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Newton–Steffensen–Type Method for Perturbed Nonsmooth Subanalytic Variational Inequalities

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Abstract. This paper is devoted to Newton–Steffensen–type method for approximating the unique solution of perturbed nonsmooth subanalytic variational inclusion in finite–dimensional spaces. We use a combination of Newton's method studied by Bolte et al. [14] for locally Lipschitz subanalytic function in order to solve nonlinear equations, with Steffensen's method [1, 2, 3, 9, 19]. Using the Lipschitz–like concept of set–valued mappings, the subanalyticity hypothesis on the involved function and some condition on divided difference operator, the superlinear convergence is established. We also present a finer convergence analysis using some ideas introduced by us in [4, 7, 8] for nonlinear equations. Finally, we present some new regula–falsi–type method for set–valued map.

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1. Introduction

This work was intented as an attempt to motivate the approximation of solution for variational inclusions in finite–dimensional spaces. For illustration example, we consider a variational inequality problem, who consists of seeking $k^\star$ in $\mathcal{K}$ such that

$$\text{For each } k \in \mathcal{K}, \quad (\Pi(k^\star), k - k^\star) \ \geq \ 0 \tag{1.1}$$

where $\mathcal{K}$ is a convex set in $\mathbb{R}^n$, Π is a function from $\mathcal{K}$ to $\mathbb{R}^n$ and $(.,.)$ is the usual scalar product on $\mathbb{R}^n$. Let $\mathcal{I}_{\mathcal{K}}$ be a convex indicator function of $\mathcal{K}$ and ∂ denotes the subdifferential operator. Then (1.1) is equivalent to the following problem

$$0 \in \Pi(k^\star) + \mathcal{R}(k^\star) \tag{1.2}$$

with $\mathcal{R} = \partial \mathcal{I}_{\mathcal{K}}$. The variational inequality problem (1.1) is equivalent to (1.3) which is a generalized equation introduced by Robinson [21]:

$$0 \in F(x) + G(x). \tag{1.3}$$

where F is a function from $\mathcal{X}$ into $\mathcal{Y}$ and G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{Y}$ and $\mathcal{X}$, $\mathcal{Y}$ are two Banach spaces.

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the inexact variational inclusion

$$0 \in F(x) + H(x) + G(x), \tag{1.4}$$

where F is a nonsmooth subanalytic function from an open subset $\mathcal{D}$ of $\mathcal{X} = \mathbb{R}^n$ into $\mathcal{X}$, H is a continuous function from $\mathcal{D}$ into $\mathcal{X}$ and G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{X}$ with closed graph.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of generalized equation (1.4). In the particular case $H = 0$ and $G = \{0\}$, (1.4) is a nonlinear equation in the form

$$F(x) = 0. \tag{1.5}$$

For example, dynamic systems are mathematically modeled by differential or difference equations and their solutions usually represent the states of the systems, which are determined by solving equation (1.5).

When F is Fréchet–differentiable in a neighborhood of the solution $x^\star$ of equation (1.4), most of the numerical approximation methods require the expensive computation of the Fréchet–derivative $F'(x)$ and the first order divided difference of operators F and H at each step respectively, for example Newton–Steffensen–type method [9, 19] for given $x_0 \in \mathcal{D}$ and $k \geq 1$:

$$0 \in F(x_k) + H(x_k) + \Big(\nabla F(x_k) + [g_1(x_k), g_2(x_k); H]\Big)(x_{k+1} - x_k) + G(x_{k+1}), \tag{1.6}$$

where $g_i : \mathcal{D} \longrightarrow \mathcal{X}$ $(i = 1, 2)$ is a continuous mapping and $[x, y; H] \in \mathcal{L}(\mathcal{X})$ is a divided difference of order one satisfying

$$[x, y; H]\,(x - y) = H(x) - H(y) \text{ for all } x, y \in \mathcal{X} \text{ with } x \neq y. \tag{1.7}$$

Note that if H is Fréchet–differentiable at x then $[x, x; H] = \nabla H(x)$ (see [5, 7]). A superlinear convergence analysis is presented in [18] for iterative method (1.6) in the particular case $g_1(x_k) = \beta\, x_k + (1 - \beta)\, x_{k-1}$, for some fixed parameter β in $[0, 1[$, using some Hölder–type conditions:

$$\begin{aligned} &\| \,[x, y; H] - [u, v; H]\, \| \leq \ \nu\ (\| x - u \|^p + \| y - v \|^p), \\ &\text{for all } x, y, u, v \in \mathcal{D},\ p \in [0, 1] \text{ and } \nu > 0, \end{aligned} \tag{1.8}$$

and

$$\| \nabla F(x) - \nabla F(y) \| \leq \sigma \ \ \| x - y \|^p, \ \text{ for } x,\ y \text{ in } \mathcal{D},\ p \in [0, 1] \text{ and } \sigma > 0. \tag{1.9}$$

We relax the assumptions (1.8) and (1.9) in [9] by using (ω, μ)–conditions:

$$\| \,[x, y; H] - [u, v; H]\, \| \leq \omega(\| x - u \|, \| y - v \|), \ \text{ for } x,\ y,\ u \text{ and } v \text{ in } D \tag{1.10}$$

and

$$\| \nabla F(x) - \nabla F(y) \| \leq \mu(\| x - y \|), \text{ for } x,\, y \,\text{in } D, \tag{1.11}$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous nondecreasing function in both arguments, and $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous nondecreasing function.

