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On Lattice Ordered Double Framed Soft Rings

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Abstract. Keeping in view the importance of lattice theory, ring theory and soft sets, in this article the notion of lattice ordered double framed soft rings is acquainted and some basic properties of the defined notion are discussed. Additional to this the behavior of different operations of lattice ordered double framed soft sets is discussed under the atmosphere of rings. Wherever necessary the concepts for the defined notion are elaborated by examples.

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1. Introduction

The algebraic structures have always been very important in the theory of pure mathematics. Although all the algebraic structure including semigroups [1], groups [2], semirings [3] and near-rings [4] have their own importance but the theory of rings [5] has an upper hand due to its extensive usage in field theory, Galois theory, linear algebra and many others. In our daily life we face many problems that are so complex and encompasses uncertainties that we cannot clearly define a way to overwhelm these hurdles. Many fields like engineering, information science, computer science, medical science and environmental science are some of the areas of life in which we usually come across such issues. Such type of problems create misperception in our mind and we cannot say anything about their solution conformably. Day by day these problems have become more complicated. To overcome such type of situations and difficulties Zadeh presented the concept of fuzzy set [6] and interval valued fuzzy sets [7], Pawlak familiarized the notion of rough sets [8] and Molodtsov initiated the notion of soft set [9]. Although the theories of fuzzy sets and rough sets have their own advantages but Aktas and cagman [10] proved that the theory of soft sets generalized these theories, this proves the upper hand of soft set theory over fuzzy set theory and rough set theory. After the introduction of soft sets this theory gained much attention of the researchers. It was Maji et al. [11], who gave operations to soft sets. They also introduced the notion of fuzzy soft sets [12]. Ali et al. [13] pointed out some inadequacies in the operations defined by Maji et al. and improved these operations. Also Ali et al. [14] gave different algebraic structures associated with their defined operations. Shabir and Naz [15] initiated the concept of soft topological spaces. Aslam et al. [16] furnished the notion of soft LA-semigroup. Further Aslam and Qureshi [17] worked on the notion of soft groups and Acar et al. [18] functioned on soft rings. Xin and Li [19] defined soft congruence relations over soft rings. Cagman and Enginoglu applied soft set theory in uni-int decision making [20]. Majumdar and Samanta introduced the notion of soft mappings. For more applications of soft sets one can see [21-28].

Keeping in view the advantages and applications of intuitionistic fuzzy sets by Atanassov [29], Jun and Ahn [30] offered the concept of double framed soft set. Hadipour [31] used Double Framed Soft Sets in BF-algebras. Jun et al. [32] introduced the notion of ideal theory of BCK/BCI-algebras based on double framed soft sets. Muhiuddin and Al-Roqi [33] introduced the notion of double framed soft hypervector spaces and discussed their basic properties. In [34] the notion of double framed soft rings is introduced and discussed. In our daily life we sometimes come across the situations when in objects under discussion have some order between them. This type of situation is deeply studied in lattice theory [35]. Keeping this situation in mind Ali et al. [36] introduced the notion of lattice ordered soft sets. In [37] Iftikhar proved some results on lattice ordered double framed soft semirings.

Up to the best of our knowledge so far the in the literature of theoretical mathematics the notion of lattice ordered double framed soft sets is not discussed in the environment of ring theory. So keeping in view the importance of this much desired study, in this paper the notion of lattice ordered double framed soft rings is familiarized. This is not only the original work but also much needed as it is equally important in lattice theory, soft set theory and in ring theory. We also discussed the basic properties of lattice order double framed soft rings and elaborated the effect of lattice order double framed soft rings over the operations of lattice order double framed soft sets. The rest of paper is organized as follows:

Section 2 of the paper consists of some preliminary notions to make the paper self-contained. In section 3 the notion of lattice ordered double framed soft rings is introduced, its basic properties are discussed and the associated results are conferred. Wherever necessary the described results are empowered by examples. In section 4 the paper is concluded and some future directions are discussed.

2. Preliminaries

In this segment we will discuss the elementary ideas of rings, Soft Sets, Double Framed Soft Sets, Lattices and Lattice ordered soft sets. From now on, R will always be a ring, "SbR(R)" be the collection of all subrings of R, the initial universe will be denoted by X, set of parameters be $\wp$, while the power set of X will be $P(X)$ and J, K, L $\subseteq \wp$. We shall denote soft set and double framed soft set by "SS" and "DFSS" respectively.

Definition 2.1. *[5] A set $R \neq \emptyset$ together with binary operations "+" and " $\bullet$" defined on R is called a ring if*
(i) $(R,$ +) is an abelian group,
(ii) $(R, .)$ is semigroup,
(iii) Distributive laws hold.

Definition 2.2. *[5] A $S \neq \emptyset \subseteq R$ is called subring of R if S is itself ring under induced binary operations of R.*

Definition 2.3. *A subset $I \neq \emptyset$ of R is Left Ideal /Right Ideal /Ideal of R*
(i) if $(I, +) \leq (R, +)$ and $\forall\, r\epsilon R,\ a\epsilon I \quad \Rightarrow ra\epsilon I$
(ii) if $(I, +) \leq (R, +)$ and $\forall\, a\epsilon I,\ r\epsilon R \Rightarrow ar\epsilon I$.
(iii) I is both left and right ideal of R.

Definition 2.4. *[9] A soft set on X is a pair $(\rho,\ J)$, $\rho : J \rightarrow P(X)$ is a mapping.*

Definition 2.5. *[18] A soft ring on X is a pair $(\rho,\ J)$, $\rho : J \rightarrow P(X)$ is a mapping, where X is a ring and $P(X)$ are subrings.*

Definition 2.6. *[30] A double framed soft set on X is a pair $< (\rho, \sigma) : \wp >$ here $\rho : \wp \rightarrow P(X)$ and $\sigma : \wp \rightarrow P(X)$ are set valued mappings.*

Definition 2.7. *[30] A double framed soft ring on X is $< (\rho, \sigma) : \wp >$, here $\rho : \wp \rightarrow P(X)$ and $\sigma : \wp \rightarrow P(X)$ are set valued mappings, where X is a ring and $P(X)$ are subrings.*

Definition 2.8. *[30] Let $J, K \subseteq X$. Then the sets J–Inclusive and K-Exclusive of a double framed soft set $< (\rho, \sigma) : \wp >$ are represented and defined as*

$$i_{\wp}(\rho,\ J) = \{t \in \wp : J \subseteq\ \rho(t)\}$$
$$e_{\wp}(\sigma,\ K) = \{t \in \wp : K \supseteq \sigma(t)\}$$

respectively. Then

$$< (\rho, \sigma) : \wp >_{(J,K)} = \{t \in \wp : J \subseteq\ \rho(t), K \supseteq\ \sigma(t)\}$$

is said to be double framed including set of the $< (\rho, \sigma) : \wp >$.
Then obviously

$$< (\rho, \sigma) : \wp >_{(A,\ B)} = i_{\wp}(\rho,\ J) \cap e_{\wp}(\sigma,\ K).$$

