Punjab University Journal of Mathematics (2021), 53(4),261-293
https://doi.org/10.52280/pujm.2021.530404

On Fuzzy Bipolar Soft Ordered Semigroups

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Received: 07 April, 2020 / Accepted: 13 March, 2021 / Published online: 20 April, 2021

Abstract. In this paper, the concept of fuzzy bipolar soft set initiated by Naz and Shabir [22] is modified and strengthened. Consequently, the basic operations on fuzzy bipolar soft sets are redefined and their algebraic properties are studied. The notion of fuzzy bipolar soft ordered semigroup is defined and, furthermore, the concepts of fuzzy bipolar soft left (resp., right, two-sided) ideals over ordered semigroups are introduced and characterized.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Ordered semigroup, fuzzy bipolar soft set, fuzzy bipolar soft ordered semigroup, fuzzy bipolar soft left (right) ideal, fuzzy bipolar soft ideal.

1. Introduction

The remarkable concept of fuzzy set was initiated by L. A. Zadeh in his seminal article [30] of 1965. Since then fuzzy set theory is widely studied and extended to various areas of mathematics. Rosenfeld [23] was the first who extended the idea of fuzzy sets to abstract algebra, and defined fuzzy subgroupoid and fuzzy subgroup. N. Kuroki [15-20] introduced the notions of fuzzy semigroups and fuzzy (left, right, bi-, interior, semiprime, generalized bi-, quasi-) ideals of semigroups. Kehayopulu and Tsingelis [4] took the lead in considering the concept of fuzzy sets in ordered groupoids (ordered semigroups). Kehayopulu and Tsingelis [5-8] defined fuzzy (left, right, bi-, interior, quasi-) ideals in ordered semigroups, and studied the decomposition of different classes of ordered semigroups. Moreover, they characterized several classes of ordered semigroups such as simple and regular ordered semigroups in terms of fuzzy sets. Shabir and Iqbal [24] introduced bipolar fuzzy left (resp., right, bi-) ideals in ordered semigroups, and characterized their various classes by the properties of these ideals. Ibrar et al. defined (α, β)-bipolar fuzzy ideals and (α, β)-bipolar fuzzy interior ideals in ordered semigroups, and characterized regular (resp., intra-regular, simple, semisimple) ordered semigroups by using properties of these generalized bipolar ideals. N. M. Khan et al. [9] characterized various classes of ordered semigroups in terms of their generalized fuzzy ideals. Muhiuddin et al. [21] studied fuzzy semiprime subsets in ordered semigroups. J. Tang et al. [27] introduced the notions of interval valued generalized fuzzy bi-ideals (resp., quasi-ideals) in ordered semigroups and examined their related properties. Moreover, they characterized bi-regular (resp., intra-regular and regular) ordered semigroups by interval valued generalized fuzzy quasi-ideals. Sana et al. [2] introduced and studied the concepts of possibility fuzzy soft ideals and possibility fuzzy soft interior ideals in ordered semigroups. In [13], the authors introduced interval valued (α, β)-fuzzy filters of ordered semigroups, and characterized fuzzy (left, right) filters of ordered semigroups in terms of interval valued $(\in, \in \vee q)$-fuzzy (left, right) filters. In [26], the authors introduced and examined the concepts of pure fuzzy and weakly pure fuzzy ideals in ordered semigroups. In [10], the authors defined the notion of (k^*, q)-quasi coincidence, and introduced $(\in, \in \vee(k^*, q_k))$-fuzzy filters (resp., fuzzy bi-filters) of an ordered semigroup. Further, they characterized different classes of ordered semigroups by these fuzzy filters. Ahn et al. [1] applied the notion of hesitant fuzzy set to ordered semigroup theory and introduced hesitant fuzzy left (resp., right, bi-, quasi-) ideals, and investigated their several properties. Similarly, many other researchers [11, 12, 14, 25, 28, 29] contributed a lot to fuzzy ordered semigroup theory through various dimensions. Besides, Zararsiz [31, 32] contributed to fuzzy set theory by studying the algebraic structure of fuzzy numbers. She also presented the methods of evaluating the similarity measures between sequences of triangular fuzzy numbers for making contributions to fuzzy risk analysis. In 2014, Naz and Shabir [22] introduced the concept of fuzzy bipolar soft set. This innovatory contribution opened a new dimension for research in the framework of fuzzy set theory.

In this paper, the notion of fuzzy bipolar soft set and the basic operations on the structure are redefined, and the algebraic properties of the notion are examined. Besides, the concept is extended to ordered semigroup theory, and the notion of fuzzy bipolar soft ordered semigroup is defined and studied. Similarly, the concepts of fuzzy bipolar soft left, right, two-sided ideals over ordered semigroups are introduced and characterized. In [22], presenting the concept of the structure, the authors consider a set of parameters E and define its Not set denoted as $\neg E$, where $\neg e = \text{not}\, e$ for each element e in E. Further, they use different subsets A, B, C of the parameters set to define fuzzy bipolar soft sets on an initial universe set X. To make things simple, we use a single set of parameters E and, for a non-empty subset A of E, define an injective function $f : A \to E$, where A and $f(A)$ are used as domains for the pair of mappings characterizing a fuzzy bipolar soft set. Hence, we consider all fuzzy bipolar soft sets over an initial universe set S with the fixed set of parameters A. In this way, the related operations become more functional and convenient for a detailed analysis of fuzzy bipolar soft set theory in various algebraic and non-algebraic structures, for example, (ordered) semigroups, (ordered) hypersemigroups, (ordered) semirings, (ordered) near-rings, (ordered) AG-groupoids, lattices and (partial) metric spaces.

2. Fuzzy bipolar soft sets

In this section, we redefine the notion of fuzzy bipolar soft set and explain it by suitable examples. Consequently, the basic operations on fuzzy bipolar soft sets such as union, intersection and product are redefined. For further details of fuzzy bipolar soft set theory and its applications in decision making problems, we may refer the readers to [22]. An ordered semigroup $(S, \cdot, \leq)$ is a set S, where $(S, \cdot)$ is a semigroup and $(S, \leq)$ is a partially ordered set (poset) such that the order relation "$\leq$" is compatible with the binary operation of multiplication "$\cdot$". Moreover, if P is a nonempty subset of S, we say that P is a subsemigroup of S if and only if: (i) $PP \subseteq P$, and ii) if $a \in P$ such that $S \ni b \leq a$, then $b \in P$. For a nonempty subset P of S, we denote by $(P]$ the subset of S defined as

$$(P] = \{s \in S \mid s \leq p \text{ for some } p \in P\}.$$

Naz and Shabir presented the notion of fuzzy bipolar soft set as follows:

Definition 2.1. *[22] A triplet (F, G, A) is called a fuzzy bipolar soft set over U, where F and G are mappings given by*

$$F : A \to FP(U) \text{ and } G : \neg A \to FP(U)$$

such that

$$0 \leq (F(e)(x) + G(\neg e)(x)) \leq 1 \ \ (\forall\, e \in A).$$

Now, in the following, we redefine the concept of fuzzy bipolar soft set.

Definition 2.2. *Assume S be an initial universe set, $\mathcal{F}(S)$ the collection of all fuzzy subsets of S and E a set of parameters with respect to S. For $A \subseteq E$, let $f : A \to E$ be an injective function. Then, a fuzzy bipolar soft (FBS) set λ_A over S is an object of the form*

$$\lambda_A = (\overset{+}{\lambda}, \overset{-}{\lambda}, A),$$

where $\overset{+}{\lambda} : A \to \mathcal{F}(S)$ and $\overset{-}{\lambda} : f(A) \to \mathcal{F}(S)$ are set-valued functions such that the condition

$$0 \leq \overset{+}{\lambda}(\varepsilon)(x) + \overset{-}{\lambda}(f(\varepsilon))(x) \leq 1$$

holds, for all $x \in S$ and $\varepsilon \in A$.

The fuzzy bipolar soft set λ_A can also be expressed as

$$\lambda_A = \{(\varepsilon, \overset{+}{\lambda}(\varepsilon), \overset{-}{\lambda}(f(\varepsilon)) \mid \varepsilon \in A\}.$$

For each $\varepsilon \in A$, the sets $\overset{+}{\lambda}(\varepsilon)$ and $\overset{-}{\lambda}(f(\varepsilon))$ are ε-approximations of the fuzzy bipoar soft set λ_A, and the values $\overset{+}{\lambda}(\varepsilon)(x)$ and $\overset{-}{\lambda}(f(\varepsilon))(x)$ respectively denote the degrees of membership and that of non-membership of $x \in S$. Besides, for all $\varepsilon \in A$ and $x \in S$, the condition

$$0 \leq \overset{+}{\lambda}(\varepsilon)(x) + \overset{-}{\lambda}(f(\varepsilon))(x) \leq 1$$

is imposed as consistency constraint.

If $\overset{+}{\lambda}(\varepsilon) = \phi$, the empty fuzzy set of S and $\overset{-}{\lambda}(f(\varepsilon)) = S$, the universal fuzzy set of S, for any $\varepsilon \in A$, then (ε, ϕ, S) will not appear in λ_A.

Note. For the sake of brevity and convenience, we shall denote $\overset{-}{\lambda}(f(\varepsilon))$ by $\overset{-}{\lambda}(\varepsilon)$, and write $(\overset{+}{\lambda}, \overset{-}{\lambda})$ instead of $(\overset{+}{\lambda}, \overset{-}{\lambda}, A)$.

Let's explain and elaborate the idea of fuzzy bipolar soft sets with the help of the following examples.

Example 2.3. *Assume $S = \mathbb{R}$ be an initial universe set, $E = \{-2, -1, 1, 2\}$ a set of parameters and $A = \{1, 2\}$ a subset of E. Let $f : A \to E$ be an injective function such that $f(1) = -1$ and $f(2) = -2$. Further, for all $\varepsilon \in A$ and $x \in S$, let's define*

$$\overset{+}{\lambda}(\varepsilon)(x) = \begin{cases} \frac{1}{2\varepsilon}, & \text{if } x \in \mathbb{Q}, \\ \frac{1}{3\varepsilon}, & \text{otherwise}, \end{cases}$$

and

$$\overset{-}{\lambda}(\varepsilon)(x) = \begin{cases} \frac{1}{3|\varepsilon|}, & \text{if } x \in \mathbb{Q}, \\ \frac{1}{2|\varepsilon|}, & \text{otherwise}. \end{cases}$$

Then, λ_A is an FBS set over S.

Example 2.4. *Let $S = \{\tau_1, \tau_2, \tau_3\}$ be an initial universe set, $E = \mathbb{Z}_3$ a set of parameters and $A = \{0, 1\}$ a subset of E. Let's define an injective function $f : A \to E$ such that $f(\varepsilon) = \varepsilon^{-1}$, for all $\varepsilon \in A$. Thus, we have $f(0) = 0$ and $f(1) = 2$. Then, an FBS set λ_A over S can be defined in terms of its fuzzy appproximate functions as follows:*

$$\overset{+}{\lambda}(0) = \{(\tau_1, 0.2), (\tau_2, 0.3), (\tau_3, 0.2)\},$$

$$\overset{+}{\lambda}(1) = \{(\tau_1, 0.4), (\tau_2, 0.5), (\tau_3, 0.4)\},$$

$$\overset{-}{\lambda}(0) = \{(\tau_1, 0.3), (\tau_2, 0.4), (\tau_3, 0.3),$$

$$\overset{-}{\lambda}(2) = \{(\tau_1, 0.3), (\tau_2, 0.2), (\tau_3, 0.3)\}.$$

The FBS set λ_A can also be expressed as

$$\lambda_A = \left\{ \begin{array}{l} (0, \{(\tau_1, 0.2), (\tau_2, 0.3), (\tau_3, 0.2)\}, \{(\tau_1, 0.3), (\tau_2, 0.4), (\tau_3, 0.3)\}), \\ (1, \{(\tau_1, 0.4), (\tau_2, 0.5), (\tau_3, 0.4)\}, \{(\tau_1, 0.3), (\tau_2, 0.2), (\tau_3, 0.3)\}) \end{array} \right\}$$

or, simply, as

$$\lambda_A = \left\{ \begin{array}{l} (0, (\tau_1, 0.2, 0.3), (\tau_2, 0.3, 0.4), (\tau_3, 0.2, 0.3)), \\ (1, (\tau_1, 0.4, 0.3), (\tau_2, 0.5, 0.2), (\tau_3, 0.4, 0.3)) \end{array} \right\}.$$