In this study, we are interested in numerical method for solving generalized equation (1.4) when the involved function F is nonsmooth and subanalytic. We proceed by replacing in method (1.6) the term $F'(x_n)$ by $\Delta F(x_n)$, where $\Delta F(x) \in \partial^\circ F(x)$, $\partial^\circ F(x)$ denotes the Clarke Jacobian of F at the point $x \in \mathcal{D}$. For approximating $x^\star$, we consider an iterative method ($x_0 \in \mathcal{D}$) and for $k \geq 1$:

$$0 \in F(x_k) + H(x_k) + \bigg(\Delta F(x_k) + [g_1(x_k), g_2(x_k); H]\bigg)(x_{k+1} - x_k) + G(x_{k+1}), \tag{1.12}$$

In the nonlinear equations case (i.e., $H = 0$ and $G = \{0\}$ in (1.4)), the method (1.12) becomes

$$0 = F(x_n) + \Delta F(x_n)\ (x_{n+1} - x_n), \quad (x_0 \in \mathcal{D}), \quad (n \geq 0), \tag{1.13}$$

which is considered in the elegant work by Bolte, Daniilidis and Lewis [14] for globally subanalytic mappings.

In the case $H = 0$ and $G \neq \{0\}$, a superlinear convergence is given in [12] under the following conditions:

$$\text{There exists } K > 0, \text{ such that } \forall x \in \mathcal{D},\ \forall \Delta F(x) \in \partial^\circ F(x),\ \| \Delta F(x) \| \leq K \tag{1.14}$$

and

$$\begin{array}{l}\text{For all } \Delta F(x^\star) \in \partial^\circ F(x^\star),\ (G + \Delta F(x^\star)\,(.-x^\star))^{-1} \text{ is } M-\text{pseudo}-\text{Lipschitz} \\ \text{around } (-F(x^\star), x^\star) \text{ with } M > 0, \text{ and}\, 2\, M\, K < 1.\end{array} \tag{1.15}$$

A finer convergence analysis than [12] is obtained in [10] under assumption (1.15) without the second part condition: $2\ M\ K < 1$ and the center–type condition:

$$\begin{array}{l}\text{There exists } K^\star > 0, \text{ such that } \| \Delta F(x) - \Delta F(x^\star) \| \leq K^\star\ \ \| x - x^\star \|, \\ \text{for all } \ \ \Delta F(x) \in \partial^\circ F(x) \text{ and } \Delta F(x^\star) \in \partial^\circ F(x^\star).\end{array} \tag{1.16}$$

Here, we are motivated by the works in [14, 10]. Using a Hölder–type conditions [5, 7], we extend the applicability of Newton–Steffensen–type method [9, 18, 19] to nonsmooth subanalytic variational inequalities. Using the Lipschitz–like concept of set–valued mappings, the subanalyticity hypothesis on the involved function, and some condition on divided difference operator, we prove that the iterative method (1.12) converges superlinearly.

The paper is organized as follows: In section 2, we collect some definitions and we recall the fixed point theorem [17]. The main results of existence and convergence for Newton–Steffensen–type algorithm (1.12) are developed in section 3. In section 4, we provide a finer convergence analysis using some ideas introduced by us in [4, 7, 8] for nonlinear equations. Finally, we present a new regula–falsi–type method for set–valued maps and some remarks.

2. Background material

In order to make the paper as self–contained as possible we reintroduce some definitions and some results on fixed point theorems [7, 8, 23]. Let us begin with some notations that will used throughout this paper. $\mathcal{X} = \mathbb{R}^n$ is equiped with the euclidian norm $\| \ .\ \|$. We denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r. The distance from a point

x to a set A in $\mathcal{X}$ is defined by $\operatorname{dist}(x,A) = \inf_{y\in A} \parallel x - y \parallel$, with the convention $\operatorname{dist}(x,\emptyset) = +\infty$ (according to the general convention $\inf \emptyset = +\infty$). Given a subset C of $\mathcal{X}$, we denote by $e(C,A)$ the Hausdorff–Pompeiu excess of C into A, defined by

$$e(C,A) = \sup_{x\in C} \operatorname{dist}(x,A),$$

with the conventions $e(\emptyset,A) = 0$ and $e(C,\emptyset) = +\infty$ whenever $C \neq \emptyset$. For a set–mapping $\Gamma : \mathcal{X} \rightrightarrows \mathcal{X}$, we denote by $\operatorname{gph}\Gamma$ the set $\{(x,y) \in \mathcal{X}\times\mathcal{X},\ y \in \Gamma(x)\}$ and $\Gamma^{-1}(y)$ the set $\{x \in \mathcal{X},\ y \in \Gamma(x)\}$. For each $n \in \mathbb{N}$, we define $\tau_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$\tau_n(x_1,x_2,\cdots,x_n) = \left(\frac{x_1}{\sqrt{1+x_1^2}}, \frac{x_2}{\sqrt{1+x_2^2}}, \cdots, \frac{x_n}{\sqrt{1+x_n^2}}\right). \tag{2.1}$$

We also need to define the pseudo–Lipschitzian concept of set–valued maps, introduced by Aubin [11] and also known as Lipschitz–like property [20]:

Definition 1. A set–valued Γ is pseudo–Lipschitz around $(\overline{x},\overline{y}) \in \operatorname{gph}\Gamma$ with modulus M if there exist constants a and b such that

$$\sup_{z\in\Gamma(y')\cap\mathbb{B}_a(\overline{y})} \operatorname{dist}(z,\Gamma(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{2.2}$$

In the term of excess, we have an equivalent definition of pseudo–Lipschitzian property replacing the inequality (2.2) by

$$e(\Gamma(y')\cap\mathbb{B}_a(\overline{y}),\Gamma(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{2.3}$$

Pseudo–Lipschitzian property play a crucial role in many aspects of variational analysis and applications [20, 23]. Let us note that the Lipschitz–like of Γ is equivalent to the metric regularity of Γ^{-1} which is a basic well–posedness property in optimization problems. Other characterization is by Mordukhovich [20] via the concept of coderivative of set–valued maps. For some characterizations and applications of the Lipschitz–like property the reader could be referred to [11, 17, 20, 22, 23] and the references given there.

