Punjab University Journal of Mathematics (2021),53(5),319-327
https://doi.org/10.52280/pujm.2021.530502

On Bohr Radius of Certain Analytic Functions with Negative Coefficients

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Received: 07 July, 2020 / Accepted: 12 March, 2021 / Published online: May 26, 2021

Abstract.: The key purpose of this paper is to investigate the Bohr radius for several subclasses of analytic functions with negative coefficients. Our investigation with the Bohr radius correlates with the classes of generalized Janowski type functions. Under this novel strategy, we develop Bohr's phenomenon for a generalized class associated with q-functions having $q \in (0, 1)$. In the applications viewpoint, our consequences have shown the applicability in the class inaugurated by Bessel functions.

AMS (MOS) Subject Classification Codes: 30C45; 30C50
Key Words: Analytic functions with negative coefficients, Open unit disk, Bohr radius, Subordination

1. Introduction

In 1914, H.Bohr [7] proved that if

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \tag{1. 1}$$

is analytic in the open unit disc $\nabla = \{z \in \mathbb{C} : |z| < 1\}$ and $|g(z)| < 1, \quad \forall z \in \mathbb{C}$.
Then

$$\sum_{n=0}^{\infty} |g_n| r^n \leq 1, \text{ where } |z| \leq \frac{1}{6}, \forall z \in \nabla.$$

The above inequality is called Bohr's Inequality. It was proved by Wiener, Reisz and Schur that the best possible value of $|z|$ for which the inequality holds is $\frac{1}{3}$. So this number is called the Bohr radius for the analytic functions from the domain ∇ to ∇. Later on the

concept was generalized on the classes of functions from ∇ to some other domain $\Theta \subseteq \mathbb{C}$ see [1, 2, 3].
The utilization of Bohr's theorem in classification of Banach algebras by Dixon[9], attracted the interest of many mathematicians in Bohr's phenomenon. The study of generalization of Bohr's theorem in different directions paved the way forward to a vast research in this area.
Recently an extension of the Bohr's inequality to the disc models of hyperbolic plane has been established by Ali and Ng in [4]. Kayumov and Ponnusamy introduced p-Bohr radius in [13], and have also given an improved version of Bohr's inequality in [14].

A class Υ containing analytic functions $g(z)$ as defined in (1.1) in the open unit disc ∇ will satisfy Bohr's phenomenon if $\exists$ an $r^\star > 0$ such that for every $g \in \Upsilon$ the following inequality is satisfied

$$\varrho(\sum_{n=0}^{\infty} |g_n z^n|, |g(0)|) = \sum_{n=1}^{\infty} |g_n z^n| \leq \varrho\,(g(0), bd(g(\nabla))), \quad \forall z \in \nabla, \tag{1. 2}$$

where ϱ is the euclidean distance and $bd(g(\nabla)$ denotes the boundary of image of ∇ under g. Eq (1.2) is called the distance formulation of Bohr's inequality. Now let A be the class of normalized analytic functions

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{1. 3}$$

where $g(0) = 0$ and $g'(0) = 1$. A function $f(z)$ is said to be subordinate to $g(z)$ in the class A, denoted by $f(z) \prec g(z)$, if there exist a function $\Psi(z)$ in A, with the conditions $\Psi(0) = 0$ and $|\Psi(z)| < 1$, such that $f(z) = g(\Psi(z))$.
For two analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ we define the Hadamard product or convolution denoted by the symbol $*$ as

$$f * g = z + \sum_{n=2}^{\infty} a_n g_n z^n.$$

The class $P[\Lambda, \Theta]$ of Janowski type functions introduced by Janowski in [12] contains analytic function

$$\zeta(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{1. 4}$$

such that

$$\zeta(z) \prec \tfrac{1+\Lambda z}{1+\Theta z}, -1 \leq \Theta < \Lambda \leq 1 \quad \text{and} \quad z \in \nabla.$$

