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Generalized Extension in the Geometry of Goncharov Motivic and Grassmannian Configuration Chain Complexes

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Abstract. The purpose of this work is to propose generalized extensions of morphisms in the geometry of Grassmannian configuration and Goncharov motivic chain complexes. This work has two major divisions: in its first part the geometry of these complexes will be extended for weight $n = 6$, secondly, the generalization of this extension for any weight N will be presented. The generalized commutative diagram will also be exhibited.

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1. Introduction

Suslin [23] introduced Grassmannian configuration chain complex of free abelian groups. Grassmannian chain complex is formed by using two types of differential morphisms between free abelian groups. Each square of the Grassmannian chain complex is commutative [14,15,23]. Leibniz defined an infinite series denoted by $Li_p(Z)$ [7] in a unit disc. For $p = 2$ Di-Logarithms function $Li_2(Z)$ [9, 18] has been studied by many mathematicians but among them the most prominent work was Abel five term functional equation.
Bloch [1] defined the polylogarithmic group denoted by $\mathcal{B}_1(F)$ for weight 1. Bloch [1] also introduced polylog group $\mathcal{B}_2(F)$ for weight 2 and defined the following chain complex called Bloch-Suslin complex

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times \ .$$

Goncharov [7–9] introduced Bloch group $\mathcal{B}_3(F)$ for weight 3 and then generalized it as $\mathcal{B}_n(F)$. Goncharov also generalized Bloch-Suslin complex to introduce the following chain complex called Goncharov's motivic complex

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^{\times} \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta} \wedge^n F^{\times}.$$

Configuration chain complexes are naturally associated with polylogarithmic groups chain complexes. Initially, Goncharov [7] connected configuration chain with the Bloch-Suslin chain complex for weight 2 then he extended this geometry for weight 3 [7]. Khalid et.al [17] generalized the work of Goncharov [7] to define geometry for any weight n. Cathelineau [2–4] introduced a variant of Goncharov chain complex in two forms, among them the first form was infinitesimal. For this form Cathelineau used $\beta_n(F)$ group whose generalized Cathelineau infinitesimal chain is given below

$$\beta_n(F) \xrightarrow{\partial_n} \begin{array}{c}\beta_{n-1}(F)\otimes F^{\times}\\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F)\end{array} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \begin{array}{c}\beta_2(F)\otimes\wedge^{n-2}F^{\times}\\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^{\times}\end{array} \xrightarrow{\partial_0} F \otimes \wedge^{n-1}F^{\times}.$$

Second form was tangential chain complex, for which Cathelineau used group $T\mathcal{B}_n(F)$ [11].

$$T\mathcal{B}_n(F) \xrightarrow{\delta_{n,\varepsilon}} \begin{array}{c}T\mathcal{B}_{n-1}(F)\otimes F^{\times}\\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F)\end{array} \cdots \xrightarrow{\delta_{1,\varepsilon}} \begin{array}{c}T\mathcal{B}_2(F)\otimes\wedge^{n-2}F^{\times}\\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^{\times}\end{array} \xrightarrow{\delta_{\varepsilon}} (F \otimes \wedge^{n-1}F^{\times}) \oplus (\wedge^n F).$$

Khalid et al. [16, 19] defined generalized geometry between Cathelineau infinitesimal and configuration chain complexes. Siddiqui [21] used derivation and group $\beta_n^D(F)$ to introduce following variant of Cathelineau infinitesimal chain complex

$$\beta_n^D(F) \xrightarrow{\partial_n^D} \begin{array}{c}\beta_{n-1}^D(F)\otimes F^{\times}\\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F)\end{array} \xrightarrow{\partial_{n-1}^D} \cdots \xrightarrow{\partial_1^D} \begin{array}{c}\beta_2^D(F)\otimes\wedge^{n-2}F^{\times}\\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^{\times}\end{array} \xrightarrow{\partial_0^D} F \otimes \wedge^{n-1}F^{\times}$$

Khalid et al. [12, 13] defined new maps to introduce geometry of configuration and variant of Cathelineau infinitesimal chain complexes for any weight n to provide generalized commutative diagrams. Recently Khalid et al. [20], introduced some extensions in geometry of Goncharov and configuration chain complexes.
In this paper, the work of [20] is now to define generalized extension in the geometry of Goncharov motivic and configuration chain complexes. Section 2 is about preliminaries of configuration chain complexes and Goncharov's generalized polylog chain complex. Section 3 presents some extensions in the geometry of configuration and Goncharov motivic polylogarithmic chain complexes up to weight 5. Section 4 covers the generalized extension of morphisms in the geometry of Goncharov motivic polylogarithmic and configuration chain complexes and also its generalized commutative diagram. The last section describes the conclusion of the entire work.

2. Configuration Complexes

Consider the following configuration chain complex

$$\begin{array}{ccccc} G_{n+4}(n+2) & \xrightarrow{d} & G_{n+3}(n+2) & \xrightarrow{d} & G_{n+2}(n+2) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{n+3}(n+1) & \xrightarrow{d} & G_{n+2}(n+1) & \xrightarrow{d} & G_{n+1}(n+1) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{n+2}(n) & \xrightarrow{d} & G_{n+1}(n) & \xrightarrow{d} & G_{n}(n) \end{array} \tag{A}$$

The group $G_n(n)$ is a free abelian group generated by all possible configuration of n points in n-dimensional vector space V^n.

Lemma 2.1. *The above diagram A is bi-complex and commutative* (see [23]).