We recall the following definition of semianalytic subsets and subanalytic functions [16, 24, 13, 14].

Definition 2.
(1) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called semianalytic if each point of $\mathbb{R}^n$ admits a neighborhood $\mathcal{V}$ for which $\mathcal{A}\cap\mathcal{V}$ assumes the following form:

$$\bigcup_{i=1}^{i=p}\bigcap_{i=1}^{i=q}\{x\in\mathcal{V} : f_{ij}(x) = 0,\ g_{ij}(x) > 0\}, \tag{2.4}$$

where the functions $f_{ij},\ g_{ij} : \mathcal{V} \longrightarrow \mathbb{R}$ are real–analytic for all $1 \leq i \leq p$, $1 \leq j \leq q$.

(2) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called subanalytic if each point of $\mathbb{R}^n$ admits a neighborhood $\mathcal{V}$ such that:

$$\mathcal{A}\cap\mathcal{V} = \{x \in \mathbb{R}^n : (x,y) \in \mathcal{B}\}, \tag{2.5}$$

where $\mathcal{B}$ is a bounded semianalytic subset of $\mathbb{R}^n\times\mathbb{R}^m$ for some $m \geq 1$.

(3) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called globally subanalytic if its image by τ_n defined by (2.1) is a subanalytic subset of $\mathbb{R}^n$.

(4) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called subanalytic, if its graph is a subanalytic subset of $\mathbb{R}^n\times\mathbb{R}^n$.

(5) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called globally subanalytic, if its graph is a globally subanalytic subset of $\mathbb{R}^n\times\mathbb{R}^n$.

We need also the following fixed point theorem [17].

Lemma 3. *Let ϕ be a set–valued map from $\mathcal{X}$ into the closed subsets of $\mathcal{X}$. We suppose that for $\eta_0 \in \mathcal{X}$, $r \geq 0$ and $0 \leq \lambda < 1$ the following properties hold*

(1) dist $(\eta_0, \phi(\eta_0)) < r\ (1 - \lambda)$.
(2) $e(\phi(y) \cap \mathbb{B}_r(\eta_0), \phi(z)) \leq \lambda\ \parallel y - z \parallel, \ \ \forall y, z \in \mathbb{B}_r(\eta_0)$.

Then ϕ has a fixed point in $\mathbb{B}_r(\eta_0)$. That is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $\mathbb{B}_r(\eta_0)$.

Finally, we recall a definition concerning directional differentiability and Clarke's Jacobian in finite dimensional spaces.

Definition 4. (1) A mapping $F\ :\ \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be directionally differentiable at $x \in \mathcal{D}$ along direction d if the following limit

$$F'(x; d) := \lim_{t \downarrow 0} \frac{F(x + t\ d) - F(x)}{t} \tag{2.6}$$

exists.

Note that every definable locally Lipschitz mapping F admits directional derivatives.

(2) For $F\ :\ \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a locally Lipschitz continuous function, the limiting Jacobian of F at $x \in \mathcal{D}$ is defined by

$$\partial F(x) = \{\mathcal{M} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\ :\ \exists\ u^k \in \mathcal{D},\ \lim_{k \longrightarrow \infty}\ F'(u^k) = \mathcal{M}\}. \tag{2.7}$$

(3) Let $F\ :\ \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a locally Lipschitz continuous function. Clarke's Jacobian of F at $x \in \mathcal{D}$ is defined by

$$\partial^\circ F(x) = \overline{\text{co}}\ \partial F(x), \tag{2.8}$$

where $\overline{\text{co}}\ \mathcal{A}$ is the closed convex envelope of $\mathcal{A} \subseteq \mathbb{R}^n$.

3. Local convergence of method (1.12)

Before presenting our main result of convergence of method (1.12), we give a variant of the result for subanalytic mappings established by Bolte, Daniilidis and Lewis [14, Lemma 3.3]:

Lemma 5. *Let $F\ :\ \mathcal{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a locally Lipschitz subanalytic function and $x \in \mathcal{D}$. Then, there exists a positive rational number γ and a constant $C_x > 0$, such that:*

$$\parallel F(y) - F(x) - \Delta(y)\ (y - x) \parallel \leq C_x\ \ \parallel y - x \parallel^{1+\gamma} \tag{3.1}$$

where $\Delta(y)$ is any element of $\partial^\circ F(y)$.

In particular case, there exists a positive rational number $\gamma^\star$ and a constant $C_{x^\star} > 0$, such that:

$$\parallel F(y) - F(x^\star) - \Delta(y)\ (y - x^\star) \parallel \leq C_{x^\star}\ \ \parallel y - x^\star \parallel^{1+\gamma^\star} \tag{3.2}$$

where $\Delta(y)$ is any element of $\partial^\circ F(y)$.

The main result of local convergence of algorithm (1.12) is as follows. We will be concerned with the existence and the convergence of the sequence defined by (1.12) to the solution $x^\star$ of (1.4). The main result of this study is as follows.

Theorem 6. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{X}$ be a locally Lipschitz subanalytic function. Let $H : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{X}$ be an operator such that for every distinct points x and y in $\mathcal{D}$, there exists a first order divided difference of H $[x, y; H]$ at these points. G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{X}$ with closed graph and $x^\star$ is a solution of (1.4).*

Assume:

($\mathcal{P}0$) *For $i = 1, 2$; g_i is α_i–Lipschitz from $\mathcal{D}$ into $\mathcal{D}$, $\alpha_i \in [0, 1)$ and $g_i(x^\star) = x^\star$;*

($\mathcal{P}1$) *There exists $\nu > 0$ such that for all x, y, u and v in $\mathcal{D}$ ($x \neq y$ and $u \neq v$)*

$$\| [x, y; H] - [u, v; H] \| \leq \nu \ (\| x - u \|^\gamma + \| y - v \|^\gamma)$$

where γ is given in Lemma 5 by (3.1).

