Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 44(2012) pp. 23-29

Jordan Left Derivations on Lie Ideals of Prime $\Gamma-$rings

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Abstract. Let M be a 2-torsion free prime Γ-ring. Let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping such that $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

AMS (MOS) Subject Classification Codes: 03E72, 54A40, 54B15

Key Words: n-torsion free, Lie ideals, Jordan left derivations, Prime Γ-rings.

1. Introduction

Let M and Γ be additive abelian groups. M is said to be a Γ-ring if there exists a mapping $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that
(a) $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)y = x\alpha y + x\beta y$,
$x\alpha(y + z) = x\alpha y + x\alpha z$,
(b) $(x\alpha y)\beta z = x\alpha(y\beta z)$,
for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.
A subset A of a Γ-ring M is a left(right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{m\alpha a : m \in M, \alpha \in \Gamma, a \in A\}, A\Gamma M$ is contained in A. The centre of M is doneted by Z(M) which is define by $Z(M) = \{m \in M : a\alpha m = m\alpha a, a \in M, \alpha \in \Gamma\}$. M is commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$ and $\alpha \in \Gamma$. M is prime if $a\Gamma M\Gamma b = 0$ with $a, b \in M$, then $a = 0$ or $b = 0$. We denote the commutator $x\alpha y - y\alpha x$ by $[x, y]_\alpha$. An additive subgroup U of M is said to be a Lie ideal of M if $[u, x]_\alpha \in U$, for all $u \in U, x \in M$ and $\alpha \in \Gamma$. M is n-torsion free if $nx = 0$, for $x \in M$ implies $x = 0$, where n is an integer. An additive mapping $d : M \to M$ is a derivation if $d(a\alpha b) = a\alpha d(b) + d(a)\alpha b$, a left derivation if $d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$, a Jordan derivation if $d(a\alpha a) = a\alpha d(a) + d(a)\alpha a$ and a Jordan left derivation if $d(a\alpha a) = 2a\alpha d(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Y.Ceven [3] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the Γ-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it. In this paper, he gave an example of Jordan left derivation for Γ-rings.

Mustafa Asci and Sahin Ceran [6] studied on a nonzero left derivation d on a prime Γ-ring M for which M is commutative with the conditions $d(U) \subseteq U$ and $d^2(U) \subseteq Z$, where U is an ideal of M and Z is the centre of M. They also proved the commutativity of M by the nonzero left derivation d_1 and right derivation d_2 on M with the conditions $d_2(U) \subseteq U$ and $d_1 d_2(U) \subseteq Z$.

In [7], M.Sapanci and A.Nakajima defined a derivation and a Jordan derivation on Γ-rings and investigated a Jordan derivation on a certain type of completely prime Γ-ring which is a derivation. They also gave examples of a derivation and a Jordan derivation of Γ-rings.

M. Bresar and J.Vukman[2] showed that the existence of a nonzero Jordan left derivation of R into X implies R is commutative, where R is a ring and X is 2-torsion free and 3-torsion free left R-module.In [8], Jun and Kim proved their results without the property 3-torsion free.

Qing Deng [4] worked on Jordan left derivations d of prime ring R of characteristic not 2 into a nonzero faithful and prime left R-module X. He proved the commutativity of R with Jordan left derivation d.

Mohammad Ashraf and Nadeem-Ur-Rehman[1] studied on Lie ideals and Jordan left derivations of prime rings.They proved that if d is an additive mapping on a 2-torsion free prime ring R satisfying $d(u^2) = 2ud(u)$, for all $u \in U$, where U is a Lie ideal of R such that $u^2 \in U$, for all $u \in U$, then $d(uv) = ud(v) + vd(u)$, for all $u \in U$.

In our paper, we reviewed the results of Mohammad Ashraf and Nadeem-Ur-Rehman[1] in gamma rings. We show that if d is an additive mapping on a 2-torsion free prime Γ-ring M such that $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, where U is a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. To complete the proof of main result in commutative sense, we take a help from the book ‘ Topics in ring theory’ of Herstein[5]. Finally, we showed that every Jordan left derivation on U is a left derivation.