2.2. **Siegel's Cross Ratio and its Properties.** Let us introduce the cross ratio of four points as

$$r(v_0, v_1, v_2, v_3) = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}$$

Siegel [22] introduced the following most important property of cross ratio:

$$1 - r(v_0, v_1, v_2, v_3) - r(v_0, v_2, v_1, v_3) = 0. \tag{2.1}$$

2.2.1. *Projected Cross Ratio Property.* Goncharov [7] defined the following projected cross ratio of four points with single projected point as

$$r(v_i|(v_0, v_2, v_1, v_3) = \frac{\triangle(v_i|v_0, v_3)\triangle(v_i|v_1, v_2)}{\triangle(v_i|v_0, v_2)\triangle(v_i|v_1, v_3)} = \frac{\triangle(v_i, v_0, v_3)\triangle(v_i, v_1, v_2)}{\triangle(v_i, v_0, v_2)\triangle(v_i, v_1, v_3)}. \tag{2.2}$$

2.2.2. *Triple Cross Ratio of Six Points.* Goncharov [7] generalized cross ratio as a triple cross ratio of six points, given by

$$r(v_0, ..., v_5) = \frac{\triangle(v_0, v_1, v_5)\triangle(v_1, v_2, v_3)\triangle(v_2, v_0, v_4)}{\triangle(v_0, v_1, v_3)\triangle(v_1, v_2, v_4)\triangle(v_2, v_0, v_5)}. \tag{2.3}$$

Theorem 2.3. *The ratio of two projected cross ratio of four points can be written in the form of triple cross ratio of six points.*

Proof. Let us assume $(v_0, ..., v_5)$ be six points with two projected points v_1 and v_2, then

$$\begin{aligned} \frac{r(v_2|v_0, v_1, v_3, v_5)}{r(v_1|v_0, v_2, v_4, v_5)} &= \frac{\triangle(v_2, v_0, v_5)\triangle(v_2, v_1, v_3)}{\triangle(v_2, v_0, v_3)\triangle(v_2, v_1, v_5)} / \frac{\triangle(v_1, v_0, v_5)\triangle(v_1, v_2, v_4)}{\triangle(v_1, v_0, v_4)\triangle(v_1, v_2, v_5)} \\ &= \frac{\triangle(v_2, v_0, v_5)\triangle(v_2, v_1, v_3)\triangle(v_1, v_0, v_4)}{\triangle(v_2, v_0, v_3)\triangle(v_1, v_0, v_5)\triangle(v_1, v_2, v_4)}. \end{aligned} \tag{2.4}$$

□

2.4. **Polylogaithmic Groups and its Complexes.** The classical p-logarithms series is expressed as $Li_p(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n^p}$, in the unit disc $x \leq 1$. For $p = 1$, $Li_1(x) = -Li(1-x)$ with generalized form $\log x + \log y = \log xy$. F is a field and $F^{\bullet\bullet} = F - \{0, 1\}$. Let $Z[\mathbf{P}^1_F/\{0, 1, \infty\}]$ is a free abelian group generated by [x]. The group $\mathcal{B}(F)$ is a quotient of $Z[\mathbf{P}^1_F]$ by its subgroup generated by Abel's five term relation $[x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-y^{-1}}{1-x^{-1}}\right]+\left[\frac{1-y}{1-x}\right]$, where $x \neq y$ and $x, y \neq 0, 1$ [5, 6, 10].

2.5. **Goncharov's Complexes.** Let $\mathcal{B}_2(F) = Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/ < \mathsf{R}_2(F) >$, where $\mathsf{R}_2(F) = \sum\limits_{i=0}^{4} (-1)^i r(v_0, ..., \hat{v}_i, ..., v_4)$ is a five term relation of cross ratio. Construct a chain

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^{\times} .$$

where morphism δ is defined as $\delta : [x]_2 \to (1-x) \wedge x$. This complex is called Bloch-Suslin complex for weight 2. For weight 3, Goncharov [7] introduced a seven-term relation of triple cross ratio given as

$$\mathsf{R}_3(F) = \sum_{i=0}^{6} (-1)^i Alt_6 \Big[r(v_0, ..., \hat{v}_i, ..., v_6) \Big]. \tag{2.5}$$

Goncharov [7] defined a group $\mathcal{B}_3(F) = Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/ < \mathsf{R}_3(F) >$. Following is a complex for weight 3

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^{\times} \xrightarrow{\delta} \wedge^3 F^{\times} .$$

Finally, Goncharov [7] introduced generalized subgroup $\mathcal{B}_n(F) = Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/ < \mathsf{R}_n(F) >$, then he introduced following generalized chain complex.

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^{\times} \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta} \frac{\wedge^n F^{\times}}{2 - torsion} \tag{2.6}$$

3. Geometry and Extension up to Weight 5 of Grassmannian and Goncharov Complexes

3.1. **Geometry for Weight 2.** As defined in [17], the geometry of Grassmannian configuration and Goncharov motivic in weight-2 is represented as

$$\begin{array}{ccccc}
G_6(3) & \xrightarrow{d} & G_5(3) & & \\
\downarrow p & & \downarrow p & & \\
G_5(2) & \xrightarrow{d} & G_4(2) & \xrightarrow{g_1^2} & \mathcal{B}_2(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_4(1) & \xrightarrow{d} & G_3(1) & \xrightarrow{g_0^2} & \wedge^2 F^{\times}
\end{array} \tag{B}$$

where,

$$g_0^2(v_0, ..., v_2) = -\triangle(v_1) \wedge \triangle(v_2) + \triangle(v_0) \wedge \triangle(v_2) - \triangle(v_0) \wedge \triangle(v_1), \tag{3.7}$$

where symbol $\triangle$ means determinant and

$$g_1^2(v_0, ..., v_3) = [r(v_0, ..., v_3)]_2. \tag{3.8}$$

Lemma 3.2. *The diagram B is bi-complex and commutative* [17].