($\mathcal{P}2$) *For all $\Delta F(x^\star) \in \partial^\circ F(x^\star)$, the set–valued map $(\Delta F(x^\star)\ (. - x^\star) + H + G)^{-1}$ is pseudo–Lipschitz around $(-F(x^\star), x^\star)$ with constants M, a and b (These constants are given in Definition 1);*

($\mathcal{P}3$) *There exists $\sigma > 0$, such that for all x and y in $\mathcal{D}$*

$$\| \Delta F(x) - \Delta F(y) \| \leq \sigma \ \| x - y \| \ \text{ for all } (\Delta F(x), \Delta F(y)) \in \partial^\circ F(x) \times \partial^\circ F(y).$$

Then, for every constant C such that

$$C \geq C_0 = M \left(C_{x^\star} + \nu \ (\alpha_1^\gamma + (1 + \alpha_2)^\gamma) \right), \tag{3.3}$$

where $C_{x^\star}$ is given in (3.2), there exists $\delta > 0$ satisfying

$$\delta < \delta_0 = \min \left\{ a; \ \sqrt[\gamma]{\frac{1}{C}}; \ \sqrt{\frac{b}{2\ \sigma}}; \ \delta_1 \right\} \tag{3.4}$$

where

$$\delta_1 = \sqrt[1+\gamma]{\frac{b}{2 \left(C_{x^\star} + 2\ \nu\ ((1 + \alpha_1)^\gamma + (1 + \alpha_2)^\gamma) \right)}}$$

such that for every starting point x_0 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$), sequence (x_k) defined by (1.12) converges to $x^\star$ and satisfies the following inequality for $k \geq 0$

$$\| x_{k+1} - x^\star \| \leq C \ \ \| x_k - x^\star \|^{1+\gamma} \tag{3.5}$$

where γ is given by (3.2).

We need to introduce some standard notations [8, 9]. First, define the set–valued map $Q : \mathcal{D} \subseteq \mathcal{X} \rightrightarrows \mathcal{X}$ by

$$Q(x) = F(x^\star) + H(x) + \Delta F(x^\star)\,(x - x^\star) + G(x). \tag{3.6}$$

For $k \in \mathbb{N}^*$ and x_k defined in (1.12), we consider the mapping

$$\begin{aligned} Z_k(x) \quad := \quad & F(x^\star) + H(x) + \Delta F(x^\star)\,(x - x^\star) - F(x_k) - H(x_k) - \\ & \left(\Delta F(x_k) + [g_1(x_k), g_2(x_k); H] \right) (x - x_k). \end{aligned} \tag{3.7}$$

Finally, define the set–valued map $\psi_k : \mathcal{D} \subseteq \mathcal{X} \rightrightarrows \mathcal{X}$ by

$$\psi_k(x) := Q^{-1}(Z_k(x)). \tag{3.8}$$

Remark 7. Hypothesis ($\mathcal{P}2$) is equivalent to Q^{-1} is pseudo–Lipschitz around $(0, x^\star)$. Indeed, simply use [17, Corollary 2, p. 486] by identifying F and f in this corollary to G and the constant function $F(x^\star)$ respectively.

The proof of theorem 6 is given by induction on k. We first state a result involving the starting point x_0. Let us note that the point x_1 is a fixed point of ψ_0 if and only if

$$0 \in F(x_0) + H(x_0) + \bigg(\Delta F(x_0) + [g_1(x_0), g_2(x_0); H]\bigg)\ (x_1 - x_0) + G(x_1).$$

Once x_k is computed, we show that the function ψ_k has a fixed point x_{k+1} in $\mathcal{X}$. This process is useful to prove the existence of a sequence (x_k) satisfying (1.12).

Remark 8. The results of this paper seem also true for a general assumption: F and H are defined in a neighborhood $\mathcal{D}$ of the solution $x^\star$ included in $\mathcal{X} = \mathbb{R}^n$ with values in $\mathcal{Y} = \mathbb{R}^m$ with $n \neq m$ and G is a set–valued map from $\mathcal{X}$ to its subsets of $\mathcal{Y}$ with closed graph.

Proposition 9. *Under the assumptions of Theorem 6 and for every distinct starting points x_0 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$), the set–valued map ψ_0 has a fixed point x_1 in $\mathbb{B}_\delta(x^\star)$ satisfying*

$$\| x_1 - x^\star \| \leq C\ \ \| x_0 - x^\star \|^{1+\gamma} \tag{3.9}$$

where C and δ are given by Theorem 6.

Proof. By hypothesis ($\mathcal{P}2$) we have

$$e(Q^{-1}(y') \cap \mathbb{B}_a(x^\star), Q^{-1}(y'')) \leq M \parallel y' - y'' \parallel,\ \forall y', y'' \in \mathbb{B}_b(0). \tag{3.10}$$

The main idea of the proof of Proposition 9 is to show that both assertions (1) and (2) of Lemma 3 hold, where $\eta_0 := x^\star$, ϕ is the function ψ_0 defined by (3.8) and where r and λ are numbers to be set.