Definition 1.1. *Let $\zeta(z)$ be of the form (1.4). Then $\zeta(z) \in P_m[\Lambda, \Theta]$, if and only if*

$$\zeta(z) = (\tfrac{m}{4} + \tfrac{1}{2})\zeta_{(1)}(z) - (\tfrac{m}{4} - \tfrac{1}{2})\zeta_{(2)}(z), \quad \forall m \geq 2,\ z \in \nabla \text{ and } -1 \leq \Theta < \Lambda \leq 1,$$

where $\zeta_{(1)}(z), \zeta_{(2)}(z) \in P[\Lambda, \Theta]$, .

For $m = 2$ we get the class of Janowski type functions.

Definition 1.2. *The function $\zeta(z) \in P^\alpha[\Lambda, \Theta]$, iff $\zeta(z) \prec (\frac{1+\Lambda z}{1+\Theta z})^\alpha$, where $0 \leq \alpha \leq 1$ and $-1 \leq \Theta < \Lambda \leq 1, z \in \nabla$. For $\alpha = 1$ we get the class $P[\Lambda, \Theta]$.*

Definition 1.3. *Let τ_1 be the class of analytic functions, defined by $\zeta(z) = 1 - \sum_{n=1}^{\infty} b_n z^n$. Then $\tau_1 P^\alpha[\Lambda, \Theta]$ and $\tau_1 P_m[\Lambda, \Theta]$ will be defined as:*

$$\tau_1 P^\alpha[\Lambda, \Theta] = \tau_1 \cap P^\alpha[\Lambda, \Theta], \tag{1. 5}$$

$$\tau_1 P_m[\Lambda, \Theta] = \tau_1 \cap P_m[\Lambda, \Theta]. \tag{1. 6}$$

A very modern and major development in the field of geometric function theory is the study of q analog of analytic functions, which caught the interest of many researchers due to its vast applications in different fields of sciences. The q-derivative operator D_q, of an analytic function $g(z) \in A$ has been introduced by Jackson in [11]. The power series representation of this operator is given by

$$D_q(z) = 1 - \sum_{n=2}^{\infty} [n]_q g_n z^{n-1}, \tag{1. 7}$$

where $g(z)$ is as defined in (1.3). Here $[n]_q = 1 + q + q^2 + ... + q^{n-1}$, $[0]_q = 0$, $[1]_q = 1$ and $\lim_{q \to 1^-} [n]_q = n$.
Let τ be the class of normalized analytic functions with negative coefficients, defined as g$\in \tau$ if and only if,

$$g(z) = z - \sum_{n=2}^{\infty} g_n z^n, \quad \forall \quad z \in \nabla. \tag{1. 8}$$

Definition 1.4. [8] *Let $g \in \tau$ be of the form defined in (1.8). Then g is said to be in the class $TST(q, \lambda, k; \Lambda, \Theta), q \in (0, 1), k \geq 0, -1 \leq \Theta < \Lambda \leq 1$ and $0 \leq \lambda \leq 1$, if and only if $\forall z \in \nabla$, the following condition is satisfied*

$$\Re \left\{ \frac{(\Theta - 1) z d_q g(z) / [(1 - \lambda) z + \lambda g(z)] - (\Lambda - 1)}{(\Theta + 1) z d_q g(z) / [(1 - \lambda) z + \lambda g(z)] - (\Lambda + 1)} \right\} > k \left| \frac{(\Theta - 1) z d_q g(z) / [(1 - \lambda) z + \lambda g(z)] - (\Lambda - 1)}{(\Theta + 1) z d_q g(z) / [(1 - \lambda) z + \lambda g(z)] - (\Lambda + 1)} - 1 \right|. \tag{1. 9}$$

2. Preliminaries

Lemma 2.1. [17] *Let $\zeta(z)$ as defined in (1.4) be in $P_m[\Lambda, \Theta]$, then for $m \geq 2$, $-1 \leq \Theta < \Lambda \leq 1$ and $z \in \nabla$,*
(i) $|\zeta(z) - \frac{1 - \Lambda\Theta r^2}{1 - \Theta^2 r^2}| \leq \frac{(\frac{m}{2})(\Lambda - \Theta) r}{1 - \Theta^2 r^2}$, *where* $|z| = r < 1$.
(ii) $|b_n| \leq (\frac{m}{2})(\Lambda - \Theta)$.