3.3. **Geometry for Weight 3.** The geometry of Grassmannian and Goncharov motivic for weight-3 is presented in [17] as follows:

$$\begin{array}{ccccc}
G_7(3) & \xrightarrow{d} & G_6(3) & & \\
\downarrow p & & \downarrow p & & \\
G_6(2) & \xrightarrow{d} & G_5(2) & \xrightarrow{g_1^3} & \mathcal{B}_2(F) \otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_5(1) & \xrightarrow{d} & G_4(1) & \xrightarrow{g_0^3} & \wedge^3 F^\times
\end{array} \quad . \tag{C}$$

where,

$$g_0^3(v_0, v_1, v_2, v_3) \to \sum_{i=j+1}^{3} (-1)^{i+1} \bigwedge_{j\neq i}^{3} \triangle(v_j) \ (mod\ 4) \tag{3.9}$$

and

$$g_1^3(v_0, v_1, ..., v_4) \to -\frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(v_0, ..., \hat{v}_i, ..., v_4)]_2 \otimes \prod_{i\neq r}^{4} \triangle(v_i, v_r) \ (mod\ 5). \tag{3.10}$$

Lemma 3.4. *The diagram C is bi-complex and commutative* [17].

3.4.1. *Extension in Geometry for Weight 3.* For weight 3, morphism g_2^3 is introduced to connect Grassmannian and Goncharov complexes and extend the commutative diagram:

$$\begin{array}{ccccc}
G_7(3) & \xrightarrow{d} & G_6(3) & \xrightarrow{g_2^3} & \mathcal{B}_3(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_6(2) & \xrightarrow{d} & G_5(2) & \xrightarrow{g_1^3} & \mathcal{B}_2(F) \otimes F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_5(1) & \xrightarrow{d} & G_4(1) & \xrightarrow{g_0^3} & \wedge^3 F^\times
\end{array} \tag{D}$$

where,

$$g_2^3(v_0, ..., v_5) = \frac{1}{15} Alt_6 \left[\frac{\triangle(v_0, v_1, v_5)\triangle(v_1, v_2, v_3)\triangle(v_2, v_0, v_4)}{\triangle(v_0, v_1, v_3)\triangle(v_1, v_2, v_4)\triangle(v_2, v_0, v_5)}\right]_3. \tag{3.11}$$

Lemma 3.5. $$g_1^3 \circ p = \delta \circ g_2^3.$$

Proof. For proof see Khalid et al. [20]. □

3.6. **Geometry for Weight 4.** Geometry for weight 4 is defined in [17] as follows

$$\begin{array}{ccccc} G_8(3) & \xrightarrow{d} & G_7(3) & & \\ \downarrow p & & \downarrow p & & \\ G_7(2) & \xrightarrow{d} & G_6(2) & \xrightarrow{g_1^4} & \mathcal{B}_2(F)\otimes\wedge^2 F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_6(1) & \xrightarrow{d} & G_5(1) & \xrightarrow{g_0^4} & \wedge^4 F^\times \end{array} \tag{E}$$

where

$$g_0^4(v_0,...,v_4) \rightarrow \sum_{i=j+1}^{4}(-1)^{i+1}\bigwedge_{j\neq i}^{4}\triangle(v_j)\ (mod\ 5) \tag{3.12}$$

and

$$g_1^4(v_0,...,v_5) = \frac{1}{6}\sum_{i\neq j}^{5}(-1)^i[r(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes \prod_{r\neq i}^{5}\triangle(v_i,v_r)\wedge\prod_{r\neq j}^{5}\triangle(v_j,v_r)\ (mod\ 6). \tag{3.13}$$

Lemma 3.7. *The diagram E is commutative* (see [17]).

3.7.1. *Extension in Geometry for Weight 4.* For this extension, two morphisms g_2^4 and g_3^4 are introduced

$$\begin{array}{ccccc} G_9(4) & \xrightarrow{d} & G_8(4) & \xrightarrow{g_3^4} & \mathcal{B}_4(F) \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_8(3) & \xrightarrow{d} & G_7(3) & \xrightarrow{g_2^4} & \mathcal{B}_3(F)\otimes F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_7(2) & \xrightarrow{d} & G_6(2) & \xrightarrow{g_1^4} & \mathcal{B}_2(F)\otimes\wedge^2 F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_6(1) & \xrightarrow{d} & G_5(1) & \xrightarrow{g_0^4} & \wedge^4 F^\times \end{array} \tag{F}$$

$$g_2^4(v_0,...,v_6) = -\frac{1}{28}\sum_{i=0}^{6}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,...,v_6)\Big]_3\otimes\prod_{r\neq i\neq j}^{6}\triangle(v_r,v_i,v_j), \tag{3.14}$$

$$g_3^4(v_0,...,v_7) = \frac{1}{66}Alt_8[r(v_0,...,v_7)]_4. \tag{3.15}$$

Lemma 3.8. $g_1^4\circ p = \delta\circ g_2^4.$

Proof. For proof see Khalid et al. [20]. □

Lemma 3.9. $g_2^4 \circ p = \delta \circ g_3^4.$

Proof. For proof see Khalid et al. [20] □

3.10. **Geometry for Weight 5.** As defined in [17], the following commutative diagram is obtained

$$\begin{array}{ccccc} G_9(3) & \xrightarrow{d} & G_8(3) & & \\ \downarrow p & & \downarrow p & & \\ G_8(2) & \xrightarrow{d} & G_7(2) & \xrightarrow{g_1^5} & \mathcal{B}_2(F)\otimes\wedge^3 F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_7(1) & \xrightarrow{d} & G_6(1) & \xrightarrow{g_0^5} & \wedge^5 F^\times \end{array} \tag{G}$$

where,

$$g_0^5(v_0,...,v_5) \to \sum_{i=j+1}^{5} (-1)^i \bigwedge_{j\neq i}^{5} \triangle(v_j) \; (\mod 6) \tag{3.16}$$

and

$$g_1^5(v_0,...,v_6) = -\frac{1}{10} \sum_{i\neq j\neq k}^{6} (-1)^i [r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2 \otimes \prod_{r\neq i}^{6} \triangle(v_i,v_r) \wedge \prod_{l\neq j}^{6} \triangle(v_j,v_r) \wedge \prod_{r\neq k}^{6} \triangle(v_k,v_r) \; (mod\ 7). \tag{3.17}$$

Lemma 3.11. *The diagram* G *is commutative* (see [17]).