According to the definition of the excess e, we have

$$dist\,(x^\star, \psi_0(x^\star)) \leq e\bigg(Q^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \psi_0(x^\star)\bigg). \tag{3.11}$$

Note that for $x \in B_\delta(x^\star)$ using ($\mathcal{P}0$) we can have for $i = 1, 2$

$$\| g_i(x) - x^\star \| = \| g_i(x) - g_i(x^\star) \| \leq \alpha_i \parallel x - x^\star \| \leq \| x - x^\star \| \leq \delta,$$

which implies $g_i(x) \in \mathbb{B}_\delta(x^\star) \subseteq \mathcal{D}$.

For all point x_0 in $B_\delta(x^\star)$ ($x_0 \neq x^\star$) we have

$$\begin{aligned} \| Z_0(x^\star) \| \ &= \ \| F(x^\star) + H(x^\star) - F(x_0) - H(x_0) - \\ &\quad \bigg(\Delta F(x_0) + [g_1(x_0), g_2(x_0); H]\bigg)(x^\star - x_0) \| \,. \end{aligned}$$

Using definitions (1.7) and (3.7) we obtain the following

$$\begin{aligned} \| Z_0(x^\star) \| \ &\leq \ \| F(x^\star) - F(x_0) - \Delta F(x_0)\ (x^\star - x_0) \| + \\ &\quad\ \| H(x^\star) - H(x_0) - [g_1(x_0), g_2(x_0); H]\ (x^\star - x_0) \| \\ &= \ \| F(x_0) - F(x^\star) - \Delta F(x_0)\ (x_0 - x^\star) \| + \\ &\quad\ \| [x^\star, x_0; H]\ (x^\star - x_0) - [g_1(x_0), g_2(x_0); H]\ (x^\star - x_0) \| \,. \end{aligned} \tag{3.12}$$

By **(**$\mathcal{P}0$**)**, **(**$\mathcal{P}1$**)** and (3.1), (3.12) becomes

$$\begin{aligned} \| Z_0(x^\star) \| \ &\leq \ C_{x^\star}\ \| x^\star - x_0 \|^{\gamma+1} + \\ &\quad\ \nu \bigg(\ \| x^\star - g_1(x_0) \|^\gamma + \| x_0 - g_2(x_0) \|^\gamma\ \bigg)\ \| x^\star - x_0 \| \\ &\leq \ \bigg(C_{x^\star} + \nu\,(\alpha_1^\gamma + (1+\alpha_2)^\gamma)\bigg)\ \| x_0 - x^\star \|^{\gamma+1}\,. \end{aligned} \tag{3.13}$$

Then (3.4) yields, $Z_0(x^\star) \in \mathbb{B}_b(0)$.

Moreover, by (2.3) and (3.11) we obtain

$$\operatorname{dist}(x^\star, \psi_0(x^\star)) \quad \leq \quad C_0 \ \parallel x_0 - x^\star \parallel^{\gamma+1} . \tag{3.14}$$

Since $C \geq C_0$ there exists $\lambda \in [0, 1[$ such that $C\ (1-\lambda) \geq C_0$ and

$$\operatorname{dist}(x^\star, \psi_0(x^\star)) \leq C\ (1-\lambda)\ \parallel x_0 - x^\star \parallel^{\gamma+1} .$$

Setting $r := r_0 = C\ \parallel x_0 - x^\star \parallel^{\gamma+1}$, we can deduce the first assertion of Lemma 3.

By (3.4) we have $r_0 \leq \delta \leq a$. Moreover for $x \in \mathbb{B}_\delta(x^\star)$ and by some intermediate estimations we have the following inequality

$$\begin{aligned} \parallel Z_0(x) \parallel &= \parallel F(x^\star) + H(x) + \Delta F(x^\star)(x - x^\star) - F(x_0) - H(x_0) - \\ &\quad \Big(\Delta F(x_0) + [g_1(x_0), g_2(x_0); H]\Big)(x - x_0) \parallel \\ &\leq \parallel F(x^\star) + \Delta F(x^\star)(x - x^\star) - F(x_0) - \Delta F(x_0)(x - x^\star + x^\star - x_0) \parallel + \\ &\quad \parallel H(x) - H(x_0) - [g_1(x_0), g_2(x_0); H]\ (x - x_0) \parallel \\ &\leq \parallel F(x_0) - F(x^\star) - \Delta F(x_0)\,(x_0 - x^\star) \parallel + \\ &\quad \parallel \Delta F(x^\star) - \Delta F(x_0) \parallel \parallel x - x^\star \parallel + \\ &\quad \parallel [x, x_0; H] - [g_1(x_0), g_2(x_0); H] \parallel \parallel x - x_0 \parallel . \end{aligned} \tag{3.15}$$

By assumptions **(**$\mathcal{P}0$**)**, **(**$\mathcal{P}1$**)**, **(**$\mathcal{P}3$**)**, and (3.1), we can estimate (3.15)

$$\begin{aligned} \parallel Z_0(x) \parallel &\leq C_{x^\star}\ \parallel x^\star - x_0 \parallel^{\gamma+1} + \sigma\ \parallel x^\star - x_0 \parallel \parallel x - x^\star \parallel + \\ &\quad \nu \left(\ \parallel x - g_1(x_0) \parallel^\gamma + \parallel x_0 - g_2(x_0) \parallel^\gamma \right)\ \parallel x - x_0 \parallel \\ &\leq \Big(C_{x^\star} + 2\ \nu\ ((1+\alpha_1)^\gamma + (1+\alpha_2)^\gamma)\Big)\ \delta^{\gamma+1} + \sigma\ \delta^2. \end{aligned} \tag{3.16}$$