Example 2.1. *Let $U = \{a, b, c \ldots z\}$ and $\wp = \{x_{1,} x_{2,}\ x_{3,} x_4\}$ and $< (\rho, \sigma) : \wp >$ be the DFSS over X, where $\rho : \wp \rightarrow P(X)$ mapping is defined by*

$$\rho(x) = \begin{cases} \{a, c, e, g, i, k, m, o, q, s, u, x, z\} & if\ x = x_{1},, \\ \{b, d, f, h, j, l, n, p, r, t, v, y\} & if\ x = x_2, \\ \{a, e, i, o, u\} & if\ x = x_3, \\ \{c, f, i, l, o, r, u, x, y, z\} & if\ x = x_4, \end{cases}$$

and $\sigma : \wp \rightarrow P(X)$ *defined is by*

$$\sigma(x) = \begin{cases} \{a,d,g,j,m,p,s,v,y\} & if\ x = x_{1,}, \\ \{b,e,h,k,n,q,t,w,z\} & if\ x = x_2, \\ \{a,c,d,g,j,m,p,s,v,y,z\} & if\ x = x_3, \\ \{c,f,i,l,o,r,u,x\} & if\ x = x_{4,}, \end{cases}$$

Now if $J = \{a,e,i,o,u\}$ *and* $K = \{a,c,d,g,j,m,p,s,v,y,z\}$*, Then* $i_\wp(\rho,\ J) = \{x_1, x_3\} = e_\wp(\sigma, K)$ *.*
Then obviously $<(\rho,\sigma):\wp>_{(J,K)} = i_\wp(\rho,\ J) \cap e_\wp(\sigma,\ K) = \{x_1, x_3\}$.

Definition 2.9. *[30] The support of DFSS* $<(\rho,\sigma) : J>$ *is represented and defined as* $Supp <(\rho,\sigma) : J> = \{m \in J,\ \rho(m) \neq \emptyset \neq \sigma(m)\}$.

Definition 2.10. *[30] Let* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : K>$ *be two DFSSs over* X*. Then we say that* $<(\rho,\sigma) : J>$ *is a DFS subset of* $<(\tau,\upsilon) : K>$*, if*

(1) $J \subseteq K$
(2) $\rho(m) \subseteq \tau(m)$ *and* $\upsilon(m) \subseteq \sigma(m)$*,* $\forall\ m \in Supp <(\rho,\sigma) : J>$ *.*

Definition 2.11. *[34] Let* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : K>$ *be two DFSRs over* R*. Then the "OR" product of DFSRs is represented and defined as* $<(\rho,\sigma) : J> \vee <(\tau,\upsilon) : K> = <(\lambda,\mu) : L>$*, where* $\lambda(m,n) = \rho(m) \cup \tau(n)$ *and* $\mu(m,n) = \sigma(m) \cap \upsilon(n) for\ all\ (m,n) \in L = J \times K$.

Example 2.2. *Consider* $X = Z_{12} = \{\overline{0},\ \overline{1}\ \overline{2}\ \overline{3} \ldots \overline{11}\}$ *here defined "*$+_{12}$*" and "*$\times_{12}$*" modulo 12 respectively. Let* $J = \{1,2\}$ *and K=* $\{3,4\}$ *and* $\rho,\sigma : J \rightarrow P(X)$ *by* $\rho(1) = \{\overline{0}\},\ \rho(2) = Z_{12},\ \sigma(1) = 3Z_{12}$ *and* $\sigma(2) = Z_{12}$ *and* $\tau,\upsilon : K \rightarrow P(X)$ *by* $\tau(3) = 2Z_{12},\ \ \tau(4) = \{\overline{0}\}, \upsilon(3) = Z_{12}$ *and* $\upsilon(4) = \{\overline{0}\}$ *then the "OR" product of* $<(\rho,\sigma) : J> \vee <(\tau,\upsilon) : K> = <(\lambda,\mu) : L>$ *is*

$$\lambda(1,3) = \rho(1) \cup \tau(3) = \{\overline{0}\} \cup 2Z_{12} = 2Z_{12},$$
$$\lambda(1,4) = \rho(1) \cup \tau(4) = \{\overline{0}\} \cup \{\overline{0}\} = \{\overline{0}\},$$
$$\lambda(2,3) = \rho(2) \cup \tau(3) = Z_{12} \cup 2Z_{12} = Z_{12},$$
$$\lambda(2,4) = \rho(2) \cup \tau(4) = Z_{12} \cup \{\overline{0}\} = Z_{12},$$
$$\mu(1,3) = \sigma(1) \cap \upsilon(3) = 3Z_{12} \cap Z_{12} = 3Z_{12},$$
$$\mu(1,4) = \sigma(1) \cap\ \upsilon(4) = 3Z_{12} \cap \{\overline{0}\} = \{\overline{0}\},$$
$$\mu(2,3) = \sigma(2) \cap\ \upsilon(3) = Z_{12} \cap Z_{12} = Z_{12}$$

and $\mu(2,4) = \sigma(2) \cap \upsilon(4) = Z_{12} \cap \{\overline{0}\} = \{\overline{0}\}$

Definition 2.12. *[34] Let* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : K>$ *be two DFSRs over* R*. Then the union of DFSRs is defined as*

$$<(\rho \cup h,\ \sigma \cap \upsilon) : J \cup K>,$$

Where $\rho \cup \tau\ : J \cup K \longmapsto P(R)$ *is defined by*

$$(\rho \cup \tau)(m) = \left\{ \begin{array}{ll} \rho(m) & if\ m\ \in J \diagdown K \\ h(m) & if\ m \in K \diagdown J \\ \rho(m) \cup\ \tau(m) & if\ m\ \in J \cap K \end{array} \right\}$$

and $\sigma \cap i : J \cup K \longmapsto P(R)$ *is defined by*

$$(\sigma \cap \upsilon)(m) = \left\{ \begin{array}{ll} \sigma(m) & if\ m \in J \diagdown K \\ i(m) & if\ m \in K \diagdown J \\ \sigma(m) \cap \upsilon(m) & if\ m \in J \cap K \end{array} \right\}$$

It is denoted as $< (\rho, \sigma) : J > \cup < (\tau, \upsilon) : K > = < (\rho \cup \tau,\ \sigma \cap \upsilon) : J \cup K >$

Definition 2.13. *[34] The restricted union of two DFSRs* $< (\rho, \sigma) : J >$ *and* $< (\tau, \upsilon) : K >$ *over* R*, is represented and defined as* $< (\rho, \sigma) : J > \cup_r < (\tau, \upsilon) : K > = < (\zeta, \eta) : L >$*, where* $\zeta(m) = \rho(m) \cup \tau(m)$ *&* $\eta(m) = \sigma(m) \cap \upsilon(m),\ \forall\ m \in L = J \cap K \neq \emptyset$*.*