3.11.1. *Extension in Geometry for Weight 5.* For this extension in geometry, three new morphisms are introduced, namely g_2^5, g_3^5 and g_4^5:

$$\begin{array}{ccccc} G_{11}(5) & \xrightarrow{d} & G_{10}(5) & \xrightarrow{g_4^5} & \mathcal{B}_5(F) \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_{10}(4) & \xrightarrow{d} & G_9(4) & \xrightarrow{g_3^5} & \mathcal{B}_4(F)\otimes F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_9(3) & \xrightarrow{d} & G_8(3) & \xrightarrow{g_2^5} & \mathcal{B}_3(F)\otimes\wedge^2 F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_8(2) & \xrightarrow{d} & G_7(2) & \xrightarrow{g_1^5} & \mathcal{B}_2(F)\otimes\wedge^3 F^\times \\ \downarrow p & & \downarrow p & & \downarrow \delta \\ G_7(1) & \xrightarrow{d} & G_6(1) & \xrightarrow{g_0^5} & \wedge^5 F^\times \end{array} \tag{H}$$

Morphisms g_2^5, g_3^5 and g_4^5 are respectively defined as

$$g_2^5(v_0,...,v_7) = \frac{1}{45}\sum_{i\neq j}^{7}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,\hat{v}_j,...,v_7)\Big]_3 \otimes$$
$$\prod_{r\neq ineqk}^{7} \triangle(v_r,v_i,v_k) \wedge \prod_{r\neq j}^{7} \triangle(v_r,v_j,v_k). \tag{3.18}$$

$$g_3^5(v_0,...,v_8) = -\frac{1}{105}\sum_{i=0}^{8}(-1)^i Alt_8\Big[r(v_0,...,\hat{v}_i,...,v_8)\Big]_4 \otimes \prod_{r\neq i\neq j\neq k}^{8} \triangle(v_r,v_i,v_j,v_k). \tag{3.19}$$

$$g_4^5(v_0,...,v_9) = \frac{1}{190}Alt_{10}\Big[r(v_0,...,v_9)\Big]_5. \tag{3.20}$$

Lemma 3.12. $g_1^5 \circ p = \delta \circ g_2^5.$

Proof. For proof see Khalid et al. [20]. □

Lemma 3.13. $g_2^5 \circ p = \delta \circ g_3^5.$

Proof. For proof see Khalid et al. [20]. □

Lemma 3.14. $g_3^5 \circ p = \delta \circ g_4^5.$

Proof. For proof see Khalid et al. [20]. □

4. Generalized Extension in Geometry

4.1. **Geometry for Weight 6.** As defined in [17], connect Grassmannian configuration complex with the sub complex of Goncharov motivic for weight 6 given as

$$\begin{array}{ccccc}
G_{10}(3) & \xrightarrow{d} & G_9(3) & & \\
\downarrow p & & \downarrow p & & \\
G_9(2) & \xrightarrow{d} & G_8(2) & \xrightarrow{g_1^6} & \mathcal{B}_2(F)\otimes\wedge^4 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_8(1) & \xrightarrow{d} & G_7(1) & \xrightarrow{g_0^6} & \wedge^6 F^\times
\end{array} \tag{I}$$

where,

$$g_0^6 : (v_0,...,v_6) \to \sum_{i=j+1}^{6}(-1)^{i+1}\bigwedge_{j\neq i}^{6}\triangle(v_j)\ (mod\ 7) \tag{4.21}$$

and

$$g_1^6 : (v_0,...,v_7) \to \frac{1}{15}\sum_{i\neq j\neq k\neq l}^{7}(-1)^i[r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,\hat{v}_l,...,v_7)]_2 \otimes \prod_{r\neq i}^{7}\triangle(v_i,v_r)$$

$$\wedge \prod_{r\neq j}^{7} \triangle(v_j, v_r) \wedge \prod_{r\neq k}^{7} \triangle(v_k, v_r) \wedge \prod_{r\neq l}^{7} \triangle(v_l, v_r)\ (mod\ 8). \quad (4.22)$$

Lemma 4.2. $$g_0^6 \circ p = \delta \circ g_1^6$$

Proof. For proof see Khalid et al. [17]. □

4.2.1. *Extension in Weight 6.* For this extension of geometry, the extended diagram (I) introduces four new morphisms g_2^6, g_3^6, g_4^6 and g_5^6, given by

$$\begin{array}{ccccc}
G_{13}(6) & \xrightarrow{d} & G_{12}(6) & \xrightarrow{g_5^6} & \mathcal{B}_6(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{12}(5) & \xrightarrow{d} & G_{11}(5) & \xrightarrow{g_4^6} & \mathcal{B}_5(F) \otimes \wedge F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{11}(4) & \xrightarrow{d} & G_{10}(4) & \xrightarrow{g_3^6} & \mathcal{B}_4(F) \otimes \wedge^2 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{10}(3) & \xrightarrow{d} & G_{9}(3) & \xrightarrow{g_2^6} & \mathcal{B}_3(F) \otimes \wedge^3 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{9}(2) & \xrightarrow{d} & G_{8}(2) & \xrightarrow{g_1^6} & \mathcal{B}_2(F) \otimes \wedge^4 F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{8}(1) & \xrightarrow{d} & G_{7}(1) & \xrightarrow{g_0^6} & \wedge^6 F^\times
\end{array} \quad \text{(J)}$$

where,

$$g_2^6(v_0, ..., v_8) = -\frac{1}{66} \sum_{i\neq j\neq k}^{8} (-1)^i Alt_6 \Big[r(v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_8) \Big]_3 \otimes$$
$$\prod_{l\neq i\neq m}^{8} \triangle(v_r, v_i, v_m) \wedge \prod_{r\neq j\neq m}^{8} \triangle(v_r, v_j, v_m) \wedge \prod_{r\neq k\neq m}^{8} \triangle(v_r, v_k, v_m). \quad (4.23)$$

$$g_3^6(v_0, ..., v_9) = \frac{1}{153} \sum_{i\neq j}^{9} (-1)^i Alt_8 \Big[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_9) \Big]_4 \otimes$$
$$\prod_{r\neq i\neq k\neq l}^{9} \triangle(v_r, v_i, v_k, v_l) \wedge \prod_{r\neq j\neq k\neq l}^{9} \triangle(v_r, v_j, v_k, v_l). \quad (4.24)$$

$$g_4^6(v_0,...,v_{10}) = -\frac{1}{276}\sum_{i=0}^{10}(-1)^i Alt_{10}\Big[r(v_0,...,\hat{v}_i,...,v_{10})\Big]_5 \otimes \prod_{r\neq i\neq j\neq k\neq l}^{10} \triangle(v_r,v_i,v_j,v_k,v_l). \tag{4.25}$$

$$g_5^6(v_0,...,v_{11}) = \frac{1}{435}Alt_{12}\Big[r(v_0,...,v_{11})\Big]_6. \tag{4.26}$$