Then by (3.4) we deduce that for all $x \in \mathbb{B}_\delta(x^\star)$ we have $Z_0(x) \in \mathbb{B}_b(0)$. Then it follows that for all $x', x'' \in \mathbb{B}_{r_0}(x^\star)$ we have

$$\begin{aligned} e(\psi_0(x') \cap B_{r_0}(x^\star), \psi_0(x'')) &\leq M\ \parallel Z_0(x') - Z_0(x'') \parallel \\ &= M\ \parallel H(x') - H(x'') + \Delta F(x^\star)\ (x' - x'') + \\ &\quad \Big(\Delta F(x_0) + [g_1(x_0), g_2(x_0); H]\Big)\ (x'' - x') \parallel \\ &\leq M\ \Big(\ \parallel (\Delta F(x_0) - \Delta F(x^\star))\ (x' - x'') \parallel + \\ &\quad \parallel [x', x''; H] - [g_1(x_0), g_2(x_0); H] \parallel \parallel (x'' - x') \parallel\ \Big). \end{aligned} \tag{3.17}$$

By **(**$\mathcal{P}1$**)** and **(**$\mathcal{P}3$**)** we deduce

$$e(\psi_0(x') \cap \mathbb{B}_{r_0}(x^\star), \psi_0(x'')) \leq M\ \Theta_\delta\ \parallel x'' - x' \parallel \tag{3.18}$$

where

$$\Theta_\delta = \sigma\ \delta + \nu\ ((1+\alpha_1)^\gamma + (1+\alpha_2)^\gamma)\ \delta^\gamma. \tag{3.19}$$

Without loss generality, we may assume that $\Theta_\delta < \dfrac{\lambda}{M}$. Since both conditions of Lemma 3 are fulfilled, we can deduce the existence of a fixed point $x_1 \in \mathbb{B}_{r_0}(x^\star)$ for the map ψ_0. □

Proof. (Proof of Theorem 6) Keep $\eta_0 = x^\star$ and for $k \geq 1$, set:

$$r := r_k = C\ \parallel x^\star - x_k \parallel^{1+\gamma} .$$

The application of Proposition 9 to the map ψ_k gives the desired result. □

Remark 10. We can enlarge the radius of convergence in Theorem 6 even further as follows: using inequalities (3.16), (3.13), we can improve δ given by (3.4) by considering the constant δ':

$$\delta' < \delta'_0 = \min\left\{a;\ \sqrt[\gamma]{\frac{1}{C}};\ \delta'_2\right\}$$

where δ'_2 is given by

$$\delta'_2 = \max\ \{\eta > 0\ :\ \Big(C_{x^\star} + 2\ \nu\ ((1+\alpha_1)^\gamma + (1+\alpha_2)^\gamma)\Big)\ \eta^{\gamma+1} + \sigma\ \eta^2 - b < 0\}.$$

Remark 11. If we replace in Theorem 6 the assumption **($\mathcal{P}3$)** by the following assumption:

($\mathcal{P}3$)' There exists $\sigma_0 > 0$, such that for all x and y in $\mathcal{D}$

$$\parallel \Delta F(x) - \Delta F(y) \parallel \leq \sigma_0\ \parallel x - y \parallel^\gamma\ \text{ for all } (\Delta F(x), \Delta F(y)) \in \partial^\circ F(x) \times \partial^\circ F(y),$$

where γ is the positive rational given by (3.1), then, it follows from the proof of Proposition 9 that constants δ and Θ_δ can be replaced by the following

$$\overline{\delta} < \overline{\delta}_0 = \min\left\{a;\ \sqrt[\gamma]{\frac{1}{C}};\ \sqrt[1+\gamma]{\frac{b}{2\ \Big(\sigma_0 + C_{x^\star} + 2\ \nu\ ((1+\alpha_1)^\gamma + (1+\alpha_2)^\gamma)\Big)}}\right\} \tag{3.20}$$

and

$$\overline{\Theta}_{\overline{\delta}} = \Big(\sigma_0 + 2\ \nu\ ((1+\alpha_1)^\gamma + (1+\alpha_2)^\gamma)\Big)\ \overline{\delta}^\gamma, \tag{3.21}$$

respectively. The estimate (3.5) of Theorem 6 seem also true.

Yakoubsohn [25] considers a regula–falsi algorithm for solving nonlinear equations. An extension of this method for perturbed generalized equations is presented in [8, 18, 19] using Hölder type condition (or ω–conditioned divided difference). Here, we consider our algorithm (1.12) by fixing $g_1(x_0)$ of the arguments of divided difference of H; more precisely, we associate to (1.4) the following algorithm ($k = 1, 2, \dots$)

$$\begin{cases} x_0 \text{ is given as starting point in } \mathcal{D} \\ 0 \in F(x_k) + H(x_k) + \Big(\Delta F(x_k) + [g_1(x_0), g_2(x_k); H]\Big)\ (x_{k+1} - x_k) + G(x_{k+1}). \end{cases} \tag{3.22}$$

We deduce the following result.

Proposition 12. *Suppose that the assumptions of Theorem 6 are satisfied. Then, for every constant $C > C_0$, where C_0 is given in Theorem 6, there exists $\zeta > 0$ such that, for every starting point x_0 in $\mathbb{B}_\zeta(x^\star)$ (x_0 and $x^\star$ distinct), the sequence (x_k) defined by (3.22) converges to $x^\star$ and satisfies*

$$\parallel x_{k+1} - x^\star \parallel \leq C\ \parallel x_k - x^\star \parallel\ \max\ \{\parallel x_k - x^\star \parallel^\gamma, \parallel x_0 - x^\star \parallel^\gamma\},$$

where γ is given by (3.1).

4. An improved local convergence

In this section, motivated by optimization considerations [5, 7] related to the resolution on nonlinear equations, we show by using more precise estimates that under less computational cost and weaker hypotheses than **($\mathcal{P}0$)**, **($\mathcal{P}1$)**, and **($\mathcal{P}3$)**: the convergence analysis of method (1.12) is improved. We can show the following results for the local convergence of method (1.12).