Throughout this paper, we shall use the mark (*) for $a\alpha b\beta c = a\beta b\alpha c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

In order to prove our main result, we shall state and prove some lemmas as primary results.

2. Primary Results

Lemma 1. *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free σ-prime Γ-ring M. Then there exists an ideal I of M such that $[I, M]_\alpha \subseteq U$ but $[I, M]_\alpha \not\subseteq Z(M)$.*

Proof. Since M is 2-torsion free and $U \not\subseteq Z(M)$, it follows from the results in [6] that $[U, U]_\alpha \neq 0$ and $[I, M]_\alpha \subseteq U$,where $I = I\alpha[U, U]_\alpha \alpha M \neq 0$ is an ideal of M generated by $[U, U]_\alpha$. Now,$U \not\subseteq Z(M)$ implies $[I, M]_\alpha \not\subseteq Z(M)$;for, if $[I, M]_\alpha \subseteq Z(M)$ then $[I, [I, M]_\alpha]_\alpha = 0$, which gives $I \subseteq Z(M)$ and, since $I \neq 0$ is an ideal of M, so $M = Z(M)$. □

Lemma 2. *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free prime Γ-ring M which satisfies the condition (*) and $a, b \in M$ such that $a\alpha U\beta b = 0$. Then $a = 0$ or $b = 0$.*

Proof. Since M is 2-torsion free prime Γ-ring, there exists an ideal I of M such that $[I, M]_\alpha \subseteq U$ but $[I, M]_\alpha \not\subseteq Z(M)$, by Lemma 1. Now, taking $u \in U$, $e \in I$ and $m \in M$, we have $[e\alpha a\alpha u, m]_\alpha \in [I, M]_\alpha \subseteq U$, and so
$0 = a\alpha[e\alpha a\alpha u, m]_\beta\beta b$
$= a\alpha[e\alpha a, m]_\alpha\beta u\beta b + a\alpha e\beta a\alpha[u, m]_\alpha\beta b$, by (*)
$= a\alpha[e\alpha a, m]_\alpha\beta u\beta b$, since $a\alpha[u, m]_\alpha \in a\alpha U\beta b$
=$a\alpha e\alpha a\alpha m\beta u\beta b - a\alpha m\alpha e\alpha a\beta u\beta b$
$= a\alpha e\alpha a\alpha m\beta u\beta b - a\alpha m\alpha e\beta a\alpha u\beta b$, by (*)
=$a\alpha e\alpha a\alpha m\beta u\beta b$, by assumption.
Thus $a\alpha I\alpha a\alpha M\beta U\beta b = 0$.
If $a \neq 0$ then $U\beta b = 0$, by the primeness of M. Now, if $u \in U$ and $m \in M$ then $u\alpha m - m\alpha u \in U$ and hence $(u\alpha m - m\alpha u)\beta b = 0$ implies $u\alpha m\beta b = 0$,that is $u\alpha M\beta b = 0$. Since $U \neq 0$, we must have $b = 0$. In the similar fashion, it can be shown that if $b \neq 0$ then $a = 0$. □

Lemma 3. *Let M be a 2-torsion free prime Γ-ring and let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then*
(a)$d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u)$. Let M satisfy (*), then
(b) $d(u\alpha v\beta u) = u\alpha u\beta d(v) + 3u\alpha v\beta d(u) - v\alpha u\beta d(u)$,
(c) $d(u\alpha v\beta w + w\alpha v\beta u) = (u\alpha w + w\alpha u)\beta d(v) + 3u\alpha v\beta d(w) + 3w\alpha v\beta d(u)$
$- v\alpha u\beta d(w) - v\alpha w\beta d(u)$,
(d) $[u, v]_\alpha\alpha u\beta d(u) = u\alpha[u, v]_\alpha\beta d(u)$
(e) $[u, v]_\alpha\beta(d(u\alpha v) - u\alpha d(v) - v\alpha d(u)) = 0$,
for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