Example 2.5. *Consider an initial universe set $S = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}$ having eight bungalows. Let $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}$ be a set of parameters, where $\varepsilon_i (i = 1, 2, 3, 4, 5, 6)$ stand for the parameters "expensive", "cheap", "traditional", "modern", "non-airconditioned" and "airconditioned", respectively. For a subset $A = \{\varepsilon_1, \varepsilon_3, \varepsilon_5\}$ of E, let's define an injective function $f : A \to E$ such that $f(\varepsilon_i) = \neg\varepsilon_i (i = 1, 3, 5)$. Here, for $i = 1, 3, 5$, the notion "$\neg\varepsilon_i$" means "not ε_i". Thus, we have*

$$\begin{aligned} f(\varepsilon_1) &= \neg\varepsilon_1 = \varepsilon_2, \\ f(\varepsilon_3) &= \neg\varepsilon_3 = \varepsilon_4, \\ f(\varepsilon_5) &= \neg\varepsilon_5 = \varepsilon_6. \end{aligned}$$

Now, let

$$\overset{+}{\lambda}(\varepsilon_1)(b) = \begin{cases} 0.3, & \text{if } b \in \{b_1, b_3, b_5, b_7\}, \\ 0.2, & \text{if } b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\lambda}(\varepsilon_3)(b) = \begin{cases} 0.4, & \text{if } b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \text{if } b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\lambda}(\varepsilon_5)(b) = \begin{cases} 0.5, & \text{if } b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \text{if } b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\lambda}(\varepsilon_2)(b) = \begin{cases} 0.6, & \text{if } b \in \{b_1, b_3, b_5, b_7\}, \\ 0.5, & \text{if } b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\lambda}(\varepsilon_4)(b) = \begin{cases} 0.5, & \text{if } b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \text{if } b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\lambda}(\varepsilon_6)(b) = \begin{cases} 0.4, & \text{if } b \in \{b_1, b_3, b_5, b_7\}, \\ 0.2, & \text{if } b \in \{b_2, b_4, b_6, b_8\}. \end{cases}$$

Then, λ_A is an FBS set over S.

Definition 2.6. *Let λ_A be an FBS set over S such that, for all $\varepsilon \in A$, we have $\overset{+}{\lambda}(\varepsilon) = S$, the universal fuzzy set of S and $\overset{-}{\lambda}(\varepsilon) = \phi$, the null fuzzy set of S. Then λ_A is called universal FBS set over S. We denote it by $S_A = (\overset{+}{S}, \overset{-}{S})$.*

Definition 2.7. *Let* λ_A *be an FBS set over* S *such that, for all* $\varepsilon \in A$*, we have* $\overset{+}{\lambda}(\varepsilon) = \phi$*, the null fuzzy set of* S *and* $\overset{-}{\lambda}(\varepsilon) = S$*, the universal fuzzy set of* S*. Then* λ_A *is called null FBS set over* S*. We denote it by* $\Phi_A = (\overset{+}{\Phi}, \overset{-}{\Phi})$*.*

Definition 2.8. *Let* λ_A *be an FBS set over* S*. The complement of* λ_A *, denoted as* $\overset{c}{\lambda}_A$*, is defined by*

$$\overset{c}{\lambda}_A = (\overset{+}{\lambda}, \overset{-}{\lambda}, A)^c = ((\overset{+}{\lambda})^c, (\overset{-}{\lambda})^c, A),$$

where

$$(\overset{+}{\lambda})^c(\varepsilon) = \overset{-}{\lambda}(\varepsilon), \quad (\overset{-}{\lambda})^c(\varepsilon) = \overset{+}{\lambda}(\varepsilon),$$

for all $\varepsilon \in A$*.*

Example 2.9. *Consider Example 2.4, where* $\lambda_A = (\overset{+}{\lambda}, \overset{-}{\lambda})$ *is an FBS set over* S*. Then, we can define* $\overset{c}{\lambda}_A$ *by virtue of Definition 2.8 as below:*

$$\overset{c}{\lambda}_A = \left\{ \begin{array}{l} (0, (\tau_1, 0.3, 0.2), (\tau_2, 0.4, 0.3), (\tau_3, 0.3, 0.2)), \\ (1, (\tau_1, 0.3, 0.4), (\tau_2, 0.2, 0.5), (\tau_3, 0.3, 0.4)) \end{array} \right\}.$$

Definition 2.10. *Let* λ_A*,* δ_A *be FBS sets over a universe* S*. We say that* λ_A *is an FBS subset of* δ_A*, denoted as* $\lambda_A \mathrel{\tilde{\preceq}} \delta_A$*, if and only if* $\overset{+}{\lambda} \leq \overset{+}{\delta}$ *and* $\overset{-}{\delta} \leq \overset{-}{\lambda}$ *if and only if* $\overset{+}{\lambda}(\varepsilon)(x) \leq \overset{+}{\delta}(\varepsilon)(x)$ *and* $\overset{-}{\delta}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(x)$*, for all* $x \in S$ *and* $\varepsilon \in A$*. Similarly,* λ_A *is said to be an FBS superset of* δ_A *if and only if* δ_A *is an FBS subset of* λ_A*.*

Definition 2.11. *For two FBS sets* λ_A *and* δ_A *over a universe* S*, we say that* λ_A *and* δ_A *are FBS equal if and only if* $\overset{+}{\lambda} = \overset{+}{\delta}$ *and* $\overset{-}{\lambda} = \overset{-}{\delta}$*. This relationship is denoted by* $\lambda_A = \delta_A$*. Further, we note that* $\overset{+}{\lambda} = \overset{+}{\delta}$ *and* $\overset{-}{\lambda} = \overset{-}{\delta}$ *if and only if* $\overset{+}{\lambda}(\varepsilon)(x) = \overset{+}{\delta}(\varepsilon)(x)$ *and* $\overset{-}{\lambda}(\varepsilon)(x) = \overset{-}{\delta}(\varepsilon)(x)$*, for all* $x \in S$ *and* $x \in A$*. Equivalently,* λ_A *and* δ_A *are FBS equal if and only if*

$$\lambda_A \mathrel{\tilde{\preceq}} \delta_A \quad \text{and} \quad \delta_A \mathrel{\tilde{\preceq}} \lambda_A.$$

Definition 2.12. *Let* λ_A *and* δ_A *be FBS sets over a common universe* S*. Then the FBS intersection of the two FBS sets is an FBS set* γ_A *over* S *defined by*

$$\overset{+}{\gamma}(\varepsilon) \quad = \quad \overset{+}{\lambda}(\varepsilon) \wedge \overset{+}{\delta}(\varepsilon),$$

and

$$\overset{-}{\gamma}(\varepsilon) \quad = \quad \overset{-}{\lambda}(\varepsilon) \vee \overset{-}{\delta}(\varepsilon),$$

for all $\varepsilon \in A$*. We denote* $\gamma_A = \lambda_A \mathbin{\tilde{\cap}} \delta_A$*, where* $\overset{+}{\gamma} = \overset{+}{\lambda} \wedge \overset{+}{\delta}$ *and* $\overset{-}{\gamma} = \overset{-}{\lambda} \vee \overset{-}{\delta}$*.*

Here, the symbols $\wedge$ and $\vee$ respectively represent the operations of fuzzy intersection and fuzzy union of two fuzzy sets. Further, for all $x \in S$ and $\varepsilon \in A$, we note that

$$\overset{+}{\gamma}(\varepsilon)(x) \quad = \quad \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)\},$$

and

$$\overset{-}{\gamma}(\varepsilon)(x) \quad = \quad \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\delta}(\varepsilon)(x)\}.$$

Example 2.13. *Consider Example 2.5, where $\lambda_A = (\overset{+}{\lambda}, \overset{-}{\lambda})$ is an FBS set over S. Let $\delta_A = (\overset{+}{\delta}, \overset{-}{\delta})$ be another FBS set over the same universe S defined as follows:*

$$\overset{+}{\delta}(\varepsilon_1)(b) \quad = \quad \begin{cases} 0.5, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\delta}(\varepsilon_3)(b) \quad = \quad \begin{cases} 0.2, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.1, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\delta}(\varepsilon_5)(b) \quad = \quad \begin{cases} 0.3, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\delta}(\varepsilon_2)(b) \quad = \quad \begin{cases} 0.4, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\delta}(\varepsilon_4)(b) \quad = \quad \begin{cases} 0.6, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

and

$$\overset{-}{\delta}(\varepsilon_6)(b) \quad = \quad \begin{cases} 0.5, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}. \end{cases}$$

Let $\gamma_A = \lambda_A \,\widetilde{\cap}\, \delta_A$. Then, γ_A is defined as follows:

$$\overset{+}{\gamma}(\varepsilon_1)(b) \quad = \quad \begin{cases} 0.3, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.2, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\gamma}(\varepsilon_3)(b) \quad = \quad \begin{cases} 0.2, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.1, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\gamma}(\varepsilon_5)(b) \quad = \quad \begin{cases} 0.3, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\gamma}(\varepsilon_2)(b) \quad = \quad \begin{cases} 0.6, & \textit{if} \;\; b \in \{b_1, b_3, b_5, b_7\}, \\ 0.5, & \textit{if} \;\; b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\gamma}(\varepsilon_4)(b) \quad = \quad \begin{cases} 0.6, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

and

$$\overset{-}{\gamma}(\varepsilon_6)(b) \quad = \quad \begin{cases} 0.5, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}. \end{cases}$$

Definition 2.14. *Let λ_A and δ_A be FBS sets over a common universe S. The FBS union of the two FBS sets is an FBS set γ_A over S defined by*

$$\overset{+}{\gamma}(\varepsilon) \quad = \quad \overset{+}{\lambda}(\varepsilon) \vee \overset{+}{\delta}(\varepsilon),$$

and

$$\overset{-}{\gamma}(\varepsilon) \quad = \quad \overset{-}{\lambda}(\varepsilon) \wedge \overset{-}{\delta}(\varepsilon),$$

for all $\varepsilon \in A$. We denote $\gamma_A = \lambda_A \mathbin{\widetilde{\cup}} \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \vee \overset{+}{\delta}$ and $\overset{-}{\gamma} = \overset{-}{\lambda} \wedge \overset{-}{\delta}$.

Here, the symbols $\vee$ and $\wedge$ respectively represent the operations of fuzzy union and fuzzy intersection of two fuzzy sets. Further, for all $x \in S$ and $\varepsilon \in A$, we note that

$$\overset{+}{\gamma}(\varepsilon)(x) \quad = \quad \max\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)\},$$

and

$$\overset{-}{\gamma}(\varepsilon)(x) \quad = \quad \min\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\delta}(\varepsilon)(x)\}.$$

Example 2.15. *Consider the FBS sets λ_A and δ_A over S defined in Examples 2.5 and 2.13, respectively. Let $\gamma_A = \lambda_A \mathbin{\widetilde{\cup}} \delta_A$. Then, we have*

$$\overset{+}{\gamma}(\varepsilon_1)(b) \quad = \quad \begin{cases} 0.5, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\gamma}(\varepsilon_3)(b) \quad = \quad \begin{cases} 0.4, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{+}{\gamma}(\varepsilon_5)(b) \quad = \quad \begin{cases} 0.5, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.4, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\gamma}(\varepsilon_2)(b) \quad = \quad \begin{cases} 0.4, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

$$\overset{-}{\gamma}(\varepsilon_4)(b) \quad = \quad \begin{cases} 0.5, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.3, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}, \end{cases}$$

and

$$\overset{-}{\gamma}(\varepsilon_6)(b) \quad = \quad \begin{cases} 0.4, & \text{if } \ b \in \{b_1, b_3, b_5, b_7\}, \\ 0.2, & \text{if } \ b \in \{b_2, b_4, b_6, b_8\}. \end{cases}$$

Definition 2.16. *Let λ_A and δ_A be FBS sets over a common universe S. Define an injective function $g : A \times A \to f(A) \times f(A)$ such that $g(\alpha,\beta) = (f(\alpha), f(\beta))$ for all (α,β) in $A\times A$. Then the And-product of the two FBS sets is an FBS set γ_A over S, where $\Lambda = A\times A$, which is defined in terms of its fuzzy approximate functions as follows:*

$$\overset{+}{\gamma}(\alpha,\beta) = \overset{+}{\lambda}(\alpha) \wedge \overset{+}{\delta}(\beta),$$

and

$$\overset{-}{\gamma}(\alpha,\beta) = \overset{-}{\lambda}(\alpha) \vee \overset{-}{\delta}(\beta),$$

for all $(\alpha,\beta) \in \Lambda$. We denote $\gamma_A = \lambda_A \mathbin{\widetilde{\wedge}} \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \wedge \overset{+}{\delta}$ and $\overset{-}{\gamma} = \overset{-}{\lambda} \vee \overset{-}{\delta}$. Further, for all $x \in S$ and $(\alpha,\beta) \in \Lambda$, we note that

$$\overset{+}{\gamma}(\alpha,\beta)(x) = \min\{\overset{+}{\lambda}(\alpha)(x), \overset{+}{\delta}(\beta)(x)\},$$

and

$$\overset{-}{\gamma}(\alpha,\beta)(x) = \max\{\overset{-}{\lambda}(\alpha)(x), \overset{-}{\delta}(\beta)(x)\}.$$

.