Definition 2.14. *[34] Let* $< (\rho, \sigma) : J >$ *and* $< (\tau, \upsilon) : K >$ *be two DFSRs over* R*. Then the "AND" product of DFSRs is represented and defined as* $< (\rho, \sigma) : J > \wedge < (\tau, \upsilon) : K > = < (\lambda, \mu) : L >$*, where* $\lambda(m, n) = \rho(m) \cap \tau(n)$ *and* $\mu(m, n) = \sigma(m) \cup \upsilon(n),\ \forall\ (m, n) \in A \times B$*..*

Example 2.3. *Consider* $Z_{15} = \{\overline{0},\ \overline{1,2,3} \ldots \overline{14}\}$ *with defined addition "*$+_{15}$*" and multiplication "*$\times_{15}$*" modulo 15 respectively. Let* $J = \{a, b\}$ *and* $K = \{c, d\}$ *and* $\rho, \sigma : J \rightarrow P(Z_{15})$ *by* $\rho(a) = 2Z_{15},\ \rho(b) = 3Z_{15},\ \sigma(a) = \left\{\overline{0}\right\}$ *and* $\sigma(b) = Z_{15}$ *and* $\tau, \upsilon : K \rightarrow P(R)$ *by* $\tau(c) = \left\{\overline{0}\right\}, \tau(d) = 5Z_{15},\ \upsilon(c) = Z_{15}$ *and* $\upsilon(d) = 3Z_{15}$ *and the "AND" product of* $< (\rho, \sigma) : J > \wedge < (\tau, \upsilon) : K > = < (\lambda, \mu) : L >$*, is*

$$\lambda(a, c) = \rho(a) \cap \tau(c) = 2Z_{15} \cap \left\{\overline{0}\right\} = \{\overline{0}\},$$

$$\lambda(a, d) = \rho(a) \cap \tau(d) = 2Z_{15} \cap 5Z_{15} = \{\overline{0}, \overline{5}, \overline{10}\},$$

$$\lambda(b, c) = \rho(b) \cap \tau(c) = 3Z_{15} \cap \left\{\overline{0}\right\} = \{\overline{0}\},$$

$$\lambda(b, d) = \rho(b) \cap \tau(d) = 3Z_{15} \cap 5Z_{15} = \left\{\overline{0}\right\},$$

$$\mu(a, c) = \sigma(a) \cup \upsilon(c) = \left\{\overline{0}\right\} \cup Z_{15} = \{\overline{0},\ \overline{1,2,3} \ldots \overline{14}\},$$

$$\mu(a, d) = \sigma(a) \cup \upsilon(d) = \left\{\overline{0}\right\} \cup 3Z_{15} = 3Z_{15},$$

$$\mu(b, c) = \sigma(b) \cup \upsilon(c) = Z_{15} \cup Z_{15} = Z_{15}$$

and $\mu(b, d) = \sigma(b) \cup \upsilon(d) = Z_{15} \cup 3Z_{15} = Z_{15}$ *.*

Definition 2.15. *[34] The restricted intersection of two DFSRs* $< (\rho, \sigma) : J >$ *and* $< (\tau, i) : K >$ *over the same universal set* R*, is represented and defined as* $< (\rho, \sigma) : J > \cap_r < (\tau, \upsilon) : K > = < (\psi, \omega) : L >$*, where* $\psi(m) = \rho(m) \cap \tau(m)$ *&* $\omega(m) = \sigma(m) \cup \upsilon(m),\ \forall\ m \in L = J \cap K \neq \emptyset$*.*

Definition 2.16. *[34] Let* $< (\rho, \sigma) : J >$ *and* $< (\tau, \upsilon) : K >$ *be two DFSRs over* R*. Then the extended intersection of DFSRs is defined as* $< (\rho \cap_e \tau,\ \sigma \cup_e \upsilon) : J \cup K >$*, where* $\rho \cap_e \tau : J \cup K \longmapsto P(R)$ *is defined as*

$$(\rho \cap_e \tau)(m) = \left\{ \begin{array}{ll} \rho(m) & if\ m \in J \diagdown K \\ h(m) & if\ m \in K \diagdown J \\ \rho(m) \cap \tau(m) & if\ m \in J \cap K \end{array} \right\}$$

and $\sigma\cup_e\upsilon : J\cup K \longmapsto P(R)$ *is defined by*

$$(\sigma\cup_e\upsilon)(m)=\left\{\begin{array}{ll}\sigma(m) & if\ m\in J\diagdown K\\ i(m) & if\ m\in K\diagdown J\\ \sigma(m)\cup\ \upsilon(m) & if\ m\in J\cap K\end{array}\right\}$$

and is denoted as $<(\rho,\sigma):J>\cap_e<(\tau,\upsilon):K>=<(\rho\cap_e\tau,\ \sigma\cup_e\upsilon):J\cup K>$

Definition 2.17. *[35] A binary relation* $\precsim$ *defined on a non-null set* X *is a partial order if the following axioms hold:*

(1) $x\precsim x, \quad \forall\, x\in X$ *(Reflexive)*
(2) $x\precsim y\ and\ y\precsim x \implies x=y \quad for\ x,y\in X$ *(Anti symmetric)*
(3) $x\precsim y\ and\ y\precsim z\Longrightarrow x\precsim z\ \forall\, x,y,z\in X$ *(Transitive)*

Definition 2.18. *[35] Let* $(L,\precsim)$ *be any partial ordered set. Then* L *is called a lattice if* $\forall\, x,y\in L$, $\{x,y\}$ *has supremum and infimum in* L.

Definition 2.19. *[36] A lattice (anti-lattice) ordered soft set over* X *is a pair* (ρ,J) *, where* $\rho:J\to P(X)$ *is a mapping and* $\forall\, x,y\in J$ *if* $x\leq y$ *then* $\rho(x)\subseteq\rho(y)(\rho(x)\supseteq\rho(y))$.

Definition 2.20. *[38] A lattice (anti-lattice) ordered soft ring over* R *is a pair* (ρ,K) *, where* $\rho:K\to P(R)$ *is a mapping and* $\forall\, x,y\in K$ *if* $x\leq y$ *then* $\rho(x)\subseteq\rho(y)(\rho(x)\supseteq\rho(y))$ *and* $\rho(x),\rho(y)\ \in$ *SbR(R).*

3. Lattice (Anti-lattice) Ordered Double Framed Soft Rings

From now onward set of parameters $\wp$ will be lattice.

Definition 3.1. *A DFSS* $<(\rho,\sigma):J>$ *over a ring* R *is said to be lattice (anti-lattice) ordered double framed soft ring over* R, *iff* $\forall\, x,y\in J$, $\rho(x),\rho(y),\sigma(x)\ \&\ \sigma(y)\in$ *SbR(R) and if* $x\preccurlyeq y$ *then* $\rho(x)\subseteq\rho(y)(\rho(x)\supseteq\rho(y))$ *and* $\sigma(x)\supseteq\sigma(y)(\sigma(x)\subseteq\sigma(y))$.

Throughout in this article lattice ordered double framed soft rings will be denoted by LODFSRs.