Theorem 13. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{X}$ be a locally Lipschitz subanalytic function. Let $H : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{X}$ be an operator such that for every distinct points x and y in $\mathcal{D}$, there exists a first order divided difference of $F\,[x, y; H]$ at these points. G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{X}$ with closed graph and $x^\star$ is a solution of (1.4).*

Assume:

($\mathcal{P}0$)* *For $i = 1, 2$; g_i is $\overline{\alpha_i}$–center–Lipschitz from $\mathcal{D}$ into $\mathcal{D}$, $\alpha_i \in [0, 1)$ and $g_i(x^\star) = x^\star$. That is*

$$\| g_i(x) - g_i(x^*) \| \leq \overline{\alpha_i} \;\; \| x - x^\star \|, \; i = 1, 2; \tag{4.1}$$

($\mathcal{P}1$)* *There exists $\overline{\nu_1} > 0$ and $\overline{\nu_2} > 0$, such that for all x, y, z in $\mathcal{D}$*

$$\| [x^\star, x; H] - [g_1(x), g_2(x); H] \| \leq \overline{\nu_1} \; (\| x^\star - g_1(x) \|^\gamma + \| x - g_2(x) \|^\gamma), \tag{4.2}$$

$$\| [y, z; H] - [g_1(x), g_2(x); H] \| \leq \overline{\nu_2} \; (\| y - g_1(x) \|^\gamma + \| z - g_2(x) \|^\gamma) \tag{4.3}$$

where γ is given in Lemma 5 by (3.1).

($\mathcal{P}2$) *For all $\Delta F(x^\star) \in \partial^\circ F(x^\star)$, the set–valued map $(\Delta F(x^\star)\;(. - x^\star) + H + G)^{-1}$ is pseudo–Lipschitz around $(-F(x^\star), x^\star)$ with constants M, a and b (These constants are given in Definition 1);*

($\mathcal{P}3$)* *There exists $\overline{\sigma} > 0$, such that for all x in $\mathcal{D}$*

$$\| \Delta F(x) - \Delta F(x^\star) \| \leq \overline{\sigma} \;\; \| x - x^\star \| \;\; \text{for all } (\Delta F(x), \Delta F(x^\star)) \in \partial^\circ F(x) \times \partial^\circ F(x^\star).$$

Then, for every $\overline{C}$ such that

$$\overline{C} \geq \overline{C_0} = M\, \Bigg(C_{x^\star} + \overline{\nu_1}\; (\overline{\alpha_1}^\gamma + (1 + \overline{\alpha_2})^\gamma) \Bigg), \tag{4.4}$$

where $C_{x^\star}$ is given in (3.1), there exists $\overline{\overline{\delta}} > 0$ satisfying

$$\overline{\overline{\delta}} < \overline{\overline{\delta_0}} = \min \left\{ a;\; \sqrt[\gamma]{\frac{1}{\overline{\overline{C}}}};\; \sqrt{\frac{b}{2\,\overline{\sigma}}};\; \overline{\delta_1} \right\} \tag{4.5}$$

where

$$\overline{\delta_1} = \sqrt[1+\gamma]{\frac{b}{2\,\Bigg(C_{x^\star} + 2\;\overline{\nu_2}\;((1 + \overline{\alpha_1})^\gamma + (1 + \overline{\alpha_2})^\gamma) \Bigg)}}$$

such that for every starting point x_0 in $\mathbb{B}_{\overline{\overline{\delta}}}(x^\star)$ (with $x_0 \neq x^\star$), the sequence (x_k) defined by (1.12) converges to $x^\star$ and satisfies the following inequality for $k \geq 0$

$$\| x_{k+1} - x^\star \| \leq \overline{C} \;\; \| x_k - x^\star \|^{1+\gamma} \,. \tag{4.6}$$

Remark 14. In general, α_i ($i = 1, 2$), ν, and σ given in Theorem 6 are not easy to compute. This is our motivation for introducing weaker hypotheses **($\mathcal{P}0$)***, **($\mathcal{P}1$)*** and **($\mathcal{P}3$)*** in Theorem 13.

Note that in general

$$\overline{\alpha_1} \leq \alpha_1, \tag{4.7}$$

$$\overline{\alpha_2} \leq \alpha_2, \tag{4.8}$$

$$\overline{\nu_1} \leq \nu, \tag{4.9}$$

$$\overline{\nu_2} \leq \nu \tag{4.10}$$

and

$$\overline{\sigma} \leq \sigma, \tag{4.11}$$

holds and $\dfrac{\alpha_1}{\overline{\alpha_1}}$, $\dfrac{\alpha_2}{\overline{\alpha_2}}$, $\dfrac{\nu}{\overline{\nu_1}}$, $\dfrac{\nu}{\overline{\nu_2}}$, $\dfrac{\sigma}{\overline{\sigma}}$ can be arbitrarily large [5, 7]. It then follows from the definitions of C, $\overline{C}$, (3.4) and (4.5) that

$$\overline{C_0} \leq C_0, \tag{4.12}$$

and

$$\delta_0 \leq \overline{\overline{\delta_0}}. \tag{4.13}$$

In practice computing parameters α_i, ν and σ requires the computation of parameters $\overline{\alpha_i}$, $\overline{\nu_i}$, and $\overline{\sigma}$ respectively ($i = 1, 2$).

An improvement of Proposition 12 is as follows.