Definition 2.17. *Let λ_A and δ_A be FBS sets over a common universe S. Define an injective function $g : A\times A \to f(A)\times f(A)$ such that $g(\alpha,\beta) = (f(\alpha), f(\beta))$ for all $(\alpha,\beta)\in A\times A$. Then the Or-product of the two FBS sets is an FBS set γ_A over S, where $\Lambda = A \times A$, which is defined in terms of its fuzzy approximate mappings as follows:*

$$\overset{+}{\gamma}(\alpha,\beta) = \overset{+}{\lambda}(\alpha) \vee \overset{+}{\delta}(\beta),$$

and

$$\overset{-}{\gamma}(\alpha,\beta) = \overset{-}{\lambda}(\alpha) \wedge \overset{-}{\delta}(\beta),$$

for all $(\alpha,\beta) \in \Lambda$. We denote $\gamma_\Lambda = \lambda_A \mathbin{\widetilde{\vee}} \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \vee \overset{+}{\delta}$, $\overset{-}{\gamma} = \overset{-}{\lambda} \wedge \overset{-}{\delta}$. Further, for all $x \in S$ and $(\alpha,\beta) \in \Lambda$, we note that

$$\overset{+}{\gamma}(\alpha,\beta)(x) = \max\{\overset{+}{\lambda}(\alpha)(x), \overset{+}{\delta}(\beta)(x)\},$$

and

$$\overset{-}{\gamma}(\alpha,\beta)(x) = \min\{\overset{-}{\lambda}(\alpha)(x), \overset{-}{\delta}(\beta)(x)\}.$$

3. THE ALGEBRAIC PROPERTIES OF FUZZY BIPOLAR SOFT SETS UNDER VARIOUS OPERATIONS

In this section, we discuss the algebraic properties of FBS sets over a common initial universe set S, and hence develop certain propositions without providing their proofs as they are straightforward. It is observed that the collection of all FBS sets over a common universe S forms a bounded distributive lattice.

Proposition 3.1. *Let λ_A and δ_A be FBS sets over a common universe S. Then the following idempotent laws hold:*

(i) $\lambda_A \,\widetilde{\cup}\, \lambda_A = \lambda_A$.
(ii) $\lambda_A \,\widetilde{\cap}\, \lambda_A = \lambda_A$.

Proposition 3.2. *Let λ_A and δ_A be FBS sets over a common universe S. Then the following commutative laws hold:*

(i) $\lambda_A \,\widetilde{\cup}\, \delta_A = \delta_A \,\widetilde{\cup}\, \lambda_A$.
(ii) $\lambda_A \,\widetilde{\cap}\, \delta_A = \delta_A \,\widetilde{\cap}\, \lambda_A$.

Proposition 3.3. *Let λ_A, δ_A and γ_A be FBS sets over a common universe S. Then the following associative laws hold:*

(i) $\lambda_A \,\widetilde{\cup}\, (\delta_A \,\widetilde{\cup}\, \gamma_A) = (\lambda_A \,\widetilde{\cup}\, \delta_A) \,\widetilde{\cup}\, \gamma_A$.
(ii) $\lambda_A \,\widetilde{\cap}\, (\delta_A \,\widetilde{\cap}\, \gamma_A) = (\lambda_A \,\widetilde{\cap}\, \delta_A) \,\widetilde{\cap}\, \gamma_A$.

Proposition 3.4. *Let λ_A and δ_A be FBS sets over a common universe S. Then the following absorption laws hold:*

(i) $\lambda_A \,\widetilde{\cap}\, (\lambda_A \,\widetilde{\cup}\, \delta_A) = \lambda_A$.
(ii) $\lambda_A \,\widetilde{\cup}\, (\lambda_A \,\widetilde{\cap}\, \delta_A) = \lambda_A$.

Proposition 3.5. *Let λ_A, δ_A and γ_A be FBS sets over a common universe S. Then the following distributive laws hold:*

(i) $\lambda_A \,\widetilde{\cap}\, (\delta_A \,\widetilde{\cup}\, \gamma_A) = (\lambda_A \,\widetilde{\cap}\, \delta_A) \,\widetilde{\cup}\, (\lambda_A \,\widetilde{\cap}\, \gamma_A)$.
(ii) $\lambda_A \,\widetilde{\cup}\, (\delta_A \,\widetilde{\cap}\, \gamma_A) = (\lambda_A \,\widetilde{\cup}\, \delta_A) \,\widetilde{\cap}\, (\lambda_A \,\widetilde{\cup}\, \gamma_A)$.

Proposition 3.6. *Let λ_A be an FBS set over a universe S. Then the following identity laws hold:*

(i) $\lambda_A \,\widetilde{\cup}\, \varPhi_A = \lambda_A$.
(ii) $\lambda_A \,\widetilde{\cup}\, S_A = S_A$.
(iii) $\lambda_A \,\widetilde{\cap}\, \varPhi_A = \varPhi_A$.
(iv) $\lambda_A \,\widetilde{\cap}\, S_A = \lambda_A$.

Proposition 3.7. *Let λ_A and δ_A be FBS sets over a common universe S. Then the following De Morgan' s laws hold:*

(i) $(\lambda_A \,\widetilde{\cup}\, \delta_A)^c = \overset{c}{\lambda}_A \,\widetilde{\cap}\, \overset{c}{\delta}_A$.
(ii) $(\lambda_A \,\widetilde{\cap}\, \delta_A)^c = \overset{c}{\lambda}_A \,\widetilde{\cup}\, \overset{c}{\delta}_A$.

Proposition 3.8. *Let λ_A be an FBS set over a universe S. Then the following laws of complementation hold:*

(i) $(\overset{c}{\lambda}_A)^c = \lambda_A$.
(ii) $\overset{c}{S}_A = \varPhi_A$.
(iii) $\overset{c}{\varPhi}_A = S_A$.

Consider an initial universe set S and a set of parameters E. Let $\mathcal{FB}(S, A)$ denotes the collection of all FBS sets over S with a fixed set of parameters A, where $A \subseteq E$. It is easy to see that the collection $\mathcal{FB}(S, A)$ is partially ordered by inclusion, where the FBS sets $\varPhi_A$ and S_A are the least and the greatest elements, respectively. So, by virtue of Propositions 3.1–3.5, we establish the following result:

Proposition 3.9. $(\mathcal{FB}(S,A), \widetilde{\cap}, \widetilde{\cup})$ *is a bounded distributive lattice with the least element* Φ_A *and the greatest element* S_A.

Note. In what follows, S always represents an ordered semigroup.

4. Fuzzy bipolar soft ordered semigroups

In this section, we define and study the notion of FBS ordered semigroup over S. It is proved that the FBS intersection (resp., And-product) of two FBS ordered semigroups over S is an FBS ordered semigroup over S. Moreover, we show by an appropriate example that the FBS union (resp., Or-product) of two FBS ordered semigroups over S is not generally an FBS ordered semigroup over S.

Now, we define an FBS ordered semigroup over S as follows:

Definition 4.1. *An FBS set* λ_A *over* S *is called an FBS ordered semigroup over* S *if and only if, for all* $\varepsilon \in A$ *and* $x, y \in S$, *the following assertions hold:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \min(\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y))$.

(ii) $\overset{-}{\lambda}(\varepsilon)(xy) \leq \max(\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y))$.

(iii) $x \leq y \Rightarrow \overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$.

Theorem 4.2. *Let* λ_A *and* δ_A *be FBS ordered semigroups over* S. *Then, so is* $\lambda_A \widetilde{\cap} \delta_A$.

Proof. Let $\gamma_A = \lambda_A \widetilde{\cap} \delta_A$. Further, let $\varepsilon \in A$ and $x, y \in S$. Then, since λ_A and δ_A are FBS ordered semigroups over S, we have

$$\begin{aligned}
\overset{+}{\gamma}(\varepsilon)(xy) &= \min(\overset{+}{\lambda}(\varepsilon)(xy), \overset{+}{\delta}(\varepsilon)(xy)) \\
&\geq \min(\min(\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)), \min(\overset{+}{\delta}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(y))) \\
&= \min(\min(\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)), \min(\overset{+}{\lambda}(\varepsilon)(y), \overset{+}{\delta}(\varepsilon)(y))) \\
&= \min(\overset{+}{\gamma}(\varepsilon)(x), \overset{+}{\gamma}(\varepsilon)(y)).
\end{aligned}$$

Similarly, we obtain

$$\overset{-}{\gamma}(\varepsilon)(xy) \leq \max(\overset{-}{\gamma}(\varepsilon)(x), \overset{-}{\gamma}(\varepsilon)(y)).$$

Now, let $a, b \in S$ such that $a \leq b$. Then, since λ_A and δ_A are FBS ordered semigroups over S, we have

$$\begin{aligned}
\overset{+}{\gamma}(\varepsilon)(a) &= \min(\overset{+}{\lambda}(\varepsilon)(a), \overset{+}{\delta}(\varepsilon)(a)) \\
&\geq \min(\overset{+}{\lambda}(\varepsilon)(b), \overset{+}{\delta}(\varepsilon)(b)) \\
&= \overset{+}{\gamma}(\varepsilon)(b),
\end{aligned}$$

and, similarly, we get $\overset{-}{\gamma}(\varepsilon)(a) \leq \overset{-}{\gamma}(\varepsilon)(b)$. Therefore, γ_A is an FBS ordered semigroup over S. This completes the proof. □

It is evident from Theorem 4.2 that the FBS intersection of two FBS ordered semigroups over S is always an FBS ordered semigroup over S. However, the FBS union of two FBS ordered semigroups over S is not generally an FBS ordered semigroup over S. We justify the claim with the help of the following example.

Example 4.3. *Consider the ordered semigroup $S = \{a, b, c, d\}$ with the multiplication "$\cdot$" and the order relation "$\leq$" given below:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	d	d
c	a	d	c	d
d	a	d	d	d

$$\leq := \{(a,a), (a,b), (a,c), (a,d), (b,b), (c,c), (d,d)\}.$$

Let $A = \{\varepsilon_1, \varepsilon_2\} \subset E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and $f : A \to E$ be an identity function. Define the FBS ordered semigroups λ_A and δ_A over S as follows:

$$\overset{+}{\lambda}(\varepsilon_1) = \{(a, 0.5), (b, 0.2), (c, 0.4), (d, 0.3)\},$$

$$\overset{+}{\lambda}(\varepsilon_2) = \{(a, 0.5), (b, 0.3), (c, 0.5), (d, 0.4)\},$$

$$\overset{-}{\lambda}(\varepsilon_1) = \{(a, 0.2), (b, 0.4), (c, 0.2), (d, 0.3)\},$$

$$\overset{-}{\lambda}(\varepsilon_2) = \{(a, 0.2), (b, 0.5), (c, 0.3), (d, 0.4)\},$$

and

$$\overset{+}{\delta}(\varepsilon_1) = \{(a, 0.4), (b, 0.4), (c, 0.2), (d, 0.3)\},$$

$$\overset{+}{\delta}(\varepsilon_2) = \{(a, 0.5), (b, 0.5), (c, 0.3), (d, 0.4)\},$$

$$\overset{-}{\delta}(\varepsilon_1) = \{(a, 0.3), (b, 0.4), (c, 0.6), (d, 0.5)\},$$

$$\overset{-}{\delta}(\varepsilon_2) = \{(a, 0.2), (b, 0.3), (c, 0.5), (d, 0.4)\}.$$

Let $\gamma_A = \lambda_A \mathbin{\widetilde{\cup}} \delta_A$. Then, γ_A is defined by

$$\overset{+}{\gamma}(\varepsilon_1) = \{(a, 0.5), (b, 0.4), (c, 0.4), (d, 0.3)\},$$

$$\overset{+}{\gamma}(\varepsilon_2) = \{(a, 0.5), (b, 0.5), (c, 0.5), (d, 0.4)\},$$

$$\overset{-}{\gamma}(\varepsilon_1) = \{(a, 0.2), (b, 0.4), (c, 0.2), (d, 0.3)\},$$

$$\overset{-}{\gamma}(\varepsilon_2) = \{(a, 0.2), (b, 0.3), (c, 0.3), (d, 0.4)\}.$$

Here, we note that

$$\overset{+}{\gamma}(\varepsilon_1)(bc) = \overset{+}{\gamma}(\varepsilon_1)(d) = 0.3 \ngeq 0.4 = \min\{\overset{+}{\gamma}(\varepsilon_1)(b), \overset{+}{\gamma}(\varepsilon_1)(c)\}$$

and, similarly,

$$\overset{-}{\gamma}(\varepsilon_1)(bc) = \overset{-}{\gamma}(\varepsilon_1)(d) = 0.3 \nleq 0.4 = \max\{\overset{-}{\gamma}(\varepsilon_1)(b), \overset{-}{\gamma}(\varepsilon_1)(c)\}.$$

Therefore, γ_A is not an FBS ordered semigroup over S.