Lemma 2.2. [10] *Let $\zeta(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and $H(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$, be analytic functions in ∇ such that, $\zeta(z) \prec H(z)$. If $H(z)$ is univalent and convex in ∇, then $|b_n| \leq |h_1|$ for all $n \geq 1$.*

Lemma 2.3. *Let $\zeta(z)$ of the form (1.4) be in $P^\alpha[\Lambda, \Theta]$, where $0 \leq \alpha \leq 1$ and $-1 \leq \Theta < \Lambda \leq 1$, then for all $z \in \nabla$,*
(i) $|b_n| \leq \alpha(\Lambda - \Theta), \quad \forall \quad n \geq 1$.
(ii) $(\frac{1 - \Lambda r}{1 - \Theta r})^\alpha \leq |\zeta(z)| \leq (\frac{1 + \Lambda r}{1 + \Theta r})^\alpha$.

Proof. (i) it is easy to show that $(\frac{1+\Lambda z}{1+\Theta z})^{\alpha}$ is convex univalent in ∇. Therefore by using Lemma [2.2], we get the required coefficient bound.
(ii) Since $\zeta(z) \prec (\frac{1+\Lambda z}{1+\Theta z})^{\alpha}$, therefore $(\zeta(z))^{\frac{1}{\alpha}} \prec (\frac{1+\Lambda z}{1+\Theta z})$. Now let $\zeta(z) = (\zeta(z))^{\frac{1}{\alpha}}$, then $\zeta(z) \in P[\Lambda, \Theta]$, and by the distortion results for Janowski type functions we know that

$$(\frac{1-\Lambda r}{1-\Theta r}) \leq |\zeta(z)| \leq (\frac{1+\Lambda r}{1+\Theta r}),$$

which gives the required result. □

Lemma 2.4. [8] *Let* $g(z) = z - \sum_{n=2}^{\infty} g_n z^n \in TST(q, \lambda, k; \Lambda, \Theta), q \in (0,1), k \geq 0, -1 \leq \Theta < \Lambda \leq 1$ *and* $0 \leq \lambda \leq 1$*, then* $\forall z \in \nabla$ *and* $|z| = r < 1$*, we have*
$r - \frac{|\Theta-\Lambda|}{2(k+1)([2]_q-\lambda)+|[2]_q(\Theta+1)-\lambda(\Lambda+1)|} r^2 \leq |g(z)| \leq r + \frac{|\Theta-\Lambda|}{2(k+1)([2]_q-\lambda)+|[2]_q(\Theta+1)-\lambda(\Lambda+1)|} r^2.$

3. Main Results

Theorem 3.1. *Let* $\zeta(z)$ *defined in (1.4) be in* $P_m[\Lambda, \Theta]$*, then for* $m \geq 2$*,* $z \in \nabla$ *and* $-1 \leq \Theta < \Lambda \leq 1$*,* $\sum_{n=1}^{\infty} |b_n z^n| \leq \varrho(1, bd(\zeta(\nabla)))$ *for* $|z| = r^{\star}$*, where* $r^{\star} = \frac{2|\Theta|-m}{2|\Theta|-2m+m\Theta^2}$ *is the Bohr radius.*