Proof. Since $u\alpha v + v\alpha u = (u + v)\alpha(u + v) - u\alpha u - v\alpha v$, we have $u\alpha v + v\alpha u \in U$, for all $u, v \in U$ and $\alpha \in \Gamma$. Then $d(u\alpha v + v\alpha u) = d((u + v)\alpha(u + v)) - d(u\alpha u) - d(v\alpha v)$ with our hypothesis yields the required result.
Replacing v by $u\beta v + v\beta u$ in (a), we have

$$d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) = 2u\alpha d(u\beta v + v\beta u) + 2(u\beta v + v\beta u)\alpha d(u). \tag{2.1}$$

Since $u\alpha v + v\alpha u \in U$, by (*) we get

$$d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) = 4u\alpha u\beta d(v) + 6u\alpha v\beta d(u) + 2v\alpha u\beta d(u). \tag{2.2}$$

On the other hand

$$d(u\alpha(u\beta v+v\beta u)+(u\beta v+v\beta u)\alpha u)= \\ d(u\alpha u\beta v+v\beta u\alpha u)+2d(u\alpha v\beta u)= \\ 2u\alpha u\beta d(v)+4v\alpha u\beta d(u)+2d(u\alpha v\beta u). \tag{2.3}$$

Combining (2.2) and (2.3) and using the condition that M is 2-torsion free, we obtain (b).
Replacing $u+w$ for u in (b) and using (*), we get

$$d((u+w)\alpha v\beta(u+w))= \\ u\alpha u\beta d(v)+w\alpha w\beta d(v)+(u\alpha w+w\alpha u)\beta d(v)+ \\ 3u\alpha v\beta d(u)+3u\alpha v\beta d(w)+3w\alpha v\beta d(u)+w\alpha v\beta d(w)- \\ v\alpha u\beta d(u)-v\alpha u\beta d(w)-v\alpha w\beta d(u)-v\alpha w\beta d(w). \tag{2.4}$$

On the other hand with (*), we have

$$d((u+w)\alpha v\beta(u+w))= \\ d(u\alpha v\beta u)+d(w\alpha v\beta w)+d(u\alpha v\beta w+w\alpha v\beta u)= \\ u\alpha u\beta d(v)+3u\alpha v\beta d(u)-v\alpha u\beta d(u)+w\alpha w\beta d(v) \\ +3w\alpha v\beta d(w)-v\alpha w\beta d(w)+d(u\alpha v\beta w+w\alpha v\beta u). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain (c).
Since $u\alpha v+v\alpha u$ and $u\alpha v-v\alpha u$ are in U, we see that $2u\alpha v\in U$, for all $u,v\in U$ and $\alpha\in\Gamma$. By hypothesis, we have $d((u\alpha v)\beta(u\alpha v))=2u\alpha v\beta d(u\alpha v)$.
Replacing w by $2u\beta v$ in (c) with (*) and the condition that M is 2-torsion free, we get

$$d(u\alpha v\beta(u\beta v)+(u\beta v)\alpha v\beta u)= \\ (u\alpha u\beta v+u\alpha v\beta u)\beta d(v)+3u\alpha v\beta d(u\beta v)+ \\ 3u\alpha v\beta v\beta d(u)-v\alpha u\beta d(u\beta v)-v\alpha u\beta v\beta d(u). \tag{2.6}$$

On the other hand with (*), we have

$$d(u\alpha v\beta(u\beta v)+(u\beta v)\alpha v\beta u)= \\ d((u\beta v)\alpha(u\beta v)+u\alpha v\beta v\beta u)= \\ 2u\alpha v\beta d(u\beta v)+2u\alpha u\beta v\beta d(v)+ \\ 3u\alpha v\beta v\beta d(u)-v\alpha v\beta u\beta d(u). \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$[u,v]_\alpha\beta d(u\beta v)= \\ u\alpha[u,v]_\beta\beta d(v)+v\alpha[u,v]_\beta\beta d(u). \tag{2.8}$$

Replacing $u+v$ for v in (2.8), we have

$$2[u,v]_\alpha\beta u\beta d(u)+[u,v]_\alpha\beta d(u\beta v)= \\ 2u\alpha[u,v]_\beta\beta d(u)+u\alpha[u,v]_\beta\beta d(v)+v\alpha[u,v]_\beta\beta d(u). \tag{2.9}$$