Theorem 4.4. *Let λ_A and δ_A be FBS ordered semigroups over S. Then, so is $\lambda_A \,\tilde{\wedge}\, \delta_A$.*

Proof. Let $\gamma_\Lambda = \lambda_A \,\tilde{\wedge}\, \delta_A$, where $\Lambda = A \times A$. Further, let $(\alpha, \beta) \in \Lambda$ and $x, y \in S$. Then, since λ_A and δ_A are FBS ordered semigroups over S, we have

$$\begin{aligned}
\overset{-}{\gamma}(\alpha,\beta)(xy) &= \max(\overset{-}{\lambda}(\alpha)(xy), \overset{-}{\delta}(\beta)(xy)) \\
&\leq \max(\max(\overset{-}{\lambda}(\alpha)(x), \overset{-}{\lambda}(\alpha)(y)), \max(\overset{-}{\delta}(\beta)(x), \overset{-}{\delta}(\beta)(y))) \\
&= \max(\max(\overset{-}{\lambda}(\alpha)(x), \overset{-}{\delta}(\beta)(x)), \max(\overset{-}{\lambda}(\alpha)(y), \overset{-}{\delta}(\beta)(y))) \\
&= \max(\overset{-}{\gamma}(\alpha,\beta)(x), \overset{-}{\gamma}(\alpha,\beta)(y)).
\end{aligned}$$

and, similarly,

$$\overset{+}{\gamma}(\alpha,\beta)(xy) \geq \min(\overset{+}{\gamma}(\alpha,\beta)(x), \overset{+}{\gamma}(\alpha,\beta)(y)).$$

Now, let $x, y \in S$ such that $x \leq y$. Then, since λ_A and δ_A are FBS ordered semigroups over S, we have

$$\begin{aligned}
\overset{-}{\gamma}(\alpha,\beta)(x) &= \max(\overset{-}{\lambda}(\alpha)(x), \overset{-}{\delta}(\beta)(x)) \\
&\leq \max(\overset{-}{\lambda}(\alpha)(y), \overset{-}{\delta}(\beta)(y)) \\
&= \overset{-}{\gamma}(\alpha,\beta)(y),
\end{aligned}$$

and, similarly,

$$\overset{+}{\gamma}(\alpha,\beta)(x) \geq \overset{+}{\gamma}(\alpha,\beta)(y).$$

Therefore, γ_Λ is an FBS ordered semigroup over S. □

It is obvious from Theorem 4.4 that the And-product of two FBS ordered semigroups over S is always an FBS ordered semigroup over S . However, the Or-product of two FBS ordered semigroups over S is not necessarily an FBS ordered semigroup over S . To justify the claim, we have the following example:

Example 4.5. *Let's reconsider Example 4.3, where λ_A and δ_A are FBS ordered semigroups over S. Let $\gamma_\Lambda = \lambda_A \,\tilde{\vee}\, \delta_A$, where $\Lambda = A \times A$. Then, we note that*

$$\begin{aligned}
\overset{+}{\gamma}(\varepsilon_1,\varepsilon_1)(bc) &= \max\{\overset{+}{\lambda}(\varepsilon_1)(d), \overset{+}{\delta}(\varepsilon_1)(d)\} \\
&= \max\{0.3, 0.3\} = 0.3,
\end{aligned}$$

$$\begin{aligned}\overset{+}{\gamma}(\varepsilon_1,\varepsilon_1)(b) &= \max\{\overset{+}{\lambda}(\varepsilon_1)(b),\overset{+}{\delta}(\varepsilon_1)(b)\}\\ &= \max\{0.2,0.4\}=0.4,\end{aligned}$$

and

$$\begin{aligned}\overset{+}{\gamma}(\varepsilon_1,\varepsilon_1)(c) &= \max\{\overset{+}{\lambda}(\varepsilon_1)(c),\overset{+}{\delta}(\varepsilon_1)(c)\}\\ &= \max\{0.4,0.2\}=0.4.\end{aligned}$$

So, we find that

$$\begin{aligned}\overset{+}{\gamma}(\varepsilon_1,\varepsilon_1)(bc) &= 0.3\\ &\ngeq 0.4\\ &= \min\{(\overset{+}{\gamma}(\varepsilon_1,\varepsilon_1)(b),\overset{+}{\gamma}(\varepsilon_1,\varepsilon_1)(c)\},\end{aligned}$$

Therefore, γ_A is not an FBS ordered semigroup over S.

4.6. **The (r,t)-level subset of a fuzzy bipolar soft set.** In this subsection, we define the (r,t)-level subset of an FBS set λ_A over S and characterize an FBS ordered semigroup λ_A over S by means of its (r,t)-level subset.

Definition 4.7. *Let λ_A be an FBS set over S. For $\varepsilon\in A$ and $r\in(0,1]$, $t\in[0,1)$, we denote by $\overset{(r,t)}{\lambda_A}(\varepsilon)$ a subset of S defined as*

$$\overset{(r,t)}{\lambda_A}(\varepsilon)=\{x\in S:\overset{+}{\lambda}(\varepsilon)(x)\geq r,\ \ \overset{-}{\lambda}(\varepsilon)(x)\leq t\}.$$

For any $\varepsilon\in A$, the subset $\overset{(r,t)}{\lambda_A}(\varepsilon)$ of S is called an (r,t)-level subset of λ_A.

Proposition 4.8. *Let λ_A be an FBS set over S. Then it is an FBS ordered semigroup over S if and only if $\overset{(r,t)}{\lambda_A}(\varepsilon)\neq\phi$ is a subsemigroup of S for all $r\in(0,1]$, $t\in[0,1)$ and $\varepsilon\in A$.*

Proof. Let λ_A be an FBS set over S and $\overset{(r,t)}{\lambda_A}(\varepsilon)\neq\phi$ an (r,t)-level subset of λ_A. First, assume that λ_A is an FBS ordered semigroup over S. We need to prove that $\overset{(r,t)}{\lambda_A}(\varepsilon)$ is a subsemigroup of S. For this, let $x,y\in\overset{(r,t)}{\lambda_A}(\varepsilon)$. Then, since λ_A is an FBS ordered semigroup over S, we have

$$\begin{aligned}\overset{+}{\lambda}(\varepsilon)(xy) &\geq \min\{\overset{+}{\lambda}(\varepsilon)(x),\overset{+}{\lambda}(\varepsilon)(y)\}\\ &= \min\{r,r\}=r,\end{aligned}$$

and

$$\begin{aligned}\overset{-}{\lambda}(\varepsilon)(xy) &\leq \max\{\overset{-}{\lambda}(\varepsilon)(x),\overset{-}{\lambda}(\varepsilon)(y)\}\\ &= \max\{t,t\}=t.\end{aligned}$$

This implies that $xy \in \overset{(r,t)}{\lambda_A}(\varepsilon)$. Now, let $a, b \in S$ such that $a \leq b$, and $b \in \overset{(r,t)}{\lambda_A}(\varepsilon)$. Then, since λ_A is an FBS ordered semigroup over S, we have

$$\overset{+}{\lambda}(\varepsilon)(a) \geq \overset{+}{\lambda}(\varepsilon)(b) \geq r,$$

$$\overset{-}{\lambda}(\varepsilon)(a) \leq \overset{-}{\lambda}(\varepsilon)(b) \leq t,$$

which implies that $a \in \overset{(r,t)}{\lambda_A}(\varepsilon)$. Thus $\overset{(r,t)}{\lambda_A}(\varepsilon)$ is a subsemigroup of S.

Conversely, assume that $\overset{(r,t)}{\lambda_A}(\varepsilon)$ be a subsemigroup of S. To prove that λ_A is an FBS ordered semigroup over S, first we need to show that

$$\overset{+}{\lambda}(\varepsilon)(xy) \geq \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\},$$

$$\overset{-}{\lambda}(\varepsilon)(xy) \leq \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\},$$

for all $x, y \in S$. On the contrary, let

$$\overset{+}{\lambda}(\varepsilon)(xy) < \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\},$$

$$\overset{-}{\lambda}(\varepsilon)(xy) > \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\},$$

for some elements $x, y \in S$. This implies that

$$\overset{+}{\lambda}(\varepsilon)(xy) < \overset{+}{\lambda}(\varepsilon)(x), \quad \overset{+}{\lambda}(\varepsilon)(xy) < \overset{+}{\lambda}(\varepsilon)(y),$$

and

$$\overset{-}{\lambda}(\varepsilon)(xy) > \overset{-}{\lambda}(\varepsilon)(x), \quad \overset{-}{\lambda}(\varepsilon)(xy) > \overset{-}{\lambda}(\varepsilon)(y).$$

Then there exist some real numbers $r_\circ \in (0, 1]$, $t_\circ \in [0, 1)$ such that

$$\overset{+}{\lambda}(\varepsilon)(xy) < r_\circ < \overset{+}{\lambda}(\varepsilon)(x), \quad \overset{+}{\lambda}(\varepsilon)(xy) < r_\circ < \overset{+}{\lambda}(\varepsilon)(y),$$

and

$$\overset{-}{\lambda}(\varepsilon)(xy) > t_\circ > \overset{-}{\lambda}(\varepsilon)(x), \quad \overset{-}{\lambda}(\varepsilon)(xy) > t_\circ > \overset{-}{\lambda}(\varepsilon)(y).$$

So, it follows that

$$\overset{+}{\lambda}(\varepsilon)(x) > r_\circ, \quad \overset{-}{\lambda}(\varepsilon)(x) < t_\circ,$$

$$\overset{+}{\lambda}(\varepsilon)(y) > r_\circ, \quad \overset{-}{\lambda}(\varepsilon)(y) < t_\circ.$$

This implies that $x, y \in \overset{(r_\circ,t_\circ)}{\lambda_A}(\varepsilon)$. Since $\overset{(r_\circ,t_\circ)}{\lambda_A}(\varepsilon)$ is a subsemigroup of S, thus, it follows that $xy \in \overset{(r_\circ,t_\circ)}{\lambda_A}(\varepsilon)$. Then, we have

$$\overset{+}{\lambda}(\varepsilon)(xy) \geq r_\circ, \quad \overset{-}{\lambda}(\varepsilon)(xy) \leq t_\circ,$$

which is a contradiction. Thus, we conclude that

$$\overset{+}{\lambda}(\varepsilon)(xy) \geq \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\},$$

and

$$\overset{-}{\lambda}(\varepsilon)(xy) \leq \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\},$$

for all $x, y \in S$. Next, let $a, b \in S$ such that $a \leq b$. Then, we need to show that

$$\overset{+}{\lambda}(\varepsilon)(a) \geq \overset{+}{\lambda}(\varepsilon)(b), \quad \overset{-}{\lambda}(\varepsilon)(a) \leq \overset{-}{\lambda}(\varepsilon)(b).$$

On the contrary, let

$$\overset{+}{\lambda}(\varepsilon)(a) < \overset{+}{\lambda}(\varepsilon)(b), \quad \overset{-}{\lambda}(\varepsilon)(a) > \overset{-}{\lambda}(\varepsilon)(b).$$

Then, there exist $r_1 \in (0, 1]$ and $t_1 \in [0, 1)$ such that

$$\overset{+}{\lambda}(\varepsilon)(a) < r_1 \leq \overset{+}{\lambda}(\varepsilon)(b),$$

$$\overset{-}{\lambda}(\varepsilon)(a) > t_1 \geq \overset{-}{\lambda}(\varepsilon)(b).$$

Then $b \in \overset{(r_1,t_1)}{\lambda_A}(\varepsilon)$ but $a \notin \overset{(r_1,t_1)}{\lambda_A}(\varepsilon)$. This is a contradiction because $\overset{(r_1,t_1)}{\lambda_A}(\varepsilon)$ is a subsemigroup of S. Thus, we conclude that

$$\overset{+}{\lambda}(\varepsilon)(a) \geq \overset{+}{\lambda}(\varepsilon)(b), \quad \overset{-}{\lambda}(\varepsilon)(a) \leq \overset{-}{\lambda}(\varepsilon)(b).$$

Therefore, λ_A is an FBS ordered semigroup over S. □

4.9. **The Cartesian product of fuzzy bipolar soft sets.** Here, we define and study the Cartesian product of two FBS sets over an ordered semigroup S. More than else, it is proved that the Cartesian product of two FBS ordered semigroups over S is an FBS ordered semigroup over $S \times S$.