Example 3.1. *Let* $(G,\ +)$ *be an abelian group and* R *be the set of all endomorphisms of* $(G,\ +)$. *Then for* $\alpha,\ \beta\in R$, *where if we define*

$$(\alpha+\beta)(x)=\ \alpha(x)+\ \beta(x)$$
$$(\alpha.\beta)(x)=\alpha(\beta(x)).$$

Then under these defined " + " and " . " R *is a ring and also* $\{O\}$, *center of* R *(denoted by* $C(R)$*) and* R *itself are subrings of* R.
Now let $J=\{x,y,z\}$ *"where* $x\preccurlyeq y\preccurlyeq z$*" be a set of parameters and* $\rho,\sigma:A\to P(R)$ *be defined by*

$$\rho(x)=\{O\},\rho(y)=C(R),\rho(z)=R,$$
$$\sigma(x)=R,\sigma(y)=C(R),\sigma(z)=\{O\},$$

Clearly $\rho(x)\subseteq\ \rho(y)\subseteq\rho(z)$ *and* $\sigma(x)\supseteq\ \sigma(y)\supseteq\ \sigma(z)$
Then $<(\rho,\sigma):J>$ *is a Lattice ordered double framed soft ring over* R.

Remark.1 In general "OR" product of two LODFSRs is not necessarily a LODFSR.

Example 3.2. *Consider the ring* $Z_{18} = \{\overline{0}, \overline{1}, \overline{2} \ldots \overline{17}\}$ *where* $\overline{0} \preccurlyeq \overline{1} \preccurlyeq \overline{2} \preccurlyeq \cdots \preccurlyeq \overline{17}$ *and let* $J = \{1,2,3,4\}$ *, where* $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4$*,* $K = \{5,6,7\}$ *and* $5 \preccurlyeq 6 \preccurlyeq 7$*, then* $< (\rho,\sigma) : J >$ *and* $< (h,\upsilon) : K >$ *be the LODFSRs over* Z_{18} *and define the set valued mappings* $\rho,\sigma : J \longrightarrow P(Z_{18})$ *by, let* $\rho(1) = \{\overline{0}\} = \rho$**(??)***,* $\rho(2) = \{\overline{0}, \overline{1}, \overline{2} \ldots \overline{17}\} = \rho$**(??)***,* $\sigma(1) = \{\overline{0}\}$*,* $\sigma(2) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}\}$*,* $\sigma(3) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{10}, \overline{12}, \overline{14}, \overline{16}\}$*,* $\sigma(4) = \{\overline{0}, \overline{1}, \overline{2} \ldots \overline{17}\}$ *and* $\tau, \upsilon : K \longrightarrow P(Z_{18})$ *by* $\tau(5) = \{\overline{0}, \overline{1}, \overline{2} \ldots \overline{17}\}$*,* $\tau(6) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}\}$*,* $\tau(7) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{10}, \overline{12}, \overline{14}, \overline{16}\}$ *and* $\upsilon(5) = \{\overline{0}, \overline{1}, \overline{2} \ldots \overline{17}\}$*,* $\upsilon(6) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{10}, \overline{12}, \overline{14}, \overline{16}\}$*,* $\upsilon(7) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}\}$ *, then clearly* $< (\rho,\sigma) : J > \ \wedge \ < (\tau,\upsilon) : K >$ *is not a LODFSR because* $\sigma(2) \cup \upsilon(6) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}\} \cup \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{10}, \overline{12}, \overline{14}, \overline{16}\} = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{9}, \overline{12}, \overline{14}, \overline{15}, \overline{16}\}$ *which is not subring.*

Theorem 3.3. *Let* $< (\rho,\sigma) : J >$ *and* $< (\tau,\upsilon) : K >$ *be two LODFSRs over the same ring* R*. Then* $< (\rho,\sigma) : J > \vee < (\tau,\upsilon) : K > = < (\lambda,\mu)$*:L>, is a LODFSR, provided that* $\rho(m) \cup \tau(n) \in \mathrm{SbR(R)}\ \forall\ (m,n) \in J \times K$.

Proof. By using Definition 2.11 "OR" product of LODFSSs $< (\rho,\sigma) : J > \ \vee < (\tau,\upsilon) : K >$ = $< (\lambda,\mu)$:L>, where $\lambda(m,n) = \rho(m) \cup \tau(n)$ and $\mu(m,n) = \sigma(m) \cap \upsilon(n)$, $\forall\ (m,n) \in J \times K$. Now as $< (\lambda,\mu)$:L$> \neq \emptyset$ be a LODFSR over R. Then $\forall\ (m,n) \in Supp < (\tau,\upsilon) : K > \neq \emptyset$, $\lambda(m,n) = \rho(m) \cup \tau(n) \neq \emptyset$ *and* $\mu(m,n) = \sigma(m) \cap \upsilon(n) \neq \emptyset$. It follows that $\rho(m)$, $\tau(n)$, $\sigma(m)$ and $\upsilon(n) \in$ SbR(R). As intersection of subrings is a subrings so $\mu(m,n) \in$ SbR(R) and $\lambda(m,n) \in$ SbR(R) as given $\sigma(m) \cup \upsilon(n) \in \mathrm{SbR(R)}$ so, $< (\lambda,\mu)$:L> is a DFSR over R, $\forall\ (m,n) \in Sup < (\lambda,\mu)$:L> . Implies that $< (\rho,\sigma) : J > \ \vee < (\tau,i) : K >$ = $< (\lambda,\mu)$:L> is a LODFSR over R.

As $< (\rho,\sigma) : J >$ and $< (\tau,\upsilon) : K >$ are two LODFSR, both J and K are partially ordered sets. Now $m \preccurlyeq_J n$ for all $m,n \in J$ then $\rho(m) \subseteq \rho(n)$, $\sigma(m) \supseteq \sigma(n)$ and $o \preccurlyeq_K p$ for all $o,p \in K$ then $\tau(o) \subseteq \tau(p)$ and $\upsilon(o) \supseteq \upsilon(p)$. Now $\preccurlyeq_L$ be a partial order relation among the elements of $J \times K$. Such that $(m,o) \preccurlyeq (n,p)$; where $(m,o), (n,p) \in J \times K$, note that this order is induced by the elements of J and K. Since $\rho(m) \subseteq \rho(n)$, $\sigma(m) \supseteq \sigma(n)$, $\tau(o) \subseteq \tau(p)$ and $\upsilon(o) \supseteq \upsilon(p)$ and $(m,o) \preccurlyeq_L (n,p)$. Then $\rho(m) \cup \tau(o) \subseteq \rho(n) \cup \tau(p)$ and $\sigma(m) \cap \upsilon(o) \supseteq \sigma(n) \cap \upsilon(p)$. It concludes that $\lambda(m,o) \subseteq \lambda(n,p)$ and $\mu(m,o) \supseteq \mu(n,p)$. As required**.**

□

Remark.2 Usually the union of two LODFSRs over a ring R is not necessarily a LODFSR over R. We explain it by the given example.