Lemma 4.3. $g_1^6 \circ p = \delta \circ g_2^6.$

Proof. Let $(v_0,...,v_8)$ be 9 points $\in G_9(3)$ and apply projection morphism p

$$p(v_0,...,v_8) = \sum_{i=0}^{8}(-1)^i(v_i|v_0,...,\hat{v}_i,...,v_8). \tag{4.27}$$

now by applying morphism g_1^6

$$g_1^6 \circ p(v_0,...,v_8) = \frac{1}{15}\sum_{i\neq j\neq k\neq l\neq m}^{8}(-1)^i[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,\hat{v}_l,\hat{v}_m,...,v_8)]_2\otimes$$
$$\prod_{r\neq j}^{8}\triangle(v_i|v_j,v_r)\wedge\prod_{r\neq k}^{8}\triangle(v_i|v_k,v_r)\wedge\prod_{r\neq l}^{8}\triangle(v_i|v_l,v_r)\wedge\prod_{r\neq m}^{8}\triangle(v_i|v_m,v_r). \tag{4.28}$$

Let us take $(v_0,...,v_8) \in G_9(3)$ again and apply morphism g_2^6

$$g_2^6(v_0,...,v_8) = -\frac{1}{66}\sum_{i\neq j\neq k\neq m}^{8}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_8)\Big]_3\otimes$$
$$\prod_{r\neq i\neq l}^{8}\triangle(v_r,v_i,v_l)\wedge\prod_{r\neq j\neq l}^{8}\triangle(v_r,v_j,v_l)\wedge\prod_{r\neq k\neq l}^{8}\triangle(v_r,v_k,v_l). \tag{4.29}$$

Then by applying differential morphism δ, which yield,

$$\delta\circ g_2^6 = -\frac{1}{66}\sum_{i\neq j\neq k\neq m}^{8}(-1)^i Alt_6\Big[r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_8)\Big]_2\otimes r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_8)$$
$$\prod_{r\neq i\neq l}^{8}\triangle(v_r,v_i,v_l)\wedge\prod_{r\neq j\neq l}^{8}\triangle(v_r,v_j,v_l)\wedge\prod_{r\neq k\neq l}^{8}\triangle(v_r,v_k,v_l). \tag{4.30}$$

Then by using Siegel, wedge, tensor and odd cycle properties, Eq. (4.30) becomes

$$\delta\circ g_2^6(v_0,...,v_8) = \frac{1}{15}\sum_{i\neq j\neq k\neq l\neq m}^{8}(-1)^i[r(v_i|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,\hat{v}_l,\hat{v}_m,...,v_8)]_2\otimes$$
$$\prod_{r\neq j}^{8}\triangle(v_i|v_j,v_r)\wedge\prod_{r\neq k}^{8}\triangle(v_i|v_k,v_r)\wedge\prod_{r\neq l}^{8}\triangle(v_i|v_l,v_r)\wedge\prod_{r\neq m}^{8}\triangle(v_i|v_m,v_r). \tag{4.31}$$

Eq.(4.28) and Eq.(4.31) shows that, $g_1^6 \circ p = \delta \circ g_2^6$. □

Lemma 4.4. $g_2^6 \circ p = \delta \circ g_3^6$.

Proof. Suppose $(v_0, ..., v_9)$ 10 points $\in G_{10}(4)$ and apply projection map p followed by morphisms g_2^6,

$$p(v_0, ..., v_9) = \sum_{i=0}^{9} (-1)^i (v_i|v_0, ..., \hat{v}_i, ..., v_9), \tag{4.32}$$

$$g_2^6 \circ p(v_0, ..., v_9) = -\frac{1}{66} \sum_{i\neq j\neq k\neq l}^{9} (-1)^i Alt_6 \Big[r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, \hat{v}_l, ..., v_9)\Big]_3 \otimes \prod_{r\neq j\neq m}^{9} \triangle(v_i|v_r, v_j, v_m) \wedge \prod_{r\neq k\neq m}^{9} \triangle(v_i|v_r, v_k, v_m) \wedge \prod_{r\neq l\neq m}^{9} \triangle(v_i|v_r, v_l, v_m). \tag{4.33}$$

Let us take $(v_0, ..., v_9) \in G_{10}(4)$ again and apply morphism g_3^6, then

$$g_3^6(v_0, ..., v_9) = \frac{1}{153} \sum_{i\neq j}^{9} (-1)^i Alt_8 \Big[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_9)\Big]_4 \otimes \prod_{k\neq i\neq l\neq m}^{9} \triangle(v_k, v_i, v_l, v_m) \wedge \prod_{k\neq j\neq l\neq m}^{9} \triangle(v_k, v_j, v_l, v_m). \tag{4.34}$$

Now compose morphism δ with above Eq.(4.34), which yields,

$$\delta \circ g_3^6 : (v_0, ..., v_9) = \frac{1}{153} \sum_{i\neq j}^{9} (-1)^i Alt_8 \Big[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_9)\Big]_3 \otimes r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_9) \prod_{k\neq i\neq l\neq m}^{9} \triangle(v_k, v_i, v_l, v_m) \wedge \prod_{k\neq j\neq l\neq m}^{9} \triangle(v_k, v_j, v_l, v_m). \tag{4.35}$$