From (2.8) and (2.9), we get (d).
Linearizing (d) on u, we have

$$[u,v]_\alpha\beta u\beta d(u)+[u,v]_\alpha\beta v\beta d(v)+[u,v]_\alpha\beta u\beta d(v)+[u,v]_\alpha\beta v\beta d(u)= \tag{2.10}$$
$$\alpha[u,v]_\beta\beta d(u)+u\alpha[u,v]_\beta\beta d(v)+v\alpha[u,v]_\beta\beta d(u)+v\alpha[u,v]_\beta\beta d(v),$$

for all $u,v\in U$ and $\alpha,\beta\in\Gamma$.
Application of (d) and (8) gives $[u,v]_\alpha\beta u\beta d(v)+[u,v]_\alpha\beta v\beta d(u)=[u,v]_\alpha\beta d(u\beta v)$ and hence $[u,v]_\alpha\beta(d(u\alpha v)-u\alpha d(v)-v\alpha d(u))=0$, for all $u,v\in U$ and $\alpha,\beta\in\Gamma$. □

Lemma 4. *Let M be a 2-torsion free Γ-ring satisfying (*) and U a Lie ideal of M such that $u\alpha u\in U$, for all $u\in U$ and $\alpha\in\Gamma$. If $d:M\to M$ is an additive mapping satisfying $d(u\alpha u)=2u\alpha d(u)$, for all $u\in U$ and $\alpha\in\Gamma$, then*
(a) $[u,v]_\alpha\beta d([u,v]_\alpha)=0$,
(b) $(u\alpha u\alpha v-2u\alpha v\alpha u+v\alpha u\alpha u)\beta d(v)=0$,
for all $u,v\in U$ and $\alpha,\beta\in\Gamma$.

Proof. By Lemma 3(a) and Lemma 3(e), we get

$$d(u\alpha v+v\alpha u)=2(u\alpha d(v)+v\alpha d(u)) \tag{2.11}$$

and

$$[u,v]_\alpha\beta(d(u\alpha v)-u\alpha d(v)-v\alpha d(u))=0. \tag{2.12}$$

Combining (2.11) and (2.12), we have

$$[u,v]_\alpha\beta(d(v\alpha u)-u\alpha d(v)-v\alpha d(u))=0. \tag{2.13}$$

Using (2.12) - (2.13), we get $[u,v]_\alpha\beta d([u,v]_\alpha)=0$, for all $u,v\in U$ and $\alpha,\beta\in\Gamma$.
For any $u,v\in U$ and $\alpha,\beta\in\Gamma$, we have $d([u,v]_\alpha\beta[u,v]_\alpha)=2[u,v]_\alpha\beta d([u,v]_\alpha)$. By (a), we have

$$d([u,v]_\alpha\beta[u,v]_\alpha)=0, \tag{2.14}$$

for all $u,v\in U$ and $\alpha,\beta\in\Gamma$.
We have $2u\alpha v\in U$, for all $u,v\in U$ and $\alpha\in\Gamma$.
Replacing u by $2u\beta v$ in $u\alpha v+v\alpha u\in U$ and $u\alpha v-v\alpha u\in U$ and adding the results and then using (*), we find that $4v\alpha u\beta v\in U$, for all $u,v\in U$ and $\alpha,\beta\in\Gamma$.
Replacing $4v\alpha u\beta v$ for v in Lemma 3(a) and since M is 2-torsoin free, we have

$$d(u\alpha(v\alpha u\beta v)+(v\alpha u\beta v)\alpha u)=2(u\alpha d(v\alpha u\beta v)+v\alpha u\beta v\alpha d(u)). \tag{2.15}$$