Definition 4.10. *Let λ_A and δ_A be FBS sets over S and $g : A \times A \to f(A) \times f(A)$ be an injective mapping defined by $g(\alpha, \beta) = (f(\alpha), f(\beta))$. Then, the Cartesian product of λ_A and δ_A is an FBS set γ_Λ over $S \times S$, where $\Lambda = A \times A$, which is defined in terms of its fuzzy approximate functions as follows:*

$$\overset{+}{\gamma}(\alpha, \beta)(x, y) = \min\{\overset{+}{\lambda}(\alpha)(x), \overset{+}{\delta}(\beta)(y)\},$$

and

$$\overset{-}{\gamma}(\alpha, \beta)(x, y) = \max\{\overset{-}{\lambda}(\alpha)(x), \overset{-}{\delta}(\beta)(y)\},$$

for all $(\alpha, \beta) \in \Lambda$ and $(x, y) \in S \times S$. We denote $\gamma_\Lambda = \lambda_A \times \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \times \overset{+}{\delta}$ and $\overset{-}{\gamma} = \overset{-}{\lambda} \times \overset{-}{\delta}$.

Theorem 4.11. *Let λ_A and δ_A be FBS ordered semigroups over S. Then $\lambda_A \times \delta_A$ is an FBS ordered semigroup over $S \times S$.*

Proof. Let $\gamma_\Lambda = \lambda_A \times \delta_A$, where $\Lambda = A \times A$. Suppose $(\alpha, \beta) \in \Lambda$ and $(a, b), (c, d)$ be any elements in $S \times S$. If $(a, b) \leq (c, d)$, then $a \leq c$ and $b \leq d$. Thus, for all $\alpha, \beta \in A$, we have

$$\overset{+}{\lambda}(\alpha)(a) \quad \geq \quad \overset{+}{\lambda}(\alpha)(c), \quad \overset{-}{\lambda}(\alpha)(a) \leq \overset{-}{\lambda}(\alpha)(c),$$

and

$$\overset{+}{\delta}(\beta)(b) \quad \geq \quad \overset{+}{\delta}(\beta)(d), \quad \overset{-}{\delta}(\beta)(b) \leq \overset{-}{\delta}(\beta)(d).$$

Now,

$$\begin{aligned} \overset{+}{\gamma}(\alpha, \beta)(a, b) &= \min\{\overset{+}{\lambda}(\alpha)(a), \overset{+}{\delta}(\beta)(b)\} \\ &\geq \min\{\overset{+}{\lambda}(\alpha)(c), \overset{+}{\delta}(\beta)(d)\} \\ &= \overset{+}{\gamma}(\alpha, \beta)((c, d)), \end{aligned}$$

and, similarly,

$$\overset{-}{\gamma}(\alpha, \beta)((a, b)) \quad \leq \quad \overset{-}{\gamma}(\alpha, \beta)((c, d)).$$

Further, we have

$$\begin{aligned} \overset{-}{\gamma}(\alpha, \beta)((a, b)(c, d)) &= \overset{-}{\gamma}(\alpha, \beta)((ac, bd)) \\ &= \max[\overset{-}{\lambda}(\alpha)(ac), \overset{-}{\delta}(\beta)(bd)] \\ &\leq \max[\max(\overset{-}{\lambda}(\alpha)(a), \overset{-}{\lambda}(\alpha)(c)), \max(\overset{-}{\delta}(\beta)(b), \overset{+}{\delta}(\beta)(d))] \\ &= \max[\max(\overset{-}{\lambda}(\alpha)(a), \overset{-}{\delta}(\beta)(b)), \max(\overset{-}{\lambda}(\alpha)(c), \overset{-}{\delta}(\beta)(d))] \\ &= \max[\overset{-}{\gamma}(\alpha, \beta)((a, b)), \overset{-}{\gamma}(\alpha, \beta)((c, d))], \end{aligned}$$

and, similarly,

$$\overset{+}{\gamma}(\alpha, \beta)((a, b)(c, d)) \quad \geq \quad \min[\overset{+}{\gamma}(\alpha, \beta)((a, b)), \overset{+}{\gamma}(\alpha, \beta)((c, d))].$$

Therefore, γ_Λ is an FBS ordered semigroup over $S \times S$. □

Definition 4.12. *Let λ_A and δ_A be FBS sets over S. For $(\alpha, \beta) \in A \times A$ and $r \in (0, 1]$, $t \in [0, 1)$, we denote by $(\lambda_A \times \delta_A)^{(r,t)}(\alpha, \beta)$ the subset of $S \times S$ which is defined as follows:*

$$(\lambda_A \times \delta_A)^{(r,t)}(\alpha, \beta) = \{(x, y) \in S \times S : \overset{+}{\lambda}(\alpha)(x), \overset{+}{\delta}(\beta)(x) \geq r, \ \overset{-}{\lambda}(\alpha)(x), \overset{-}{\delta}(\beta)(x) \leq t\}.$$

Equivalently, we have

$$(\lambda_A \times \delta_A)^{(r,t)}(\alpha, \beta) \quad = \quad \lambda_A^{(r,t)}(\alpha) \times \lambda_A^{(r,t)}(\beta) \ .$$

For any $(\alpha, \beta) \in A \times A$, the subset $(\lambda_A \times \delta_A)^{(r,t)}(\alpha, \beta)$ of $S \times S$ is called an (r, t)-level subset of $\lambda_A \times \delta_A$.

Proposition 4.13. *Let λ_A and δ_A be FBS ordered semigroups over S. Then $\overset{(r,t)}{(\lambda_A \times \delta_A)}(\alpha, \beta)$ $(\neq \phi)$ is a subsemigroup of $S \times S$ for all $r \in (0, 1]$, $t \in [0, 1)$ and $(\alpha, \beta) \in A \times A$.*

Proof. Let $\gamma_\Lambda = \lambda_A \times \delta_A$, where $\Lambda = A \times A$. Then, by Theorem 4.11, we see that γ_Λ is an FBS ordered semigroup over $S \times S$. We need to prove that $\overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)(\neq \phi)$ is a subsemigroup of $S \times S$, for all $r \in (0, 1]$, $t \in [0, 1)$ and $(\alpha, \beta) \in \Lambda$. For this, let $(a, b), (c, d) \in \overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$. Then, since γ_Λ is an FBS ordered semigroup over $S \times S$, we have

$$\begin{aligned} \overset{+}{\gamma}(\alpha, \beta)((a, b)(c, d)) &\geq \min[\overset{+}{\gamma}(\alpha, \beta)((a, b)), \overset{+}{\gamma}(\alpha, \beta)((c, d))] \\ &\geq \min(r, r) = r, \end{aligned}$$

and

$$\begin{aligned} \overset{-}{\gamma}(\alpha, \beta)((a, b)(c, d)) &\leq \max[\overset{-}{\gamma}(\alpha, \beta)((a, b)), \overset{+}{\gamma}(\alpha, \beta)((c, d))] \\ &\leq \max(t, t) = t. \end{aligned}$$

This implies that $(a, b)(c, d) \in \overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$. Further, let $(e, f), (g, h) \in S \times S$ such that $(e, f) \leq (g, h)$ and $(g, h) \in \overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$. Then, since γ_Λ is an FBS ordered semigroup over $S \times S$, we have

$$\begin{aligned} \overset{+}{\gamma}(\alpha, \beta)((e, f)) &\geq \overset{+}{\gamma}(\alpha, \beta)((g, h)) \geq r, \\ \overset{-}{\gamma}(\alpha, \beta))((e, f)) &\leq \overset{-}{\gamma}(\alpha, \beta)((g, h)) \leq t. \end{aligned}$$

This implies that $(e, f) \in \overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$. Therefore, $\overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$ is a subsemigroup of $S \times S$. $\square$

Proposition 4.14. *Let λ_A and δ_A be FBS sets over S. Then $\lambda_A \times \delta_A$ is an FBS ordered semigroup over $S \times S$ if and only if $\overset{(r,t)}{(\lambda_A \times \delta_A)}(\alpha, \beta)(\neq \phi)$ is a subsemigroup of $S \times S$ for all $r \in (0, 1]$, $t \in [0, 1)$ and $(\alpha, \beta) \in A \times A$.*

Proof. Let $\gamma_\Lambda = \lambda_A \times \delta_A$, where $\Lambda = A \times A$. First, assume that γ_Λ is an FBS ordered semigroup over $S \times S$. Then, as we proceeded in Proposition 4.13, it is proved that $\overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$ is a subsemigroup of $S \times S$ for all $r \in (0, 1]$, $t \in [0, 1)$ and $(\alpha, \beta) \in \Lambda$.

Conversely, assume that $\overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$ is a subsemigroup of $S \times S$ for all $r \in (0, 1]$, $t \in [0, 1)$ and $(\alpha, \beta) \in \Lambda$. We have to prove that γ_Λ is an FBS ordered semigroup over $S \times S$. Let (e, f), $(g, h) \in S \times S$. If $(e, f) \leq (g, h)$, then

$$\begin{aligned} \overset{+}{\gamma}(\alpha, \beta)((e, f)) &\geq \overset{+}{\gamma}(\alpha, \beta)((g, h)) \geq r, \\ \overset{-}{\gamma}(\alpha, \beta)((e, f)) &\leq \overset{-}{\gamma}(\alpha, \beta)((g, h)) \leq t. \end{aligned}$$

In fact: Let $\overset{+}{\gamma}(\alpha, \beta)((g, h)) = r_1 \in (0, 1]$ and $\overset{-}{\gamma}(\alpha, \beta)((g, h)) = t_1 \in [0, 1)$. Then (g, h) $\in \overset{(r_1,t_1)}{\gamma_\Lambda}(\alpha, \beta)$ which implies that $(e, f) \in \overset{(r_1,t_1)}{\gamma_\Lambda}(\alpha, \beta)$ because $\overset{(r_1,t_1)}{\gamma_\Lambda}(\alpha, \beta)$ is a subsemigroup of

$S \times S$. Thus, it follows that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)) \quad \geq \quad r_1 = \overset{+}{\gamma}(\alpha,\beta)((g,h)),$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)) \quad \leq \quad t_1 = \overset{-}{\gamma}(\alpha,\beta)((g,h)).$$

Next, for all (e,f), $(g,h) \in S \times S$, we need to show that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \quad \geq \quad \min[\overset{+}{\gamma}(\alpha,\beta)((e,f)), \overset{+}{\gamma}(\alpha,\beta)((g,h))],$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \quad \leq \quad \max[\overset{-}{\gamma}(\alpha,\beta)((e,f)), \overset{-}{\gamma}(\alpha,\beta)((g,h))].$$

On the contrary, suppose that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \quad < \quad \min[\overset{+}{\gamma}(\alpha,\beta)((e,f)), \overset{+}{\gamma}(\alpha,\beta)((g,h))],$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \quad > \quad \max[\overset{-}{\gamma}(\alpha,\beta)((e,f)), \overset{-}{\gamma}(\alpha,\beta)((g,h))].$$

Then there exist $r_0 \in (0,1]$ and $t_0 \in [0,1)$ such that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \quad < r_0 < \quad \min[\overset{+}{\gamma}(\alpha,\beta)((e,f)), \overset{+}{\gamma}(\alpha,\beta)((g,h))],$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \quad > t_0 > \quad \max[\overset{-}{\gamma}(\alpha,\beta)((e,f)), \overset{-}{\gamma}(\alpha,\beta)((g,h))].$$

This implies that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)) > r_0, \quad \overset{+}{\gamma}(\alpha,\beta)((g,h)) > r_0,$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)) < t_0, \quad \overset{-}{\gamma}(\alpha,\beta)((g,h)) < t_0.$$

Thus, it follows that $(e,f),(g,h) \in \overset{(r_0,t_0)}{\gamma_\Lambda}(\alpha,\beta)$. Since, by the hypothesis, $\overset{(r_0,t_0)}{\gamma_\Lambda}(\alpha,\beta)$ is a subsemigroup of $S \times S$, thus $(e,f)(g,h) \in \overset{(r_0,t_0)}{\gamma_\Lambda}(\alpha,\beta)$. This further implies that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \geq r_0,$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \leq t_0,$$

which is a contradiction. Thus, we conclude that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \geq \min[\overset{+}{\gamma}(\alpha,\beta)((e,f)), \overset{+}{\gamma}(\alpha,\beta)((g,h))],$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \leq \max[\overset{-}{\gamma}(\alpha,\beta)((e,f)), \overset{+}{\gamma}(\alpha,\beta)((g,h))].$$

Therefore, γ_Λ is an FBS ordered semigroup over $S \times S$. □

4.15. **Fuzzy bipolar soft characteristic function.** In this subsection, we define a fuzzy bipolar soft characteristic function and characterize a subsemigroup P of S by means of its fuzzy bipolar soft characteristic function.