Proof. By Lemma[2.1], we can see that the the circle $|\zeta(z) - \frac{1-\Lambda\Theta}{1-\Theta^2}| = \frac{m}{2}(\frac{\Lambda-\Theta}{1-\Theta^2})$ is the boundary of the disc containing $\zeta(\nabla)$. The diametric end points of this disc are

$$\frac{(1-\Lambda\Theta)-\frac{m}{2}(\Lambda-\Theta)}{1-\Theta^2} \text{ and } \frac{(1-\Lambda\Theta)+\frac{m}{2}(\Lambda-\Theta)}{1-\Theta^2}.$$

So the distance of the boundary of the disc $\zeta(\nabla)$ from $\zeta(0)$ is

$$\begin{aligned}\varrho(\zeta(0), bd(\zeta(\nabla))) &= min\left\{\left|1 - \frac{(1-\Lambda\Theta)-\frac{m}{2}(\Lambda-\Theta)}{1-\Theta^2}\right|, \left|1 - \frac{(1-\Lambda\Theta)+\frac{m}{2}(\Lambda-\Theta)}{1-\Theta^2}\right|\right\}, \\ &= \frac{(\frac{m}{2}-|\Theta|)(\Lambda-\Theta)}{1-\Theta^2}. \qquad (3.10)\end{aligned}$$

Now since

$$\sum_{n=1}^{\infty} |b_n z^n| \leq \sum_{n=1}^{\infty} r^n b_n,$$

then by using Lemma 2.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty} r^n b_n &\leq \frac{m}{2}(\Lambda-\Theta)\frac{r}{1-r}, \\ &\leq \frac{(\frac{m}{2}-|\Theta|)(\Lambda-\Theta)}{1-\Theta^2}, \qquad \forall \quad 0 < r < r^{\star} \\ &< \varrho(\zeta(0), bd(\zeta(\nabla))),\end{aligned}$$

where $r^{\star}$ can be obtained by solving the equation

$$\frac{m}{2}(\Lambda-\Theta)\frac{r^{\star}}{1-r^{\star}} = \frac{(\frac{m}{2}-|\Theta|)(\Lambda-\Theta)}{1-\Theta^2}.$$

Hence Bohr's inequality holds for all $0 < r < r^{\star}$. □

Corollary 3.2. *For m=2 in* $P_m[\Lambda, \Theta]$*, we get* $r^{\star} = \frac{1}{|\Theta|+2}$*, for the class* $P[\Lambda, \Theta]$ *obtained by Ali in* [5].

Theorem 3.3. *Let $\zeta(z) \in P^{\alpha}[\Lambda, \Theta]$, where $0 \leq \alpha \leq 1$ and $-1 \leq \Theta < \Lambda \leq 1$. Then the Bohr's Inequality holds for all $0 < r < r^{\star}$, where*

$$r^{\star} = \frac{1}{1 + \alpha(\Lambda - \Theta)^{1-\alpha}(1 + |\Theta|^{\alpha})}. \tag{3. 11}$$

Proof. By using Lemma [2.2], we find that the end points of the diameter of the boundary of $\zeta(\nabla)$ are $(\frac{1-\Lambda}{1-\Theta})^{\alpha}$ and $(\frac{1+\Lambda}{1+\Theta})^{\alpha}$. Therefore, we can easily see that

$$\varrho(1, \zeta(\nabla)) = \tfrac{(\Lambda-\Theta)^{\alpha}}{1-|\Theta^{\alpha}|}.$$

It is easy to verify that $r^{\star}$ given in eq [3.11], is the root of the equation

$$\alpha(\Lambda - \Theta)\tfrac{r}{1-r} = \tfrac{(\Lambda-\Theta)^{\alpha}}{1-|\Theta^{\alpha}|}.$$

Now as

$$\begin{aligned} \sum_{n=1}^{\infty} |b_n z^n| &\leq \alpha(\Lambda - \Theta)\frac{r}{1-r}, \\ &\leq \frac{(\Lambda - \Theta)^{\alpha}}{1 - |\Theta^{\alpha}|}, \qquad \forall \quad 0 < r < r^{\star} \\ &\leq \varrho(1, \zeta(\nabla)). \end{aligned} \tag{3. 12}$$

