.