Example 3.4. *Consider the ring* $R = Z_{20} = \{\overline{0}, \overline{1}, \overline{2} \ldots \overline{19}\}$ *where* $\overline{0} \preccurlyeq \overline{1} \preccurlyeq \overline{2} \preccurlyeq \cdots \preccurlyeq \overline{19}$ *and let* $J = \{\overline{0}, \overline{2}, \overline{4}\}$*. Now define the mappings* $\rho,\sigma : J \to P(R)$ *by* $\rho(\overline{0}) = \{\overline{0}\}$*,* $\rho(\overline{2}) = 5Z_{20}$*,* $\rho(\overline{4}) = 2Z_{20}$*,* $\sigma(\overline{0}) = 5Z_{20}$*,* $\sigma(\overline{2}) = 2Z_{20}$*,* $\sigma(\overline{4}) = Z_{20}$ *and* $\tau,\upsilon : J \to P(R)$ *by* $\tau(\overline{0}) = 5Z_{20}$*,* $\tau(\overline{2}) = 2Z_{20}$*,* $\tau(\overline{4}) = \{\overline{0}\}$*,* $\upsilon(\overline{0}) = 2Z_{20}$*,* $\upsilon(\overline{2}) = 2Z_{20}$*,* $\upsilon(\overline{4}) = Z_{20}$*. Then obviously* $< (\rho,\sigma) : J >$ *and* $< (\tau,\upsilon) : J >$ *are LODFSRs over* R*. Now by the Definition 2.17, we have* $< (\rho,\sigma) : J > \cup_{\mathfrak{e}} < (\tau,\upsilon) : J > = <$

$(\rho\cup_{\mathfrak{e}}\tau,\ \ g\cap_{\mathfrak{e}}\upsilon) : J> = <(\lambda,\mu) : J>$ *where*

$$\lambda\left(\overline{0}\right) = \rho\left(\overline{0}\right)\cup\tau\left(\overline{0}\right) = \left\{\overline{0}\right\}\cup 5Z_{20} = 5Z_{20}$$
$$\lambda\left(\overline{2}\right) = \rho\left(\overline{2}\right)\cup\tau\left(\overline{2}\right) = 5Z_{20}\cup 2Z_{20} = \{\overline{0},\overline{2},\overline{4},5,\overline{6},\overline{8},\overline{10},\overline{12}\,,\overline{14}\,,\overline{15},\overline{16},\overline{18}\}$$
$$\lambda\left(\overline{4}\right) = \rho\left(\overline{4}\right)\cup\tau\left(\overline{4}\right) = 2Z_{20}\cup\left\{\overline{0}\right\} = 2Z_{20}$$
$$\mu\left(\overline{0}\right) = \sigma\left(\overline{0}\right)\cap\upsilon\left(\overline{0}\right) = 5Z_{20}\cap 2Z_{20} = 10Z_{20}$$
$$\mu\left(\overline{2}\right) = \sigma\left(\overline{2}\right)\cap\upsilon\left(\overline{2}\right) = 2Z_{20}\cap 2Z_{20} = 2Z_{20}$$
$$\mu\left(\overline{4}\right) = \sigma\left(\overline{4}\right)\cap\upsilon\left(\overline{4}\right) = Z_{20}\cap Z_{20} = Z_{20}$$

Here $\lambda\left(\overline{2}\right) = \{\overline{0},\overline{2},\overline{4},5,\overline{6},\overline{8},\overline{10},\overline{12}\,,\overline{14}\,,\overline{15},\overline{16},\overline{18}\}$ *is not a subring. So we conclude that union of two LODFSRs is not necessarily a double framed soft ring.*

Theorem 3.5. *The union of LODFSRs* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : J>$ *over a same ring* R *is LODFSR over* R*, provided that* $\rho(m)$ *is a subring of* $\tau(m)$ *or* $\tau(m)$ *is a subring of* $\rho(m)$, $\forall m\in J$.

Proof. Suppose $\rho\,(m)$ is a subring of $\tau\,(m)$ and $\tau\,(m)$ is a subring of $\rho\,(m)$, then by either case $\rho\,(m)\cup\tau(m)$ is a subring of R. Now consider $(\rho\cup\tau)\,(m)$. If $m\in J\diagdown K$, then $(\rho\cup\tau)\,(m) = \rho(m)$ and if $m\in K\diagdown J$, then $(\rho\cup\tau)\,(m) = \tau\,(m)$. In either case $(\rho\cup\tau)\,(m)\in \mathrm{SbR(R)}$. Now if $m\in J\cap K$ then $(\rho\cup\tau)\,(m) = \rho\,(m)\cup\tau\,(m)$ and $(m)\cup\tau(m)\in\mathrm{SbR(R)}$. Then again $(\rho\cup\tau)\,(m)\in\mathrm{SbR(R)}$. Next consider $(\sigma\cap\upsilon)\,(m)$. If $m\in J\diagdown K$, then $(\sigma\cap\upsilon)\,(m) = \sigma(m)$ and if $m\in K\diagdown J$, then $(\sigma\cap\upsilon)\,(m) = \upsilon(m)\in \mathrm{SbR(R)}$. Furthermore if $m\in J\cap K$, then $(\sigma\cap\upsilon)\,(m)\in\mathrm{SbR(R)}$ because $(\sigma\cap\upsilon)\,(m) = \sigma(m)\cap\upsilon(m)$ then again $(\sigma\cap\upsilon)\,(m)\in\mathrm{SbR(R)}$. Hence $<(\rho,\sigma) : J>\cup <(\tau,\upsilon) : B>$ is LODFSR over R.

As $<(\rho,\sigma) : J>$ and $<(\tau,\upsilon) : K>$ are two LODFSRs over R so for all $m,n\ \in J$ so that $m\preccurlyeq n$ then $\rho\,(m)\subseteq\rho\,(n)$, $\sigma(m)\supseteq\sigma(n)$ similarly for all $m,n\ \in K$ so that $m\preccurlyeq n$ then $h\,(m)\subseteq\tau\,(n)$ and$\upsilon(m)\supseteq\upsilon(n)$. So $\rho\,(m)\cup\tau\,(m)\subseteq\rho\,(n)\cup\ \tau\,(n)$ and $\sigma(m)\cap\upsilon(m)\supseteq\sigma(n)\cap\upsilon(n)$. □

Remark.3 In general restricted union of two LODFSRs is not necessarily a LODFSR.