After simplification, Eq.(4.35) becomes

$$\delta \circ g_3^6(v_0, ..., v_9) = -\frac{1}{66} \sum_{i\neq j\neq k\neq l}^{9} (-1)^i Alt_6 \Big[r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, \hat{v}_l, ..., v_9)\Big]_3 \otimes \prod_{r\neq j\neq m}^{9} \triangle(v_i|v_r, v_j, v_m) \wedge \prod_{r\neq k\neq m}^{9} \triangle(v_i|v_r, v_k, v_m) \wedge \prod_{r\neq l\neq m}^{9} \triangle(v_i|v_r, v_l, v_m). \tag{4.36}$$

Eq.(4.33) and Eq.(4.36) shows that, $g_2^6 \circ p = \delta \circ g_3^6$. □

Lemma 4.5. $g_3^6 \circ p = \delta \circ g_4^6.$

Proof. Let us assume $(v_0, ..., v_{10})$ be eleven points $\in G_{11}(5)$ and apply map p followed by morphism g_3^6,

$$p(v_0, ..., v_{10}) = \sum_{i=0}^{10} (-1)^i (v_i | v_0, ..., \hat{v}_i, ..., v_{10}). \tag{4.37}$$

$$g_3^6 \circ p(v_0, ..., v_{10}) = \frac{1}{153} \sum_{i \neq j \neq k}^{10} (-1)^i Alt_8 \Big[r(v_i | v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_{10}) \Big]_4 \otimes \prod_{r \neq j \neq l \neq m}^{10} \triangle(v_i | v_r, v_j, v_l, v_m) \wedge \prod_{r \neq k \neq l \neq m}^{10} \triangle(v_i | v_r, v_k, v_l, v_m). \tag{4.38}$$

Let us take $(v_0, ..., v_{10}) \in G_{11}(5)$ again, apply morphism g_4^6 followed by morphisms δ

$$g_4^6(v_0, ..., v_{10}) = -\frac{1}{276} \sum_{i=0}^{10} (-1)^i Alt_{10} \Big[r(v_0, ..., \hat{v}_i, ..., v_{10}) \Big]_5 \otimes \prod_{r \neq i \neq j \neq k \neq l}^{10} \triangle(v_r, v_i, v_j, v_k, v_l), \tag{4.39}$$

$$\delta \circ g_4^6(v_0, ..., v_{10}) = -\frac{1}{276} \sum_{i=0}^{10} (-1)^i Alt_{10} \Big[r(v_0, ..., \hat{v}_i, ..., v_{10}) \Big]_4 \otimes r(v_0, ..., \hat{v}_i, ..., v_{10}) \wedge \prod_{r \neq i \neq j \neq k \neq l}^{10} \triangle(v_r, v_i, v_j, v_k, v_l). \tag{4.40}$$

After applying all properties Siegel, wedge, tensor and odd cycle, the following is obtained

$$\delta \circ g_4^6(v_0, ..., v_{10}) = \frac{1}{153} \sum_{i \neq j \neq k}^{10} (-1)^i Alt_8 \Big[r(v_i | v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_{10}) \Big]_4 \otimes \prod_{r \neq j \neq l \neq m}^{10} \triangle(v_i | v_r, v_j, v_l, v_m) \wedge \prod_{r \neq k \neq l \neq m}^{10} \triangle(v_i | v_r, v_k, v_l, v_m). \tag{4.41}$$

From Eq.(4.38) and Eq.(4.41), it is observed that, $g_3^6 \circ p = \delta \circ g_4^6$. □

Lemma 4.6. $g_4^6 \circ p = \delta \circ g_5^6.$

Proof. Suppose $(v_0, ..., v_{11})$ 12 points $\in G_{12}(6)$ and apply map p followed by morphism g_4^6,

$$p(v_0, ..., v_{11}) = \sum_{i=0}^{11} (-1)^i (v_i | v_0, ..., \hat{v}_i, ..., v_{11}), \tag{4.42}$$

$$g_4^6 \circ p(v_0, ..., v_{11}) = -\frac{1}{276}\sum_{i\neq j}^{11}(-1)^i Alt_{10}\Big[r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_{11})\Big]_5 \otimes \prod_{r\neq j\neq k\neq l\neq m}^{11} \triangle(v_i|v_r, v_j, v_k, v_l, v_m). \quad (4.43)$$

Let us take $(v_0, ..., v_{11}) \in G_{12}(6)$ again and apply morphism g_5^6

$$g_5^6(v_0, ..., v_{11}) = \frac{1}{435} Alt_{12}\Big[r(v_0, ..., v_{11})\Big]_6. \quad (4.44)$$

Now by applying morphism δ, then

$$\delta \circ g_5^6 : (v_0, ..., v_{11}) = \frac{1}{435} Alt_{12}\Big[r(v_0, ..., v_{11})\Big]_5 \otimes r(v_0, ..., v_{11}). \quad (4.45)$$

After applying Siegel, wedge, tensor and odd cycle properties, Eq. (4.45) becomes

$$\delta \circ g_5^6(v_0, ..., v_{11}) = -\frac{1}{276}\sum_{i\neq j}^{11}(-1)^i Alt_{10}\Big[r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_{11})\Big]_5 \otimes \prod_{r\neq j\neq k\neq l\neq m}^{11} \triangle(v_i|v_r, v_j, v_k, v_l, v_m). \quad (4.46)$$