The equality holds for $r = r^{\star}$. □

Corollary 3.4. *(i)For $\alpha = 1$, we have the class of Janowski type functions.*
(ii) For $\alpha = \frac{1}{2}$, we have $r^{\star} = \frac{1}{1+\frac{1}{2}(\Lambda-\Theta)^{\frac{1}{2}}(1+|\Theta|^{\frac{1}{2}})}$.
(iii) For $\Lambda = 1, \Theta = -1$, $r^{\star} = \frac{1}{1+\alpha(2)^{\alpha-1}(1+(-1)^{\alpha})}$ for the class P^{α}. And if we also take $\alpha = 1$ we get the result for the class of caratheodory functions.
(iv) For $\Lambda = 1 - 2\gamma, \Theta = -1$, we get the class $P^{\alpha}(\gamma)$ with $r^{\star} = \frac{1}{1+\alpha(2-2\gamma)^{\alpha-1}(1+(-1)^{\alpha})}$.

Theorem 3.5. *Let $\zeta(z) = 1 - \sum_{n=1}^{\infty} b_n z^n \in \tau_1 P_m[\Lambda, \Theta]$, then*
(i)$\sum_{n=1}^{\infty} |b_n| \leq \frac{\frac{m}{2}(\Lambda-\Theta)}{1-\frac{m}{2}\Theta}$.
(ii) $\sum_{n=1}^{\infty} |b_n z^n| \leq \varrho(1, bd(\zeta(\nabla)))$, for all $r < 1$.

Proof. (i) Since

$$|1 - \zeta(z)| \leq |\tfrac{m}{2}(\Lambda - \Theta\zeta(z))|,$$

so at $z = r$

$$(1 - \tfrac{m}{2}\Theta) \textstyle\sum_{n=1}^{\infty} |b_n z^n| \leq \tfrac{m}{2}(\Lambda - \Theta).$$

Now as $Re(\zeta(z)) > 0$, therefore $\sum_{n=1}^{\infty} |b_n z^n| < 1$ in ∇. So we have

$$\textstyle\sum_{n=1}^{\infty} |b_n||r^n| \leq \frac{\frac{m}{2}(\Lambda-\Theta)}{1-\frac{m}{2}\Theta}.$$

Hence when $r \longrightarrow 1$, we get (i).
(ii) From (i) we can see that the diametric end points of $bd(\zeta(\nabla))$ are

$$d_1 = \tfrac{1-\frac{m}{2}\Lambda}{1-\frac{m}{2}\Theta} \text{ and } d_2 = \tfrac{1+\frac{m}{2}\Lambda-m\Theta}{1-\frac{m}{2}\Theta}.$$

Therefore, the distance between the boundary of $\zeta(\nabla)$ and $\zeta(0)$ is

$$\varrho(1, bd(\zeta(\nabla))) = min\{|1-d_1|, |1-d_2|\} = \frac{\frac{m}{2}(\Lambda-\Theta)}{1-\frac{m}{2}\Theta}.$$

Now we see that

$$\begin{aligned}\sum_{n=1}^{\infty}|b_n z^n| &\leq \frac{\frac{m}{2}(\Lambda-\Theta)}{1-\frac{m}{2}\Theta}r,\\ &\leq \frac{\frac{m}{2}(\Lambda-\Theta)}{1-\frac{m}{2}\Theta},\\ &\leq \varrho(1, bd(\zeta(\nabla))), \quad \forall \quad r<1,\end{aligned}$$

we get the equality for $r=1$. □

The result is best possible for the function

$$\zeta_o(z) = 1 - \frac{\frac{m}{2}(\Lambda-\Theta)}{1-\frac{m}{2}\Theta}z.$$

Theorem 3.6. *Let* $\zeta(z) = 1 - \sum_{n=1}^{\infty} b_n z^n \in \tau_1 P^{\alpha}[\Lambda, \Theta]$, *then*
(i)$|\zeta(z)-1| \leq \frac{\alpha(\Lambda-\Theta)}{1-\alpha\Theta}r$, *for all* $z \in \nabla$.
(ii)$\zeta(z)$ *satisfies Bohr's inequality with Bohr radius* $r^\star = 1$.