Using (2.15) in (2.14) and then (*), we have

$$\begin{aligned}
&0=\\
&d(u\alpha(v\alpha u\beta v)+(v\alpha u\beta v)\alpha u)-d(u\alpha(v\alpha v)\beta u)-d(v\alpha(u\alpha u)\beta v)=\\
&2(u\alpha d(v\alpha u\beta v)+v\alpha u\beta v\alpha d(u))-u\alpha u\beta d(v\alpha v)\\
&-3u\alpha v\alpha v\beta d(u)+v\alpha v\alpha u\beta d(u)-v\alpha v\beta d(u\alpha u)\\
&-3v\alpha u\alpha u\beta d(v)+u\alpha u\alpha v\beta d(v)=\\
&-3(u\alpha u\alpha v-2u\alpha v\alpha u+v\alpha u\alpha u)\beta d(v)\\
&-(u\alpha v\alpha v-2v\alpha u\alpha v+v\alpha v\alpha u)\beta d(u)
\end{aligned}$$

and hence

$$3(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) + \\ (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\beta d(u) = 0, \tag{2.16}$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.
Replacing u by $u + v$ in Lemma3(d), we get

$$(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) - \\ (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\beta d(u) = 0, \tag{2.17}$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.
Combining (2.16) and (2.17), we obtain

$$(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) = 0. \tag{2.18}$$

By (2.17) and (2.18), we arrive at (b). □

3. MAIN RESULT

The main result of this paper states as follows.

Theorem 5. *Let M be a 2-torsion free prime Γ-ring satisfying (*) and U a Lie ideal of M such that $u\alpha u \in U$, for all $u, v \in U$ and $\alpha \in \Gamma$. If $d : M \to M$ is an additive mapping such that $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof. Suppose U is a commutative Lie ideal of M. Let $a \in U$ and $x \in M$. Then $[a, x]_\alpha \in U$ and so commutes with a .Now, for $x, y \in M$, we have $a\beta[a, x\alpha y]_\alpha = [a, x\alpha y]_\alpha\beta a$, for all $\alpha, \beta \in \Gamma$. Expanding $[a, x\alpha y]_\alpha$ as $[a, x]_\alpha\alpha y + x\alpha[a, y]_\alpha$ and using that a commutes with this, with $[a, x]_\alpha$ and with $[a, y]_\alpha$, we have $2[a, x]_\alpha\alpha[a, y]_\alpha = 0$ and so $[a, x]_\alpha\alpha[a, y]_\alpha = 0$, since M is 2-torsion free. Replacing y by $a\beta x$ in $[a, x]_\alpha\alpha[a, y]_\alpha = 0$ and then using (*), we have $[a, x]_\alpha\alpha M\beta[a, x]_\alpha = 0$, for all $x \in M$ and $\alpha, \beta \in \Gamma$. Since M is prime, $[a, x]_\alpha = 0$ and so $U \subset Z(M)$. Hence by Lemma 3(a), we have $2d(u\alpha v) = 2(u\alpha d(v) + v\alpha d(u))$. Since M is 2-torsion free, $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$.
We assume that U is a noncommutative Lie ideal of M.
Now, replacing u by $[u_1, w]_\alpha$ in Lemma 3(d), we get

$$([u_1, w]_\alpha\alpha[u_1, w]_\alpha\alpha v - 2[u_1, w]_\alpha\alpha v\alpha[u_1, w]_\alpha \\ + v\alpha[u_1, w]_\alpha\alpha[u_1, w]_\alpha)\beta d([u_1, w]_\alpha) = 0, \tag{3.1}$$

for all $u, v, u_1, w \in U$ and $\alpha, \beta \in \Gamma$.
Using Lemma 4(a) in (3.1), we get $[u_1, w]_\alpha\alpha[u_1, w]_\alpha\alpha v\beta d([u_1, w]_\alpha) = 0$
and so $[u_1, w]_\alpha\alpha[u_1, w]_\alpha\alpha U\beta d([u_1, w]_\alpha) = 0$.
Hence by Lemma 2, either $[u_1, w]_\alpha\alpha[u_1, w]_\alpha = 0$ or $d([u_1, w]_\alpha) = 0$.
If $d([u_1, w]_\alpha) = 0$ i.e, $d(u_1\alpha w) = d(w\alpha u_1)$, for all $u_1, w \in U$ and $\alpha \in \Gamma$, then by Lemma 3(a) and the fact that M is 2-torsion free, we get $d(u_1\alpha w) = u_1\alpha d(w) + w\alpha d(u_1)$.
On the other hand let $[u_1, w]_\alpha\alpha[u_1, w]_\alpha = 0$, for some $u_1, w \in U$ and $\alpha \in \Gamma$.
Replacing v by $[u_1, w]_\alpha$ in Lemma 4(b), we get