Proposition 15. *Suppose that the assumptions of Theorem 13 are satisfied. Then, for every constant $\overline{C} > \overline{C_0}$, where $\overline{C_0}$ is given in Theorem 13, there exists $\overline{\zeta} > 0$ such that, for every starting point x_0 in $\mathbb{B}_{\overline{\zeta}}(x^\star)$ (x_0 and $x^\star$ distinct), the sequence (x_k) defined by (3.22) converges to $x^\star$ and satisfies*

$$\| x_{k+1} - x^\star \| \leq \overline{C} \ \ \| x_k - x^\star \| \ \ \max \{\| x_k - x^\star \|^\gamma, \| x_0 - x^\star \|^\gamma\}.$$

In Theorem 13 and Proposition 15, we use the constant γ given by Lemma 5 (see (3.1)). In the following results, we can improve Theorem 13 and Proposition 15 by using only the center–estimate (3.2):

Corollary 16. *Suppose that the same hypotheses of Theorem 13 by replacing γ by $\gamma^\star$ in* **(**$\mathcal{P}1$**)****, where $\gamma^\star$ is given by (3.2) are satisfied. Then, for every constant $\overline{\overline{C}}$ such that*

$$\overline{\overline{C}} \geq \overline{\overline{C_0}} = M \left(C_{x^\star} + \overline{\nu_1} \ (\overline{\alpha_1}^{\gamma^\star} + (1 + \overline{\alpha_2})^{\gamma^\star}) \right), \tag{4.14}$$

there exists $\kappa > 0$ satisfying

$$\kappa < \kappa_0 = \min \left\{ a; \ \sqrt[\gamma^\star]{\frac{1}{\overline{\overline{C}}}}; \ \sqrt{\frac{b}{2 \ \overline{\sigma}}}; \ \overline{\overline{\delta_1}} \right\}$$

where

$$\overline{\overline{\delta_1}} = \sqrt[1+\gamma^\star]{\frac{b}{2 \left(C_{x^\star} + 2 \ \overline{\nu_2} \ ((1 + \overline{\alpha_1})^{\gamma^\star} + (1 + \overline{\alpha_2})^{\gamma^\star}) \right)}}$$

such that, for every starting point x_0 in $\mathbb{B}_\kappa(x^\star)$ (with $x_0 \neq x^\star$), the sequence (x_k) defined by (1.12) converges to $x^\star$ and satisfies the following inequality for $k \geq 0$

$$\| x_{k+1} - x^\star \| \leq \overline{\overline{C}} \ \ \| x_k - x^\star \|^{1+\gamma^\star} \ . \tag{4.15}$$

Corollary 17. *Suppose that the assumptions of Corollary 16 are satisfied. Then, for every constant $\overline{\overline{C}} > \overline{\overline{C_0}}$, where $\overline{\overline{C_0}}$ is given in Corollary 16, there exists $\overline{\overline{\zeta}} > 0$ such that, for every starting point x_0 in $\mathbb{B}_{\overline{\overline{\zeta}}}(x^\star)$ (x_0 and $x^\star$ distinct), the sequence (x_k) defined by (3.22) converges to $x^\star$ and satisfies*

$$\| x_{k+1} - x^\star \| \leq \overline{\overline{C}} \ \ \| x_k - x^\star \| \ \ \max \{\| x_k - x^\star \|^{\gamma^\star}, \| x_0 - x^\star \|^{\gamma^\star}\}.$$

Remark 18. Remark 10 can be applied for enlarging the different radius of convergence even further in Theorem 13, Proposition 15 and Corollaries 16, 17.

Remark 19. There exist a real nonsmooth functions verifying the assumptions ($\mathcal{P}_3$) or ($\mathcal{P}_3$)' or ($\mathcal{P}_3$)*. Note that $(\mathcal{P}_3) \Longrightarrow (\mathcal{P}_3)^\star$. For example, we take the classical convex funtion $F : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $F(x) = |x|$. We know that Clarke's subdifferential coincides with the subdifferential in the meaning of convex analysis:

$$\partial F(x) = \begin{cases} \{-1\} & \text{if} \quad x < 0 \\ (-1,1) & \text{if} \quad x = 0 \\ \{1\} & \text{if} \quad x > 0. \end{cases}$$

It is clear that F is not Fréchet–differentiable at 0 and for all $\sigma > 0$, if $x = y = 0$, there exist $\Delta F(x) = 1 \in \partial^\circ F(0)$ and $\Delta F(y) = -1 \in \partial^\circ F(0)$ such that $\| \Delta F(x) - \Delta F(y) \| = |1 - (-1)| = 2 > \sigma\, |x - y|$; i.e., ($\mathcal{P}_3$) is not satisfied. We can easily check that if $x > 0$ and $y > 0$ (or $x < 0$ and $y < 0$), ($\mathcal{P}_3$) is satisfied. The construction of examples of a class of functions F satisfying ($\mathcal{P}_3$) or ($\mathcal{P}_3$)' or ($\mathcal{P}_3$)* is very difficult. The search of sufficient (and necessary) conditions that provide ($\mathcal{P}_3$) or ($\mathcal{P}_3$)' or ($\mathcal{P}_3$)* is even more difficult. The novelty of our work is an introduction of a new method for solving a perturbed nonsmooth subanalytic variational inequalities. In relevant paper of Bolte et al. [15], we can find some constructions of examples for subanalytic Lipschitz continuous functions. In a future paper using ideas in [13, 14, 15], we will recover the numerical examples concerning our method (1.12) in the context of nonsmooth subanalytic variational inequalities.

Conclusion

We provided a Newton–Steffensen–type method for approximating an unique solution for perturbed nonsmooth subanalytic generalized equations in finite–dimensional spaces. Under some ideas given in [4, 7, 8] for nonlinear equations, we also provided a finer nonsmooth analysis in Sect. 4 with a finer error estimate on the distances $\| x_n - x^\star \|$ $(n \geq 1)$ using some center–type conditions and Lipschitz–like concept for set–valued maps. The use of center–type conditions is very important in computational mathematics [5, 7].

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