Definition 4.16. *Let P be a non-empty subset of S. Then an FBS set of the form*

$$\overset{P}{\chi}_A = (\overset{+}{\chi}_P, \overset{-}{\chi}_P, A)$$

over S, where

$$\overset{+}{\chi}_P(\varepsilon)(x) = \begin{cases} 1, & \textit{if } x \in P, \\ 0, & \textit{if } x \notin P, \end{cases}$$

and

$$\overset{-}{\chi}_P(\varepsilon)(x) = \begin{cases} 0, & \textit{if } x \in P, \\ 1, & \textit{if } x \notin P, \end{cases}$$

for all $\varepsilon \in A$ and $x \in S$, is called an FBS characteristic function of P.

Proposition 4.17. *Let P be a nonempty subset of S. Then the following axioms are equivalent on S:*

(i) *P is a subsemigroup of S.*

(ii) *The FBS characteristic function $\overset{P}{\chi}_A$ of P is an FBS ordered semigroup over S.*

Proof. First assume that P is a subsemigroup of S. Let $\overset{P}{\chi}_A$ be the FBS characteristic function of P. Suppose that $x, y \in S$ and $\varepsilon \in A$. Firstly, we prove that

$$\overset{+}{\chi}_P(\varepsilon)(xy) \geq \min\{\overset{+}{\chi}_P(\varepsilon)(x), \overset{+}{\chi}_P(\varepsilon)(y)\},$$

$$\overset{-}{\chi}_P(\varepsilon)(xy) \leq \max\{\overset{-}{\chi}_P(\varepsilon)(x), \overset{-}{\chi}_P(\varepsilon)(y)\}.$$

For this we discuss the following two cases:

Case (a). Let $x \in P$ and $y \in P$, then $xy \in P$. Thus, we have

$$\overset{+}{\chi}_P(\varepsilon)(xy) = 1, \quad \overset{-}{\chi}_P(\varepsilon)(xy) = 0,$$

which implies that

$$\overset{+}{\chi}_P(\varepsilon)(xy) \geq \min\{\overset{+}{\chi}_P(\varepsilon)(x), \overset{+}{\chi}_P(\varepsilon)(y)\},$$

$$\overset{-}{\chi}_P(\varepsilon)(xy) \leq \max\{\overset{-}{\chi}_P(\varepsilon)(x), \overset{-}{\chi}_P(\varepsilon)(y)\}.$$

Case (b). Let $x \notin P$ or $y \notin P$. Then

$$\overset{+}{\chi}_P(\varepsilon)(x) = 0, \ \ \overset{-}{\chi}_P(\varepsilon)(x) = 1,$$

or

$$\overset{+}{\chi}_P(\varepsilon)(y) = 0, \ \ \overset{-}{\chi}_P(\varepsilon)(y) = 1.$$

This implies that

$$\overset{+}{\chi}_P(\varepsilon)(xy) \geq 0 = \min\{\overset{+}{\chi}_P(\varepsilon)(x), \overset{+}{\chi}_P(\varepsilon)(y)\},$$

and

$$\overset{-}{\chi}_P(\varepsilon)(xy) \leq 1 = \max\{\overset{-}{\chi}_P(\varepsilon)(x), \overset{-}{\chi}_P(\varepsilon)(y)\}.$$

Thus, in both the cases, we have

$$\overset{+}{\chi}_P(\varepsilon)(xy) \geq \min\{\overset{+}{\chi}_P(\varepsilon)(x), \overset{+}{\chi}_P(\varepsilon)(y)\},$$

$$\overset{-}{\chi}_P(\varepsilon)(xy) \leq \max\{\overset{-}{\chi}_P(\varepsilon)(x), \overset{-}{\chi}_P(\varepsilon)(y)\}.$$

Secondly, let $x, y \in S$ such that $x \leq y$. If $y \in P$, then $x \in P$ which implies that

$$\overset{+}{\chi}_P(\varepsilon)(x) = 1, \quad \overset{-}{\chi}_P(\varepsilon)(x) = 0.$$

Thus, in this case, we have

$$\overset{+}{\chi}_P(\varepsilon)(x) \geq \overset{+}{\chi}_P(\varepsilon)(y), \quad \overset{-}{\chi}_P(\varepsilon)(x) \leq \overset{-}{\chi}_P(\varepsilon)(y).$$

If $y \notin P$, then

$$\overset{+}{\chi}_P(\varepsilon)(y) = 0, \quad \overset{-}{\chi}_P(\varepsilon)(y) = 1.$$

So, in this case, we also have

$$\overset{+}{\chi}_P(\varepsilon)(x) \geq \overset{+}{\chi}_P(\varepsilon)(y), \quad \overset{-}{\chi}_P(\varepsilon)(x) \leq \overset{-}{\chi}_P(\varepsilon)(y).$$

Consequently, $\overset{P}{\chi}_A$ is an FBS ordered semigroup over S. Conversely, assume that Axiom (ii) holds. Let $x, y \in S$ and $\varepsilon \in A$. Suppose that $x, y \in P$. Then, we have

$$\overset{+}{\chi}_P(\varepsilon)(x) = \overset{+}{\chi}_P(\varepsilon)(y) = 1,$$

$$\overset{-}{\chi}_P(\varepsilon)(x) = \overset{-}{\chi}_P(\varepsilon)(y) = 0.$$

Moreover, by the hypothesis, we have

$$\begin{aligned}\overset{+}{\chi}_P(\varepsilon)(xy) &\geq \min\{\overset{+}{\chi}_P(\varepsilon)(x), \overset{+}{\chi}_P(\varepsilon)(y)\} \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\overset{-}{\chi}_P(\varepsilon)(xy) &\leq \max\{\overset{-}{\chi}_P(\varepsilon)(x), \overset{-}{\chi}_P(\varepsilon)(y)\} \\ &= 0.\end{aligned}$$

This implies that $xy \in P$. Now, let $x \leq y$ such that $y \in P$. Then, we have

$$\overset{+}{\chi}_P(\varepsilon)(y) = 1, \quad \overset{-}{\chi}_P(\varepsilon)(y) = 0.$$

Since $\overset{P}{\chi}_A$ is an FBS ordered semigroup over S, thus, it follows that

$$\overset{+}{\chi}_P(\varepsilon)(x) \geq \overset{+}{\chi}_P(\varepsilon)(y) = 1,$$

and

$$\overset{-}{\chi}_P(\varepsilon)(x) \leq \overset{-}{\chi}_P(\varepsilon)(y) = 0.$$

This implies that $x \in P$. Therefore, P is a subsemigroup of S. $\square$

4.18. **The product of fuzzy bipolar soft sets.** In this subsection, we define the product of two fuzzy bipolar soft sets over an ordered semigroup S, and study some characteristics of fuzzy bipolar soft ordered semigroups in terms of the product of two fuzzy bipolar soft sets.

For $x \in S$, let's define the set X_x as follows:

$$X_x = \{(a,b) \in S \times S \mid x \leq ab\}.$$

Definition 4.19. *Let λ_A and δ_A be FBS sets over S. Let $p, q, x \in S$ and ε be an element in A. Then, the product of λ_A and δ_A is defined to be the FBS set γ_A over S, where*

$$\overset{+}{\gamma}(\varepsilon)(x) = \begin{cases} \bigvee_{(p,q)\in X_x} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{\delta}(\varepsilon)(q)\}, & \text{if } X_x \neq \phi, \\ 0, & \text{if } X_x = \phi, \end{cases}$$

and

$$\overset{-}{\gamma}(\varepsilon)(x) = \begin{cases} \bigwedge_{(p,q)\in X_x} \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{\delta}(\varepsilon)(q)\}, & \text{if } X_x \neq \phi, \\ 1, & \text{if } X_x = \phi. \end{cases}$$

We denote $\gamma_A = \lambda_A \circ \delta_A$, where $\overset{+}{\gamma} = \overset{+}{\lambda} \circ \overset{+}{\delta}$ and $\overset{-}{\gamma} = \overset{-}{\lambda} \circ \overset{-}{\delta}$.

Remark 4.20. *Consider an initial universe set S and a set of parameters E. Let $\mathcal{FB}(S,A)$ denotes the set of all FBS sets over S with a fixed set of parameters A, where $A \subseteq E$. Define the order relation "$\tilde{\preceq}$" on $\mathcal{FB}(S,A)$ as follows:*

$$\lambda_A \tilde{\preceq} \delta_A \Leftrightarrow \overset{+}{\lambda}(\varepsilon)(x) \leq \overset{+}{\delta}(\varepsilon)(x),\ \overset{-}{\delta}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(x),$$

for all $x \in S$ and $\varepsilon \in A$. Clearly, $(\mathcal{FB}(S,A), \circ, \tilde{\preceq})$ is an ordered semigroup.

Proposition 4.21. *Let λ_A be an FBS set over S. Then λ_A is an FBS ordered semigroup over S if and only if the following conditions hold:*

(i) $\lambda_A \circ \lambda_A \tilde{\preceq} \lambda_A$.
(ii) *If $x \leq y$, then, for all $\varepsilon \in A$ and $x, y \in S$, we have*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y).$$

Proof. First assume that λ_A is an FBS ordered semigroup over S. Let $\varepsilon \in A$ and $a, x, y \in S$. If $X_a = \phi$, then

$$(\overset{+}{\lambda} \circ \overset{+}{\lambda})(\varepsilon)(a) = 0 \leq \overset{+}{\lambda}(\varepsilon)(a),$$

$$(\overset{-}{\lambda} \circ \overset{-}{\lambda})(\varepsilon)(a) = 1 \geq \overset{-}{\lambda}(\varepsilon)(a).$$

Let $X_a \neq \phi$, then there exist some elements $x, y \in S$ such that $a \leq xy$. Thus, we have

$$(\overset{+}{\lambda} \circ \overset{+}{\lambda})(\varepsilon)(a) = \bigvee_{(x,y)\in X_a} \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\}$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{\lambda})(\varepsilon)(a) = \bigwedge_{(x,y)\in X_a} \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}.$$

Since λ_A is an FBS ordered semigroup over S, we have

$$\overset{+}{\lambda}(\varepsilon)(xy) \geq \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\},$$

and

$$\overset{-}{\lambda}(\varepsilon)(xy) \leq \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}.$$

In particular, for all $x, y \in S$ such that $(x, y) \in X_a$, we have

$$\overset{+}{\lambda}(\varepsilon)(xy) \geq \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\},$$

and

$$\overset{-}{\lambda}(\varepsilon)(xy) \leq \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}.$$

Thus,

$$\begin{aligned}\overset{+}{\lambda}(\varepsilon)(a) &\geq \overset{+}{\lambda}(\varepsilon)(xy)\\ &\geq \bigvee_{(x,y)\in X_a} \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\}\\ &= (\overset{+}{\lambda} \circ \overset{+}{\lambda})(\varepsilon)(a),\end{aligned}$$

and

$$\begin{aligned}\overset{-}{\lambda}(\varepsilon)(a) &\leq \overset{-}{\lambda}(\varepsilon)(xy)\\ &\leq \bigwedge_{(x,y)\in X_a} \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}\\ &= (\overset{-}{\lambda} \circ \overset{-}{\lambda})(\varepsilon)(a).\end{aligned}$$

Thus, Condition (i) follows and, by Definition 4.1, Condition (ii) holds. Conversely, assume that Conditions (i) and (ii) hold. Let $\varepsilon \in A$ and $x, y \in S$. Then, since $\lambda_A \circ \lambda_A \mathrel{\tilde{\preceq}} \lambda_A$, we have

$$\begin{aligned}\overset{+}{\lambda}(\varepsilon)(xy) &\geq (\overset{+}{\lambda} \circ \overset{+}{\lambda})(\varepsilon)(xy)\\ &\geq \min\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\},\end{aligned}$$

and

$$\begin{aligned}\overset{-}{\lambda}(\varepsilon)(xy) &\leq (\overset{-}{\lambda} \circ \overset{-}{\lambda})(\varepsilon)(xy)\\ &\leq \max\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}.\end{aligned}$$

Therefore, λ_A is an FBS ordered semigroup over S. □

5. Some properties of fuzzy bipolar soft left/right and fuzzy bipolar soft two-sided ideals

In this section, we introduce the notions of FBS left, right and two-sided ideals over ordered semigroups, and study some properties of these FBS ideals. It is proved, among others, that the FBS union (resp., intersection) and the Or-product (resp., And-product) of two FBS left (resp., right, two-sided) ideals over S are FBS left (resp., right, two-sided) ideals over S.