Example 3.6. *Consider the ring* $R = Z_{14} = \{\overline{0},\ \overline{1},\overline{2},\ \overline{3}\ldots\overline{13}\}$ *where* $\overline{0}\preccurlyeq\overline{1}\preccurlyeq\overline{2}\preccurlyeq\overline{3}\cdots\preccurlyeq\overline{13}$ *and let* $J = \left\{\overline{0},\ \overline{2},\overline{4},\overline{6},\overline{8}\right\}$, *where* $\overline{0}\preccurlyeq\overline{2}\preccurlyeq\overline{4}\preccurlyeq\overline{6}\preccurlyeq\overline{8}$, K= $\left\{\ \overline{1},\overline{3},\ \overline{5},\overline{7},\overline{9}\right\}$ $where\ \overline{1}\preccurlyeq\overline{3}\preccurlyeq\overline{5}\preccurlyeq\overline{7}\preccurlyeq\overline{9}$, $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : K>$ *be the LODFSRs over* R *and define the set valued mappings* $\rho,\sigma : J\longrightarrow P(R)$ *by, let* $\rho\left(\overline{0}\right) = \left\{\overline{0}\right\}$, $\rho\left(\overline{2}\right) = \left\{\overline{0},\ \overline{2},\overline{4},\overline{6},\overline{8}\right\}$, $\rho\left(\overline{4}\right) = \{\overline{0},\ \overline{1},\overline{2},\ \overline{3}\ldots\overline{13}\}$, $\rho\left(\overline{6}\right) = \left\{\overline{0},\ \overline{7}\right\}$, $\rho\left(\overline{8}\right) = \{\overline{0}\ \overline{1},\overline{2},\ \overline{3}\ldots\overline{13}\}$, $\sigma\left(\overline{0}\right) = \sigma\left(\overline{4}\right) = \sigma\left(\overline{8}\right) = \left\{\overline{0}\right\}$, $\sigma\left(\overline{2}\right) = \sigma(\overline{6}) = \{\overline{0},\overline{1},\overline{2},\overline{3}\ldots\overline{13}\}\}$ *and* $\tau,\upsilon : K\longrightarrow P(R)$ *by* $\tau\left(\overline{1}\right) = \left\{\overline{0}\right\}$, $\tau\left(\overline{3}\right) = \left\{\overline{0},\ \overline{7}\right\}$, $\tau\left(\overline{5}\right) = \{\overline{0},\overline{1},\overline{2},\overline{3}\ldots\overline{13}\}$, $\tau\left(\overline{7}\right) = \left\{\overline{0},\ \overline{2},\overline{4},\overline{6},\overline{8}\right\}$, $\tau\left(\overline{9}\right) = \{\overline{0},\ \overline{1},\overline{2},\ \overline{3}\ldots\overline{13}\}$ *and* $\upsilon\left(\overline{1}\right) = \upsilon\left(\overline{5}\right) = \upsilon\left(\overline{9}\right) = \{\overline{0},\ \overline{1},\overline{2},\ \overline{3}\ldots\overline{13}\}$, $\upsilon\left(\overline{3}\right) = \upsilon\left(\overline{7}\right) = \left\{\overline{0}\right\}$, *then* $<(\rho,\sigma) : J>\ \cup_r\ <(\tau,\upsilon) : K>$ *is not a LODFSR because* $\rho\left(\overline{2}\right)\cup\tau\left(\overline{3}\right) = \left\{\overline{0},\ \overline{2},\overline{4},\overline{6},\overline{8}\right\}\cup\left\{\overline{0},\ \overline{7}\right\} = \{\overline{0},\ \overline{2},\overline{4},\overline{6},\overline{7},\overline{8}\}$ *which is not subring of R.*

Theorem 3.7. *Let* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : J>$ *be two LODFSRs over the same ring R, Then the restricted union of* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : J> = <(\lambda,\mu) : L>$ *is a LODFSR over* R *if* $<(\lambda,\mu) : L>\neq\phi$ *and* $\rho\,(m)\cup\tau\,(m)\in\mathrm{SbR(R)}$, $\forall\ m\in J$.

Proof. By using Definition 2.13, $<(\rho,\sigma):J> \cup_r <(\tau,\upsilon):J> = <(\lambda,\mu):L>$, $\lambda(m)=\rho(m)\cup\tau(m)$ *and* $\mu(m)=\sigma(m)\cap\upsilon(m)$, $\forall\, m\in J$. Let $<(\lambda,\mu):L>$ be a LODFSS over R. If $m\in Supp <(\lambda,\mu):L>$ *then* $\lambda(m)=\rho(m)\cup\tau(m)$ is non empty and $\mu(m)=\sigma(m)\cap\upsilon(m)$ is also non empty. So (m), $\tau(m)$, $\sigma(m)$ *and* $\upsilon(m)$ $\in \mathrm{SbR(R)}$. As intersection of subrings is a subring so $\mu(m)\in\mathrm{SbR(R)}$and if $\rho(m)\cup\tau(m)\in\mathrm{SbR(R)}, \forall\, m\in Supp <(\lambda,\mu):L>$ then $<(\rho,\sigma):J> \cup_r <(\tau,\upsilon):J> = <(\lambda,\mu):L>$ is a LODFSRs over R. As $<(\rho,\sigma):J>$ and $<(\tau,\upsilon):J>$ are two LODFSRs over R, thus J is a partially ordered set, so for all $m,n \in J$ so that $m\preccurlyeq n$ then $\rho(m)\subseteq\rho(n)$, $\sigma(m)\supseteq\sigma(n)$, similarly for all $m,n\in J$ so that $m\preccurlyeq n$ then $\tau(m)\subseteq\tau(n)$ and $\upsilon(m)\supseteq\upsilon(n)$. Therefore $m,n\in J$ s.t. $m\preccurlyeq n$ implies that $\rho(m)\subseteq$ ñ(n), $\sigma(m)\supseteq\sigma(n)$ and $\tau(m)\subseteq\tau(n)$ and $\upsilon(m)\supseteq\upsilon(n)$ Thus $\rho(m)\cup\tau(m)\subseteq\rho(n)\cup\tau(n)$ and $\sigma(m)\cap\upsilon(m)\supseteq\sigma(n)\cap i(n)$. This implies that $\lambda(m)\subseteq\lambda(n)$ and $\mu(m)\supseteq\mu(n)$ as required. □

Remark.4 In general "AND" product of two LODFSRs is not necessarily a LODFSR.

Example 3.8. *Consider the ring* $Z_{21}=\{\overline{0},\overline{1},\overline{2},\overline{3}\ldots\overline{20}\}$ *where* $\overline{0}\preccurlyeq\overline{1}\preccurlyeq\overline{2}\preccurlyeq\overline{3}\cdots\preccurlyeq\overline{20}$ *and let* $J=\{0,2,4,6,8\}$ *where* $0\preccurlyeq 2\preccurlyeq 4\preccurlyeq 6\preccurlyeq 8$ *and* $K=\{0,5\}$ *and* $0\preccurlyeq 5$ *and* $<(\rho,\sigma):J>$ *and* $<(h,\upsilon):K>$ *be the LODFSRs over* Z_{21}, *define the set valued mappings* $\rho,\sigma: J\longrightarrow P(Z_{21})$ *by let* $\rho(0)=\{\overline{0},\overline{1},\overline{2},\overline{3}\ldots\overline{20}\}$, $\rho(2)=\{\overline{0},\overline{3},\overline{6},\overline{9,12,15,18}\}$, $\rho(4)=\{\overline{0},\overline{3},\overline{6},\overline{9,12,15,18}\}$, $\rho(6)=\{\overline{0,7},\overline{14}\}$, $\rho(8)=\{\overline{0,7},\overline{14}\}$, $\sigma(0)=\{\overline{0}\}$, $\sigma(2)=\{\overline{0}\}$, $\sigma(4)=\{\overline{0}\}$, $\sigma(6)=\{\overline{0},\overline{3},\overline{6},\overline{9,12,15,18}\}$, $\sigma(8)=\{\overline{0},\overline{3},\overline{6},\overline{9,12,15,18}\}$ *and* $\tau,\upsilon: K\longrightarrow P(Z_{21})$ *by* $\tau(0)=\{\overline{0,7},\overline{14}\}$, $\tau(5)=\{\overline{0},\overline{3},\overline{6},\overline{9,12,15,18}\}$, *and* $\upsilon(0)=\{\overline{0}\}$, $\upsilon(5)=\{\overline{0,7},\overline{14}\}$, *then clearly* $<(\rho,\sigma):J> \wedge <(\tau,\upsilon):K>$ *is not a LODFSR because* $\sigma(6)\cup\upsilon(5)=\{\overline{0},\overline{3},\overline{6},\overline{9,12,15,18}\}\cup\{\overline{0,7},\overline{14}\}=\{\overline{0},\overline{3},\overline{6},\overline{7},\overline{9},\overline{12},\overline{14},\overline{15},\overline{18}\}$ *which is not subring of* R.