Eq.(4.43) and Eq.(4.46) shows that, $g_4^6 \circ p = \delta \circ g_5^6$. □

4.7. **Geometry for Any Weight N.** As defined in [17], two morphisms g_0^n and g_1^n are generalized as follows

$$\begin{array}{ccccc}
G_{N+4}(3) & \xrightarrow{d} & G_{N+3}(3) & & \\
\downarrow p & & \downarrow p & & \\
G_{N+3}(2) & \xrightarrow{d} & G_{N+2}(2) & \xrightarrow{g_1^N} & \mathcal{B}_2(F)\otimes \wedge^{N-2}F^\times \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{N+2}(1) & \xrightarrow{d} & G_{N+1}(1) & \xrightarrow{g_0^N} & \wedge^N F^\times
\end{array} \quad (N \geq 2) \quad \text{(K)}$$

where,

$$g_0^N(v_0, ..., v_N) \to \sum_{i=j+1}^{N}(-1)^i \bigwedge_{j\neq i}^{N} \triangle(v_j)\ (mod\ N+1). \quad (4.47)$$

and

$$g_1^N(v_0, ..., v_{N+1}) = \frac{1}{{}^N C_2}(-1)^N \sum_{\substack{i_0=0 \\ i_1=i_0+1 \\ \vdots \\ i_{N-3}=i_0+N-3}}^{N+1} (-1)^i [r(v_0, ..., \hat{v}_{i_0}, \hat{v}_{i_1}, ...,$$

$$\hat{v}_{i_{N-3}}, ..., v_{N+1})]_2 \otimes \prod_{j\neq i_0}^{N+1} \triangle(v_{i_0}, v_j)\wedge$$
$$\prod_{j\neq i_1}^{N+1} \triangle(v_{i_1}, v_j)\wedge$$
$$.$$
$$.$$
$$.$$
$$\wedge \prod_{j\neq i_{N-3}}^{N+1} \triangle(v_{i_{N-3}}, v_j)\ (mod\ N+2). \qquad (4.48)$$

Theorem 4.8. *The above diagram K is commutative.*

Proof. For proof see [17]. □

4.8.1. *Generalized Extension in Geometry for Weight N.* For generalized extension in geometry, connect the two generalized chain complexes Goncharov and Grassmannian complexes to obtain

$$\begin{array}{ccccc}
G_{2n+1}(n) & \xrightarrow{d} & G_{2n}(n) & \xrightarrow{g^n_{n-1}} & \mathcal{B}_n(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{2n}(n-1) & \xrightarrow{d} & G_{2n-1}(n-1) & \xrightarrow{g^n_{n-2}} & \mathcal{B}_{n-1}(F)\otimes F^{\times} \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{2n-1}(n-2) & \xrightarrow{d} & G_{2n-2}(n-2) & \xrightarrow{g^n_{n-3}} & \mathcal{B}_{n-2}(F)\otimes \wedge^2 F^{\times} \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
\vdots & & \vdots & & \vdots \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{n+3}(2) & \xrightarrow{d} & G_{n+2}(2) & \xrightarrow{g^n_1} & \mathcal{B}_2(F)\otimes \wedge^{n-2} F^{\times} \\
\downarrow p & & \downarrow p & & \downarrow \delta \\
G_{n+2}(1) & \xrightarrow{d} & G_{n+1}(1) & \xrightarrow{g^n_0} & \wedge^n F,^{\times}
\end{array} \qquad \text{(L)}$$

such that

$$g^n_{n-3}(v_0, ..., v_{2n-3}) = (-1)^{2n-2}\frac{1}{{}_{n(n-3)}C_2}\sum_{i=0}^{2n-2}(-1)^i Alt_{2n-4}\Big[r(v_0, ..., \hat{v_{i_0}}, \hat{v_{j_0}}, ...,$$
$$v_{2n-2})\Big]_{n-2} \otimes \prod_{i_0\neq i_1\neq i_2\neq ...\neq i_{n-2}}^{2n-2} \triangle(v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_{n-2}})$$

$$\wedge \prod_{j_0 \neq j_1 \neq j_2 \neq \ldots \neq j_{n-2}}^{2n-2} \triangle(v_{j_0}, v_{j_1}, v_{j_2}, ..., v_{j_{n-2}}). \quad (4.49)$$

$$g^n_{n-2}(v_0, ..., v_{2n-2}) = (-1)^{2n-1} \frac{1}{{}_{n(n-2)}C_2} \sum_{i=0}^{2n-2} (-1)^i Alt_{2n-2}\Big[r(v_0, ..., \hat{v_{i_0}}, ..., v_{2n-2})\Big]_{n-1} \otimes \prod_{i_0 \neq i_1 \neq i_2 \neq \ldots \neq i_{n-2}}^{2n-2} \triangle(v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_{n-2}}) \quad (4.50)$$

$$g^n_{n-1}(v_0, ..., v_{2n-1}) = (-1)^{2n} \frac{1}{{}_{n(n-1)}C_2} Alt_{2n}\Big[r(v_0, ..., v_{2n-1})\Big]_n. \quad (4.51)$$

Theorem 4.9. $g^n_{n-3} \circ p = \delta \circ g^2_{n-2}$.

Proof. Let $(v_0, ..., v_{2n-2}) \in G_{2n-1}(n)$, apply morphism p

$$p(v_0, ..., v_{2n-2}) = \sum_{i=0}^{2n-2} (-1)^i (v_i | v_0, ..., \hat{v}_i, ..., v_{2n-2}). \quad (4.52)$$

Now apply morphism g^n_{n-3}, which yields:

$$g^n_{n-2} \circ p = (-1)^{2n-2} \frac{1}{{}_{n(n-3)}C_2} \sum_{i=0}^{2n-2} (-1)^i Alt_{2n-4}\Big[r(v_0, ..., \hat{v}_i, \hat{v_{i_0}}, \hat{v_{j_0}}, ..., v_{2n-2})\Big]_{n-2} \otimes \prod_{i_0 \neq i_1 \neq i_2 \neq \ldots \neq i_{n-2}}^{2n-2} \triangle(v_i | v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_{n-2}}) \wedge \prod_{j_0 \neq j_1 \neq j_2 \neq \ldots \neq j_{n-2}}^{2n-2} \triangle(v_i | v_{j_0}, v_{j_1}, v_{j_2}, ..., v_{j_{n-2}}). \quad (4.53)$$