Definition 5.1. *Let λ_A be an FBS set over S. Then, it is called an FBS left ideal over S if and only if, for all $x, y \in S$ and $\varepsilon \in A$, the following assertions hold:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(xy) \leq \overset{-}{\lambda}(\varepsilon)(y)$.

(ii) $x \leq y \Rightarrow \overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$.

Definition 5.2. *Let λ_A be an FBS set over S. Then, it is called an FBS right ideal over S if and only if, for all $x, y \in S$ and $\varepsilon \in A$, the following assertions hold:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \overset{+}{\lambda}(\varepsilon)(x), \quad \overset{-}{\lambda}(\varepsilon)(xy) \leq \overset{-}{\lambda}(\varepsilon)(x)$.

(ii) $x \leq y \Rightarrow \overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$.

Definition 5.3. *Let λ_A be an FBS set over S. If it is both an FBS left and an FBS right ideal over S, then it is called an FBS two-sided ideal or, simply, an FBS ideal over S. Equivalently, λ_A is an FBS ideal over S if and only if, for all $x, y \in S$ and $\varepsilon \in A$, the following assertions hold:*

(i) $\overset{+}{\lambda}(\varepsilon)(xy) \geq \max\{\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)\}$.

(ii) $\overset{-}{\lambda}(\varepsilon)(xy) \leq \min\{\overset{-}{\lambda}(\varepsilon)(x), \overset{-}{\lambda}(\varepsilon)(y)\}$.

(iii) $x \leq y \Rightarrow \overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y)$.

The proof of the following theorem is straightforward.

Theorem 5.4. *Let λ_A and δ_A be FBS left (resp., right, two-sided) ideals over S. Then, we have*

(i) $\lambda_A \,\widetilde{\cup}\, \delta_A$ *is an FBS left (resp., right, two-sided) ideal over S.*

(ii) $\lambda_A \,\widetilde{\cap}\, \delta_A$ *is an FBS left (resp., right, two-sided) ideal over S.*

(iii) $\lambda_A \,\widetilde{\vee}\, \delta_A$ *is an FBS left (resp., right, two-sided) ideal over S.*

(iv) $\lambda_A \,\widetilde{\wedge}\, \delta_A$ *is an FBS left (resp., right, two-sided) ideal over S.*

(v) $\lambda_A \circ \delta_A$ *is an FBS left (resp., right, two-sided) ideal over S.*

(vi) $\lambda_A \times \delta_A$ *is an FBS left (resp., right, two-sided) ideal over $S \times S$.*

Proposition 5.5. *Let λ_A and δ_A be FBS left (resp., right, two-sided) ideals over S. Then $\overset{(r,t)}{(\lambda_A \times \delta_A)}(\alpha, \beta) (\neq \phi)$ is an FBS left (resp., right, two-sided) ideal over $S \times S$ for all $r \in (0, 1]$, $t \in [0, 1)$ and $(\alpha, \beta) \in A \times A$.*

Proof. Let $\gamma_\Lambda = \lambda_A \times \delta_A$, where $\Lambda = A \times A$. Then, by Theorem 5.4, we see that γ_Λ is an FBS left ideal over $S \times S$. We have to prove that $\overset{(r,t)}{\gamma_\Lambda}(\alpha, \beta)$ is a left ideal of $S \times S$, for all $r \in (0,1]$, $t \in [0,1)$ and $(\alpha,\beta) \in \varLambda$. Let $(a,b) \in \overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)$ and $(c,d) \in S \times S$. Then, since $\gamma_{\varLambda}$ is an FBS left ideal over $S \times S$, we have

$$\begin{aligned}\overset{+}{\gamma}(\alpha,\beta)((c,d)(a,b)) &\geq \overset{+}{\gamma}(\alpha,\beta)(a,b)\\ &\geq r,\end{aligned}$$

and

$$\begin{aligned}\overset{-}{\gamma}(\alpha,\beta)((c,d)(a,b)) &\leq \overset{-}{\gamma}(\alpha,\beta)(a,b)\\ &\leq t.\end{aligned}$$

This implies that $(c,d)(a,b) \in \overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)$. Now, let $(e,f) \in S \times S$ and $(g,h) \in \overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)$ such that $(e,f) \leq (g,h)$. Then, since $\gamma_{\varLambda}$ is an FBS left ideal over $S \times S$, we have

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)) \geq \overset{+}{\gamma}(\alpha,\beta)((g,h)) \geq r,$$

and

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)) \leq \overset{-}{\gamma}(\alpha,\beta)((g,h)) \leq t.$$

This implies that $(e,f) \in \overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)$. Therefore, $\overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)$ is a left ideal of $S \times S$. The other parts of the proposition can be proved similarly. □

Proposition 5.6. *Let λ_A and δ_A be FBS sets over S. Then $\lambda_A \times \delta_A$ is an FBS left (resp., right, two-sided) ideal over $S \times S$ if and only if $\overset{(r,t)}{(\lambda_A \times \delta_A)}(\alpha,\beta)(\neq \phi)$ is an FBS left (resp., right, two-sided) ideal over $S \times S$ for all $r \in (0,1]$, $t \in [0,1)$ and $(\alpha,\beta) \in A \times A$.*

Proof. Let $\gamma_{\varLambda} = \lambda_A \times \delta_A$, where $\varLambda = A \times A$. First assume that $\gamma_{\varLambda}$ is an FBS left ideal over $S \times S$. Then, as we proceeded in Proposition 5.5, it is proved that $\overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)$ is a left ideal of $S \times S$, for all $r \in (0,1]$, $t \in [0,1)$ and $(\alpha,\beta) \in \varLambda$.

Conversely, assume that, for all $r \in (0,1]$, $t \in [0,1)$ and $(\alpha,\beta) \in \varLambda$, the (r,t)-level subset $\overset{(r,t)}{\gamma_{\varLambda}}(\alpha,\beta)(\neq \phi)$ of $\gamma_{\varLambda}$ is a left ideal of $S \times S$. We need to prove that $\gamma_{\varLambda}$ is an FBS left ideal over $S \times S$. Let $(a,b),(c,d) \in S \times S$. If $(a,b) \leq (c,d)$, then

$$\overset{+}{\gamma}(\alpha,\beta)((a,b)) \geq \overset{+}{\gamma}(\alpha,\beta)((c,d)),$$

$$\overset{-}{\gamma}(\alpha,\beta)((a,b)) \leq \overset{-}{\gamma}(\alpha,\beta)((c,d)).$$

In fact: Let $\overset{+}{\gamma}(\alpha,\beta)(c,d) = r_1$ and $\overset{-}{\gamma}(\alpha,\beta)(c,d) = t_1$. Then $(c,d) \in \overset{(r_1,t_1)}{\gamma_{\varLambda}}(\alpha,\beta)$. Since $\overset{(r_1,t_1)}{\gamma_{\varLambda}}(\alpha,\beta)$ is a left ideal of $S \times S$ and $(a,b) \leq (c,d)$, then $(a,b) \in \overset{(r_1,t_1)}{\gamma_{\varLambda}}(\alpha,\beta)$. Thus, it foliows that

$$\overset{+}{\gamma}(\alpha,\beta)((a,b)) \geq r_1 = \overset{+}{\gamma}(\alpha,\beta)((c,d)),$$

$$\overset{-}{\gamma}(\alpha,\beta)((a,b)) \leq t_1 = \overset{-}{\gamma}(\alpha,\beta)((c,d)).$$

Next, we have to prove that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \geq \overset{+}{\gamma}(\alpha,\beta)((g,h)),$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \leq \overset{-}{\gamma}(\alpha,\beta)((g,h)),$$

for all (e,f) and $(g,h) \in S \times S$. On the contrary, suppose that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) < \overset{+}{\gamma}(\alpha,\beta)((g,h)),$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) > \overset{-}{\gamma}(\alpha,\beta)((g,h)).$$

Then there exist $r_0 \in (0,1]$ and $t_0 \in [0,1)$ such that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) < r_0 < \overset{+}{\gamma}(\alpha,\beta)((g,h)),$$

and

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) > t_0 > \overset{-}{\gamma}(\alpha,\beta)((g,h)).$$

So, it follows that $(g,h) \in \overset{(r_0,t_0)}{\gamma_A}(\alpha,\beta)$. Since, by the hypothesis, $\overset{(r_0,t_0)}{\gamma_A}(\alpha,\beta)$ is a left ideal of $S \times S$, thus $(e,f)(g,h) \in \overset{(r_0,t_0)}{\gamma_A}(\alpha,\beta)$. Then, we have

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \geq r_0,$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \leq t_0,$$

which is a contradiction. Thus, it follows that

$$\overset{+}{\gamma}(\alpha,\beta)((e,f)(g,h)) \geq \overset{+}{\gamma}(\alpha,\beta)((g,h)),$$

$$\overset{-}{\gamma}(\alpha,\beta)((e,f)(g,h)) \leq \overset{-}{\gamma}(\alpha,\beta)((g,h)).$$

Therefore, γ_A is an FBS left ideal over $S \times S$. The other parts of the proposition can be proved similarly. □

Proposition 5.7. *Let λ_A, δ_A, γ_A and ϑ_A be FBS sets over S such that $\lambda_A \overset{\sim}{\preceq} \delta_A$ and $\gamma_A \overset{\sim}{\preceq} \vartheta_A$. Then, we have*

$$\lambda_A \circ \gamma_A \quad \overset{\sim}{\preceq} \quad \delta_A \circ \vartheta_A.$$

Proof. Let $\varepsilon \in A$ and $a \in S$. If $X_a = \phi$, then, we have

$$(\overset{+}{\lambda} \circ \overset{+}{\gamma})(\varepsilon)(a) \quad = \quad 0 \leq 0 = (\overset{+}{\delta} \circ \overset{+}{\vartheta})(\varepsilon)(a),$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{\gamma})(\varepsilon)(a) \quad = \quad 1 \geq 1 = (\overset{-}{\delta} \circ \overset{-}{\vartheta})(\varepsilon)(a).$$

Let $X_a \neq \phi$. Then, for some $(p,q) \in X_a$, we have

$$\text{(i)} \qquad (\overset{+}{\lambda} \circ \overset{+}{\gamma})(\varepsilon)(a) = \bigvee_{(p,q)\in X_a} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{\gamma}(\varepsilon)(q)\},$$

$$\text{(ii)} \qquad (\overset{-}{\lambda} \circ \overset{-}{\gamma})(\varepsilon)(a) = \bigwedge_{(p,q)\in X_a} \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{\gamma}(\varepsilon)(q)\},$$

$$\text{(iii)} \qquad (\overset{+}{\delta} \circ \overset{+}{\vartheta})(\varepsilon)(a) = \bigvee_{(p,q)\in X_a} \min\{\overset{+}{\delta}(\varepsilon)(p), \overset{+}{\vartheta}(\varepsilon)(q)\},$$

and

$$\text{(iv)} \qquad (\overset{-}{\delta} \circ \overset{-}{\vartheta})(\varepsilon)(a) = \bigwedge_{(p,q)\in X_a} \max\{\overset{-}{\delta}(\varepsilon)(p), \overset{-}{\vartheta}(\varepsilon)(q)\}.$$

Since $p, q \in S$ and $\lambda_A \mathrel{\tilde{\preceq}} \delta_A,\ \gamma_A \mathrel{\tilde{\preceq}} \vartheta_A$, thus, we have

$$\overset{+}{\lambda}(\varepsilon)(p) \leq \overset{+}{\delta}(\varepsilon)(p), \quad \overset{-}{\lambda}(\varepsilon)(p) \geq \overset{-}{\delta}(\varepsilon)(p),$$

and

$$\overset{+}{\gamma}(\varepsilon)(q) \leq \overset{+}{\vartheta}(\varepsilon)(q), \quad \overset{-}{\gamma}(\varepsilon)(q) \geq \overset{-}{\vartheta}(\varepsilon)(q).$$