Theorem 3.9. *Let* $<(\rho,\sigma):J>$ *and* $<(\tau,\upsilon):K>$ *be two LODFSRs over* R. *Then* $<(\rho,\sigma):J> \wedge <(\tau,\upsilon):K> = <(\lambda,\mu)$*:L>, is a LODFSR, provided that* $\sigma(m)\cup\upsilon(n)\in$ *SbR(R),* $\forall\ (m,n)\in A\times B$.

Proof. By using Definition 2.14 "AND" product of LODFSRs $<(\rho,\sigma):J> \wedge <(\tau,\upsilon):K> = <(\lambda,\mu)$:L>, $\lambda(m,n)=\rho(m)\cap\tau(n)$ and $\mu(m,n)=\sigma(m)\cup\upsilon(n)$, $\forall\ (m,n)\in J\times K$. Now as $<(\lambda,\mu)$:L> $\neq\emptyset$ be a LODFSR. Then $\forall\ (m,n)\in Supp <(\lambda,\mu)$:L>$\neq\emptyset$, $\lambda(m,n)=\rho(m)\cap\tau(n)\neq\emptyset$ *and* $\mu(m,n)=\sigma(m)\cup\upsilon(n)\neq\emptyset$. Which concluded that $\rho(m)$, $\tau(n)$, $\sigma(m)$ and $\upsilon(n)\in$ SbR(R). As intersection of subrings is a subrings so $\lambda(m,n)\in$ SbR(R) and $\mu(m,n)\in$ SbR(R) as given that $\sigma(m)\cup\upsilon(n)\in$ SbR(R), if so, $<(\lambda,\mu)$:L> is a double framed soft ring over R, $\forall\,(m,n)\in Supp <(\lambda,\mu)$:L> . .
$\Rightarrow <(\rho,\sigma):J> \wedge <(\tau,\upsilon):K> = <(\lambda,\mu)$: L> is a LODFSR over R. As $<(\rho,\sigma):J>$ and $<(\tau,\upsilon):K>$ are two LODFSR, both A and B are partially ordered sets. Now $m\preccurlyeq_A n$ for all $m,n\in J$ then $\rho(m)\subseteq\rho(n)$, $\sigma(m)\supseteq\sigma(n)$ and $o\preccurlyeq_B p$ for all $o,p\in K$ then $\tau(o)\subseteq\tau(p)$ and $\upsilon(o)\supseteq\upsilon(p)$. Now $\preccurlyeq_C$ be a partial order relation among the elements of $J\times K$. Such that $(m,o)\preccurlyeq_C(n,p)$; where (m,o) , $(n,p)\in J\times K$, note that this order is induced by the elements of J and K. Since $\rho(m)\subseteq\rho(n)$, $\sigma(m)\supseteq\sigma(n)$, $\tau(o)\subseteq\tau(p)$ and $\upsilon(o)\supseteq\upsilon(p)$ and $(m,o)\preccurlyeq_C(n,p)$. Then $\rho(m)\cap\tau(o)\subseteq\rho(n)\cap\tau(p)$ and $\sigma(m)\cup\upsilon(o)\supseteq\sigma(n)\cup\upsilon(p)$. It concludes that $\lambda(m,o)\subseteq\lambda(n,p)$ and $\mu(m,o)\supseteq$

$\mu(n,p)$. As required.

□

Remark.5 In general restricted intersection of two LODFSRs is not necessarily a LODFSR.

Example 3.10. *Consider the ring* $Z_{24} = \{\overline{0}, \overline{1}, \overline{2} \dots \overline{23}\}$ *where* $\overline{0} \preccurlyeq \overline{1} \preccurlyeq \overline{2} \preccurlyeq \cdots \preccurlyeq \overline{23}$ *and let* $J = \{1,2,3,4\}$, *where* $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4$, $K = \{5,6,7\}$ *where* $5 \preccurlyeq 6 \preccurlyeq 7$ *and* $<(\rho,\sigma) : J>$ *and* $<(h,\upsilon) : K>$ *be the LODFSRs over* Z_{24} *and define the set valued mappings* $\rho, \sigma : J \longrightarrow P(Z_{18})$ *by, let* $\rho(1) = \{\overline{0}, \overline{1}, \overline{2} \dots \overline{23}\}$, $\rho(2) = \{\overline{0}\}$, $\rho(3) = \{\overline{0}, \overline{1}, \overline{2} \dots \overline{23}\}$, $\rho(4) = \{\overline{0}\}$, $\sigma(1) = \{\overline{0}\}$, $\sigma(2) = \{\overline{0}, \overline{24}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}$, $\sigma(3) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\}$, $\sigma(4) = \{\overline{0}, \overline{1}, \overline{2} \dots \overline{23}\}$ *and* $\tau, \upsilon : K \longrightarrow P(Z_6)$ *by* $\tau(5) = \{\overline{0}, \overline{1}, \overline{2} \dots \overline{23}\}$, $\tau(6) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\}$, $\tau(7) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}\}$ *and* $\upsilon(5) = \{\overline{0}, \overline{1}, \overline{2} \dots \overline{23}\}$, $\upsilon(6) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\}$, $\upsilon(7) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\}$, *then* $<(\rho,\sigma) : J> \cap_r <(\tau,\upsilon) : K>$ *is not a LODFSR because* $\sigma(2) \cup \upsilon(6) = \{\overline{0}, \overline{24}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\} \cup \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\} = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{9}, \overline{10}, \overline{12}, \overline{14}, \overline{15}, \overline{16}, \overline{18}, \overline{20}, \overline{21}, \overline{22}\}$ *which is not subring.*

Proposition 3.1. *Let* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : J>$ *be two DFSRs over the same ring R, Then the restricted intersection of* $<(\rho,\sigma) : J> \cap_r <(\tau,\upsilon) : J> = <(\lambda,\mu) : L>$ *is a DFSR over* R *provided that it is non-null and* $\sigma(m) \cup \upsilon(m) \in \mathrm{SbR(R)}$, $\forall\, m \in J$.