Let us take $(v_0, ..., v_{2n-2}) \in G_{2n-1}(n)$ again, apply morphism g^n_{n-2} followed by δ, then

$$g^n_{n-2}(v_0, ..., v_{2n-2}) = (-1)^{2n-1} \frac{1}{{}_{n(n-2)}C_2} \sum_{i=0}^{2n-2} (-1)^i Alt_{2n-2}\Big[r(v_0, ..., \hat{v_{i_0}}, ..., v_{2n-2})\Big]_{n-1} \otimes \prod_{i_0 \neq i_1 \neq i_2 \neq \ldots \neq i_{n-2}}^{2n-2} \triangle(v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_{n-2}}). \quad (4.54)$$

now by applying morphism δ, then

$$\delta \circ g^n_{n-2}(v_0, ..., v_{2n-2}) = (-1)^{2n-1} \frac{1}{{}_{n(n-2)}C_2} \sum_{i=0}^{2n-2} (-1)^i Alt_{2n-2}\Big[r(v_0, ..., \hat{v_{i_0}}, ..., v_{2n-2})\Big]_{n-2} \otimes r(v_0, ..., \hat{v_{i_0}}, ..., v_{2n-2}) \wedge \prod_{i_0 \neq i_1 \neq i_2 \neq \ldots \neq i_{n-2}}^{2n-2} \triangle(v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_{n-2}}). \quad (4.55)$$

By using Siegel, wedge, tensor and odd cycle properties, Eq. (4.55) becomes

$$\delta\circ g^n_{n-2} = (-1)^{2n-2}\frac{1}{{}_{n(n-3)}C_2}\sum_{i=0}^{2n-2}(-1)^i Alt_{2n-4}\Big[r(v_0,...,\hat{v}_i,\hat{v_{i_0}},\hat{v_{j_0}},...,v_{2n-2})\Big]_{n-2}\otimes$$
$$\prod_{i_0\neq i_1\neq i_2\neq\ldots\neq i_{n-2}}^{2n-2}\triangle(v_i|v_{i_0},v_{i_1},v_{i_2},...,v_{i_{n-2}})\wedge$$
$$\prod_{j_0\neq j_1\neq j_2\neq\ldots\neq j_{n-2}}^{2n-2}\triangle(v_i|v_{j_0},v_{j_1},v_{j_2},...,v_{j_{n-2}}). \tag{4.56}$$

From Eq.(4.55) and Eq.(4.56) it is observed that, $g^n_{n-3}\circ p=\delta\circ g^2_{n-2}$. □

Theorem 4.10. *The following upper right square of the generalized diagram (L) is commutative.*

$$\begin{array}{ccc} G_{2n}(n) & \xrightarrow{g^n_{n-1}} & \mathcal{B}_n(F) \\ \downarrow p & & \downarrow \delta \\ G_{2n-1}(n-1) & \xrightarrow{g^n_{n-2}} & \mathcal{B}_n(F)\otimes F^\times \end{array} \tag{M}$$

Proof. Suppose $(v_0,...,v_{2n-1})$ denotes $2n$ points $\in G_{2n}(n)$, apply morphism p

$$p(v_0,...,v_{2n-1})=\sum_{i=0}^{2n-1}(-1)^i(v_i|v_0,...,\hat{v}_i,...,v_{2n-1}). \tag{4.57}$$

Now by applying morphisms g^n_{n-2}, to obtain

$$g^n_{n-2}\circ p = (-1)^{2n-1}\frac{1}{{}_{n(n-2)}C_2}\sum_{i=0}^{2n-2}(-1)^i Alt_{2n-2}\Big[r(v_i|v_0,...,\hat{v_{i_0}},\hat{v}_i,...,v_{2n-2})\Big]_{n-1}\otimes$$
$$\prod_{i_0\neq i_1\neq i_2\neq\ldots\neq i_{n-2}}^{2n-2}\triangle(v_i|v_{i_0},v_{i_1},v_{i_2},...,v_{i_{n-2}}). \tag{4.58}$$

Let us take $(v_0,...,v_{2n-1})\in G_{2n}(n)$ again, apply morphism g^n_{n-1} followed by the morphism δ,

$$g^n_{n-1}:(v_0,...,v_{2n-1})=(-1)^{2n}\frac{1}{{}_{n(n-1)}C_2}Alt_{2n}\Big[r(v_0,...,v_{2n-1})\Big]_n. \tag{4.59}$$

$$\delta\circ g^n_{n-1}:(v_0,...,v_{2n-1})=(-1)^{2n}\frac{1}{{}_{n(n-1)}C_2}Alt_{2n}\Big[r(v_0,...,v_{2n-1})\Big]_{n-1}\otimes r(v_0,...,v_{2n-1}). \tag{4.60}$$

Simplifying it by using Siegel, wedge, tensor and odd cycle properties, Eq. (4.60) becomes

$$\delta\circ g^n_{n-1}=(-1)^{2n-1}\frac{1}{{}_{n(n-2)}C_2}\sum_{i=0}^{2n-2}(-1)^i Alt_{2n-2}\Big[r(v_i|v_0,...,\hat{v_{i_0}},\hat{v}_i,...,v_{2n-2})\Big]_{n-1}\otimes$$

$$\prod_{i_0 \neq i_1 \neq i_2 \neq \ldots \neq i_{n-2}}^{2n-2} \triangle(v_i | v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_{n-2}}). \tag{4.61}$$

From Eq.(4.58) and Eq.(4.61), the upper right square of the generalized diagram L is commutative. □

5. Conclusion

In this paper, a generalized extension of geometry between configuration and Goncharov motivic chain complexes is proposed to produce a generalized commutative diagram. Previous researches were based on generalizing morphisms between Goncharov motivic and configuration chain complexes of weight g_0^n and g_1^n only, but this work generalized all morphisms between Goncharov motivic and configuration chain complexes. After generalizing the geometry of Grassmannian configuration and Goncharov classical polylog chain complexes, this topic is replete for delightful details. The generalizations presented in this paper will help future researchers discover the geometry of different modern form of chain complexes.

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