$$(u\alpha u\alpha[u_1, w]_\alpha)\beta d([u_1, w]_\alpha) \\ -2(u\alpha[u_1, w]_\alpha\alpha u)\beta d([u_1, w]_\alpha) + ([u_1, w]_\alpha\alpha u\alpha u)\beta d([u_1, w]_\alpha) = 0. \tag{3.2}$$

Applying Lemma 4(a) in (3.2), we have

$$([u_1, w]_\alpha \alpha u \alpha u)\beta d([u_1, w]_\alpha) - 2(u\alpha[u_1, w]_\alpha \alpha u)\beta d([u_1, w]_\alpha) = 0, \tag{3.3}$$

for all $u \in U$ and $\alpha, \beta \in \Gamma$.
Linearizing (3.3) on u and using Lemma 4(b), we have

$$\begin{aligned}([u_1, w]_\alpha \alpha u \alpha v)\beta d([u_1, w]_\alpha) + ([u_1, w]_\alpha \alpha v \alpha u)\beta d([u_1, w]_\alpha) \\ -2((u\alpha[u_1, w]_\alpha \alpha v) + (v\alpha[u_1, w]_\alpha \alpha u))\beta d([u_1, w]_\alpha) = 0,\end{aligned} \tag{3.4}$$

for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.
Replacing u by $2u\beta v_1$ in (3.4)and then using the fact the M is 2-torsion free and (*), we have

$$\begin{aligned}[u_1, w]_\alpha \alpha u \beta v_1 \alpha v \beta d([u_1, w]_\alpha) + [u_1, w]_\alpha \alpha v \beta u \alpha v_1 \beta d([u_1, w]_\alpha) \\ -2(u\alpha v_1 \beta[u_1, w]_\alpha \alpha v + v\alpha[u_1, w]_\alpha \alpha u \beta v_1)\beta d([u_1, w]_\alpha) = 0.\end{aligned} \tag{3.5}$$

Further, replacing v_1 by $[u_1, w]_\alpha$ in (3.5) and then using Lemma 4(b), $[u_1, w]_\alpha \alpha [u_1, w]_\alpha = 0$ and (*),
we get $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha \alpha v \beta d([u_1, w]_\alpha) = 0$
i.e.,$([u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha)\alpha U \beta d([u_1, w]_\alpha) = 0$, for all $u \in U$ and $\alpha, \beta \in \Gamma$. By Lemma 2, either $d([u_1, w]_\alpha) = 0$ or $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha = 0$.
If $d([u_1, w]_\alpha) = 0$, then by the same argument as above we get the required result. On the other hand , if $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha = 0$, for all $u \in U$ and $\alpha, \beta \in \Gamma$, then by Lemma 2, we have $[u_1, w]_\alpha = 0$. Further, by Lemma 3(a) and the fact that M is 2-torsion free, we have $d(u_1 \alpha w) = u_1 \alpha d(w) + w\alpha d(u_1)$. Hence in both cases, we find that $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. The proof is thus complete. □

Corollary 6. *Let M be a 2-torsion free prime Γ-rins and $d : M \to M$ a Jordan left derivation. Then d is a left derivation on M.*

Proof. If M is commutative, then $u\alpha v = v\alpha u$, for all $u, v \in M$ and $\alpha \in \Gamma$, and so by Lemma 3(a), we have $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in M$ and $\alpha \in \Gamma$. If M is noncommutative, then by Theorem 5, we have $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in M$ and $\alpha \in \Gamma$. □

Acknowledgement. The authors are grateful to the referees for their valuable instructions.

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