Proof. By using Definition 2.15, $<(\rho,\sigma) : J> \cap_r <(\tau,\upsilon) : J> = <(\lambda,\mu) : L>$, $\lambda(m) = \rho(m) \cap \tau(m)$ *and* $\mu(m) = \sigma(m) \cup \upsilon(m)$ $\forall\, m \in J$. Let $<(\lambda,\mu) : L>$ be a DFSR over R. If $m \in Supp <(\lambda,\mu) : L>$ *then* $\lambda(m) = \rho(m) \cap \tau(m)$ is non empty and $\mu(m) = \sigma(m) \cup \upsilon(m)$ is also non empty. So $\rho(m)$, $\tau(m)$, $\sigma(m)$ *and* $\upsilon(m) \in \mathrm{SbR(R)}$. As intersection of subrings is a subring. So $\lambda(m) \in \mathrm{SbR(R)}$ and if $\sigma(m) \cup \upsilon(m) \in \mathrm{SbR(R)}, \forall\, m \in Supp <(\lambda,\mu) : L>$ and so $<(\rho,\sigma) : J> \cap_r <(\tau,\upsilon) : J> = <(\lambda,\mu) : L>$ is a DFSR over R.
As $<(\rho,\sigma) : J>$ and $<(\tau,\upsilon) : J>$ are two LODFSRs over R, thus J is a partially ordered set, so for all $m, n \in J$ so that $m \preccurlyeq n$ then $\rho(m) \subseteq \rho(n)$, $\sigma(m) \supseteq \sigma(n)$ similarly for all $m, n \in J$ so that $m \preccurlyeq n$ then $h(m) \subseteq h(n)$ and $\upsilon(m) \supseteq \upsilon(n)$. Therefore $m, n \in J$ s.t. $m \preccurlyeq n$ implies that $\rho(m) \subseteq \rho(n)$, $\sigma(m) \supseteq \sigma(n)$ and $\tau(m) \subseteq \tau(n)$ and $\upsilon(m) \supseteq \upsilon(n)$ Thus $\rho(m) \cap \tau(m) \subseteq \rho(n) \cap \tau(n)$ and $g(m) \cup \upsilon(m) \supseteq \sigma(n) \cup \upsilon(n)$. This implies that $\lambda(m) \supseteq \lambda(n)$ and $\mu(m) \subseteq \mu(n)$ as required.

□

Remark.6 In general extended intersection of two LODFSRs is not necessarily a LODFSR.

Example 3.11. *Consider the ring* $Z_{12} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$ *where* $\overline{0} \preccurlyeq \overline{1} \preccurlyeq \overline{2} \preccurlyeq \overline{3} \cdots \preccurlyeq \overline{11}$ *and let* $J = K = \{1,2,3\}$ *where* $1 \preccurlyeq 2 \preccurlyeq 3$ *and* $<(\rho,\sigma) : J>$ *and* $<(\tau,\upsilon) : K>$ *be the LODFSRs over* Z_{12}, *define the set valued mappings* $\rho, \sigma : J \longrightarrow P(Z_{12})$ *by let* $\rho(1) = \{\overline{0}\}$, $\rho(2) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, $\rho(3) = \{\overline{0}\}$, $\sigma(1) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, $\sigma(2) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$, $\sigma(3) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, *and* $\tau, \upsilon : K \longrightarrow P(Z_6)$ *by* $\tau(1) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, $\tau(2) = \{\overline{0}\}$, $\tau(3) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, *and* $\upsilon(1) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, $\upsilon(2) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$, $\upsilon(3) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3} \dots \overline{11}\}$, *then* $<(\rho,\sigma) : J> \cap_e <(\tau,\upsilon) : K>$ *is not DFSR because* $\sigma(2) \cup \upsilon(2) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\} \cup \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\} = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}, \overline{10}\}$ *which is not subring.*

Theorem 3.12. *The extended intersection of LODFSRs* $<(\rho,\sigma):J>$ *and* $<(\tau,\upsilon):K>$ *over a same ring* R *is LODFSR over* R*, provided that* $\sigma(m)$ *is a subring of* $\upsilon(m)$ *or* $\upsilon(m)$ *is a subring of* $\sigma(m)$, $\forall m\in J\cap K$.

Proof. Suppose $\sigma(m)$ is a subring of $\upsilon(m)$ and $\upsilon(m)$ is a subring of $\sigma(m)$, then by either case $\sigma(m)\cup\upsilon(m)\ \in \mathrm{SbR(R)}$. Now consider $(\rho\cap\tau)(m)$, if $m\in J\diagdown K$, then $(\rho\cap\tau)(m)=\rho(m)$ and if $m\in K\diagdown J$, then $(\rho\cap\tau)(m)=\tau(m)$. In either case $(\rho\cap\tau)(m)\in\mathrm{SbR}(\mathrm{R})$. Now if $m\in J\cap K$ then $(\rho\cap\tau)(m)=\rho(m)\cap\tau(m)$, then again $(\rho\cap\tau)(m)\in\mathrm{SbR(R)}$. Next consider $(\sigma\cup\upsilon)(m)$, if $m\in J\diagdown K$ then $(\sigma\cup\upsilon)(m)=\sigma(m)$ and if $m\in K\diagdown J$, then $(\sigma\cup\upsilon)(m)=\upsilon(m)\in\mathrm{SbR(R)}$. Furthermore if $m\in J\cap K$, then $(\sigma\cup\upsilon)(m)\in\mathrm{SbR(R)}$ because $(\sigma\cup\upsilon)(m)=\sigma(m)\cup\upsilon(m)$ and $\sigma(m)\cup\upsilon(m)$ $\in\mathrm{SbR(R)}$. Hence $<(\rho,\sigma):J>\cap_e\ <(\tau,\upsilon):K>$ is LODFSR over R. As $<(\rho,\sigma):J>$ and $<(\tau,\upsilon):K>$ are two LODFSRs over R, thus J is a partially ordered set, so for all $m,n\ \in J$ so that $m\preccurlyeq n$ then $\rho(m)\subseteq\rho(n)$, $\sigma(m)\supseteq\sigma(n)$ similarly for all $m,n\ \in K$ so that $m\preccurlyeq n$ then $\tau(m)\subseteq\tau(n)$ and $\upsilon(m)\supseteq\upsilon(n)$. Thus $\rho(m)\cap h(m)\subseteq\rho(n)\cap\ h(n)$ and $\sigma(m)\cup\upsilon(m)\supseteq\sigma(n)\cup\upsilon(n)$. Which completes the proof. □

4. CONCLUSION

In this article we have introduced the notion of lattice ordered double framed soft rings. We discussed that how the operations of union, intersection, AND product and OR product of double framed soft sets are affected in the environment of lattice ordered double framed soft rings. Similarly the behavior of these operations can be checked in the environment of lattice ordered double framed soft ideals in rings, anti-lattice ordered double framed soft rings and anti-lattice ordered double framed soft ideals in rings. This study can further be carried for other algebraic structures.

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