These inequalities imply that

$$\text{(v)} \qquad \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{\gamma}(\varepsilon)(q)\} \leq \min\{\overset{+}{\delta}(\varepsilon)(p), \overset{+}{\vartheta}(\varepsilon)(q)\},$$

and

$$\text{(vi)} \qquad \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{\gamma}(\varepsilon)(q)\} \geq \max\{\overset{-}{\delta}(\varepsilon)(p), \overset{-}{\vartheta}(\varepsilon)(q)\}.$$

So, from (i), (ii), (iii), (iv), (v) and (vi), it follows that

$$\begin{aligned}(\overset{+}{\lambda} \circ \overset{+}{\gamma})(\varepsilon)(a) &= \bigvee_{(p,q)\in X_a} \min\{\overset{+}{\lambda}(\varepsilon)(p), \overset{+}{\gamma}(\varepsilon)(q)\} \\ &\leq \bigvee_{(p,q)\in X_a} \min\{\overset{+}{\delta}(\varepsilon)(p), \overset{+}{\vartheta}(\varepsilon)(q)\} \\ &= (\overset{+}{\delta} \circ \overset{+}{\vartheta})(\varepsilon)(a),\end{aligned}$$

and

$$\begin{aligned}(\overset{-}{\lambda} \circ \overset{-}{\gamma})(\varepsilon)(a) &= \bigwedge_{(p,q)\in X_a} \max\{\overset{-}{\lambda}(\varepsilon)(p), \overset{-}{\gamma}(\varepsilon)(q)\} \\ &\geq \bigwedge_{(p,q)\in X_a} \max\{\overset{-}{\delta}(\varepsilon)(p), \overset{-}{\vartheta}(\varepsilon)(q)\} \\ &= (\overset{-}{\delta} \circ \overset{-}{\vartheta})(\varepsilon)(a).\end{aligned}$$

Therefore, we have $\lambda_A \circ \gamma_A \mathrel{\widetilde{\preceq}} \delta_A \circ \vartheta_A$. Thus the proof is completed. □

Proposition 5.8. *Let λ_A be an FBS set over S. Then, it is an FBS left ideal over S if and only if the following conditions hold:*

(i) $S_A \circ \lambda_A \mathrel{\widetilde{\preceq}} \lambda_A$.

(ii) *If $x \leq y$, then*

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$ and $x, y \in S$.

Proof. First assume that λ_A is an FBS left ideal over S. Let $a \in S$ and $\varepsilon \in A$. If $X_a = \phi$, then

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(a) = 0 \leq \overset{+}{\lambda}(\varepsilon)(a),$$

$$(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(a) = 1 \geq \overset{-}{\lambda}(\varepsilon)(a).$$

If $X_a \neq \phi$, then there exists (y, z) in X_a such that $a \leq yz$. Then, since λ_A is an FBS left ideal over S, we have

$$\begin{aligned}
(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(a) &= \bigwedge_{(y,z)\in X_a} \max[\overset{-}{S}(\varepsilon)(y), \overset{-}{\lambda}(\varepsilon)(z)] \\
&\geq \bigwedge_{(y,z)\in X_a} \max[\overset{-}{S}(\varepsilon)(y), \overset{-}{\lambda}(\varepsilon)(yz)] \\
&\geq \bigwedge_{(y,z)\in X_a} \max[\overset{-}{S}(\varepsilon)(y), \overset{-}{\lambda}(\varepsilon)(a)] \\
&= \overset{-}{\lambda}(\varepsilon)(a),
\end{aligned}$$

and, similarly,

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(a) \leq \overset{+}{\lambda}(\varepsilon)(a).$$

Therefore, $S_A \circ \lambda_A \mathrel{\widetilde{\preceq}} \lambda_A$. Thus, Condition (i) holds. Moreover, by Definition 5.1, we see that Condition (ii) holds.

Conversely, assume that Conditions (i) and (ii) hold. Let $x, y \in S$ and $\varepsilon \in A$. Then, we have

$$\begin{aligned}
\overset{+}{\lambda}(\varepsilon)(xy) &\geq (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xy) \\
&= \bigvee_{(a,b)\in X_{xy}} \min[\overset{+}{S}(\varepsilon)(a), \overset{+}{\lambda}(\varepsilon)(b)] \\
&\geq \min[\overset{+}{S}(\varepsilon)(x), \overset{+}{\lambda}(\varepsilon)(y)] \\
&= \overset{+}{\lambda}(\varepsilon)(y).
\end{aligned}$$

Similarly, we obtain

$$\overset{-}{\lambda}(\varepsilon)(xy) \quad \leq \quad \overset{-}{\lambda}(\varepsilon)(y).$$

Therefore, λ_A is an FBS left ideal over S. □

Similarly, we establish the following proposition:

Proposition 5.9. *Let λ_A be an FBS set over S. Then, it is an FBS right ideal over S if and only if the following conditions hold:*

(i) $\lambda_A \circ S_A \mathrel{\widetilde{\preceq}} \lambda_A$.
(ii) *If $x \leq y$, then*

$$\overset{+}{\lambda}(\varepsilon)(x) \quad \geq \quad \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$ and $x, y \in S$.

Proof. It is straightforward. □

The combined effect of Propositions 5.8 and 5.9 is formulated as follows:

Proposition 5.10. *Let λ_A be an FBS set over S. Then, it is an FBS ideal over S if and only if the following conditions hold:*

(i) $S_A \circ \lambda_A \mathrel{\widetilde{\preceq}} \lambda_A$.
(ii) $\lambda_A \circ S_A \mathrel{\widetilde{\preceq}} \lambda_A$.
(iii) *If $x \leq y$, then*

$$\overset{+}{\lambda}(\varepsilon)(x) \quad \geq \quad \overset{+}{\lambda}(\varepsilon)(y), \quad \overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

for all $\varepsilon \in A$ and $x, y \in S$.

Proof. It is straightforward. □

Proposition 5.11. *Let λ_A be an FBS left (resp., right, two-sided) ideal over S. Then, we have*

$$\lambda_A \circ \lambda_A \quad \widetilde{\preceq} \quad \lambda_A.$$

Proof. Since we have $\lambda_A \mathrel{\widetilde{\preceq}} S_A$ and $\lambda_A \mathrel{\widetilde{\preceq}} \lambda_A$, then, by Proposition 5.7, it follows that $\lambda_A \circ \lambda_A \mathrel{\widetilde{\preceq}} S_A \circ \lambda_A$. Moreover, by Proposition 5.8, we have $S_A \circ \lambda_A \mathrel{\widetilde{\preceq}} \lambda_A$ because λ_A is an FBS left ideal over S. This implies that $\lambda_A \circ \lambda_A \mathrel{\widetilde{\preceq}} \lambda_A$. Similarly, the other parts of the proposition can be proved. □

Proposition 5.12. *Let λ_A be an FBS set over S. Then $S_A \circ \lambda_A$ is an FBS left ideal over S.*

Proof. Let $\varepsilon \in A$ and $x, y \in S$. If $X_y = \phi$, then, we have

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y) = 0 \leq (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xy),$$

and

$$(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y) = 1 \geq (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(xy).$$

Let $X_y \neq \phi$. Then, for each $(a,b) \in X_y$, we have $y \leq ab \Rightarrow xy \leq (xa)b$. This means that $(xa,b) \in X_{xy}$. Moreover, $\overset{-}{S}(\varepsilon)(xa)$ = 0 = $\overset{-}{S}(\varepsilon)(a)$. Thus, we have

$$\begin{aligned}(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y) &= \bigwedge_{(a,b)\in X_y} \max[\overset{-}{S}(\varepsilon)(a), \overset{-}{\lambda}(\varepsilon)(b)] \\ &\geq \bigwedge_{(c,d)\in X_{xy}} \max[\overset{-}{S}(\varepsilon)(c), \overset{-}{\lambda}(\varepsilon)(d)] \\ &= (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(xy),\end{aligned}$$

and, similarly,

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y) \quad \leq \quad (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(xy).$$

Let $x \leq y$ and $(a,b) \in X_y$. Then

$$y \leq ab \Rightarrow x \leq ab \Rightarrow (a,b) \in X_x.$$

So $X_y \subseteq X_x$. If $X_x = \phi$, then $X_y = \phi$. Thus, we have

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x) = 0 = (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y),$$

and

$$(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x) = 1 = (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y).$$

On the other hand, if $X_y \neq \phi$, then $X_x \neq \phi$. Thus, we have

$$\begin{aligned}(\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(y) &= \bigwedge_{(a,b)\in X_y} \max[\overset{-}{S}(\varepsilon)(a), \overset{-}{\lambda}(\varepsilon)(b)] \\ &\geq \bigwedge_{(c,d)\in X_x} \max[\overset{-}{S}(\varepsilon)(c), \overset{-}{\lambda}(\varepsilon)(d)] \\ &= (\overset{-}{S} \circ \overset{-}{\lambda})(\varepsilon)(x),\end{aligned}$$

and, similarly,

$$(\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(y) \quad \leq \quad (\overset{+}{S} \circ \overset{+}{\lambda})(\varepsilon)(x).$$

Therefore, $S_A \circ \lambda_A$ is an FBS left ideal over S. □

Similarly, we establish the following proposition:

Proposition 5.13. *Let λ_A be an FBS set over S. Then $\lambda_A \circ S_A$ is an FBS right ideal over S.*

Proof. It is straightforward. □

Proposition 5.14. *Let λ_A and δ_A be FBS right and FBS left ideals over S, respectively. Then, we have*

$$\lambda_A \circ \delta_A \mathrel{\tilde{\preceq}} \lambda_A \mathbin{\tilde{\cap}} \delta_A.$$

Proof. Let $\varepsilon \in A$ and $x \in S$. If $X_x = \phi$, then, since λ_A and δ_A are respectively FBS right and FBS left ideals over S, we have

$$(\overset{+}{\lambda} \circ \overset{+}{\delta})(\varepsilon)(x) = 0 \leq (\overset{+}{\lambda} \wedge \overset{+}{\delta})(\varepsilon)(x),$$

and

$$(\overset{-}{\lambda} \circ \overset{-}{\delta})(\varepsilon)(x) = 1 \geq (\overset{-}{\lambda} \vee \overset{-}{\delta})(\varepsilon)(x).$$

Let $X_x \neq \phi$. Then $(y, z) \in X_x$, that is, $x \leq yz$ for some $y, z \in S$. Thus, we have

$$\overset{+}{\lambda}(\varepsilon)(x) \geq \overset{+}{\lambda}(\varepsilon)(yz) \geq \overset{+}{\lambda}(\varepsilon)(y),$$

$$\overset{-}{\lambda}(\varepsilon)(x) \leq \overset{-}{\lambda}(\varepsilon)(yz) \leq \overset{-}{\lambda}(\varepsilon)(y),$$

$$\overset{+}{\delta}(\varepsilon)(x) \geq \overset{+}{\delta}(\varepsilon)(yz) \geq \overset{+}{\delta}(\varepsilon)(z),$$

and

$$\overset{-}{\delta}(\varepsilon)(x) \leq \overset{-}{\delta}(\varepsilon)(yz) \leq \overset{-}{\delta}(\varepsilon)(z).$$

So, it follows that

$$\begin{aligned}
(\overset{+}{\lambda} \circ \overset{+}{\delta})(\varepsilon)(x) &= \bigvee_{(y,z)\in X_x} \min[\overset{+}{\lambda}(\varepsilon)(y), \overset{+}{\delta}(\varepsilon)(z)] \\
&\leq \bigvee_{(y,z)\in X_x} \min[\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)] \\
&= \min[\overset{+}{\lambda}(\varepsilon)(x), \overset{+}{\delta}(\varepsilon)(x)] \\
&= (\overset{+}{\lambda} \wedge \overset{+}{\delta})(\varepsilon)(x).
\end{aligned}$$

Similarly, we obtain

$$(\overset{-}{\lambda} \circ \overset{-}{\delta})(\varepsilon)(x) \quad \geq \quad (\overset{-}{\lambda} \vee \overset{-}{\delta})(\varepsilon)(x).$$

Therefore, we have $\lambda_A \circ \delta_A \mathrel{\widetilde{\preceq}} \lambda_A \mathbin{\widetilde{\cap}} \delta_A$. □

6. CONCLUSION

In this article, we redefined the concept of FBS sets and studied their algebraic properties. The notion of FBS ordered semigroup was initiated and some characteristics of the structure were examined. Similarly, the concepts of FBS left (resp., right, two-sided) ideals over ordered semigroups were introduced and characterized. In future, we plan to expose further the ideal theory in ordered semigroups in terms of FBS sets. So, we are going to study FBS interior ideals, FBS (generalized) bi-ideals and FBS quasi- ideals in ordered semigroups.

Acknowledgements. The authors are very greatful to the refrees and the Editor in Chief for their valuable comments and suggestions for improving the paper.

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