Proof. The proof is same as of theorem[3.3]. The result is sharp for the function

$$\zeta_1(z) = 1 - \frac{\alpha(\Lambda-\Theta)}{1-\alpha\Theta}z.$$

□

It is interesting to note that the Bohr radius for the class of Janowski type functions with negative coefficients and its generalized classes $\tau_1 P_m[\Lambda, \Theta]$ and $\tau_1 P^{\alpha}[\Lambda, \Theta]$ is 1.

Theorem 3.7. *Let* $g \in TST(q, \lambda, k; \Lambda, \Theta), q \in (0,1), k \geq 0, -1 \leq \Theta < \Lambda \leq 1$ *and* $0 \leq \lambda \leq 1$, *then for all* $z \in \nabla$, $g(z)$ *satisfies Bohr's phenomenon for*

$$r^\star = \frac{2(s+|t-v|-|w|)}{s+|t-v|+\sqrt{[s+|t-v|]^2+4|w|(s+|t-v|-|w|)}}, \tag{3. 13}$$

where $s = 2(k+1)([2]_q - \lambda), t = [2]_q(\Theta+1), v = \lambda(\Lambda+1)$ *and* $w = \Theta - \Lambda$.

Proof. Let $g(z) = z - \sum_{n=2}^{\infty} g_n z^n$, then by using Lemma[2.4], we get

$$\varrho(g(0), bd(g(\nabla))) \geq 1 - \frac{|\Theta-\Lambda|}{2(k+1)([2]_q-\lambda)+|[2]_q(\Theta+1)-\lambda(\Lambda+1)|}.$$

Now substituting $s = 2(k+1)([2]_q - \lambda), t = [2]_q(\Theta+1), v = \lambda(\Lambda+1)$ and $w = \Theta - \Lambda$, we get

$$\varrho(g(0), bd(g(\nabla))) \geq 1 - \frac{|w|}{s+|t-v|}.$$

By taking

$$g_o(z) = z + \frac{|w|}{s+|t-v|}z^2, \tag{3. 14}$$

then again by Lemma[2.4], we can write

$$-g_o(-r) \leq |g(z)| \leq g_o(r), \text{ where } |z| = r < 1.$$

Let $r^\star$ be the root of the equation $g_o(r) + g_o(-1) = 0$, then we will get the expression (3.13). Now consider

$$\begin{aligned} |z| + \sum_{n=2}^{\infty} |g_n z^n| &\leq r + \sum_{n=2}^{\infty} g_n r^n, \\ &\leq r + \frac{|w|}{s + |t - v|} r^2, \\ &\leq 1 - \frac{|w|}{s + |t - v|}, \qquad \forall \quad 0 \leq r \leq r^\star, \\ &\leq \varrho(0, bg(g(\nabla))). \end{aligned}$$

The equality holds for $r = r^\star$. The result is sharp for the function $g_o(z)$ defined in eq (3.14) . □

Corollary 3.8. *For $\lambda = 1, \Lambda = 1 - 2\gamma, 0 \leq \gamma < 1, \Theta = -1$ and $q \longrightarrow 1^-$, we obtain the class $\tau \cap S_p(k, \gamma)$, where $S_p(k, \gamma)$ is the class of k-uniformly starlike functions of order γ see* [16]. *So applying theorem 3.5, we have $s = 2(k + 1), t = 0, v = 2 - 2\gamma$ and $w = 2\gamma - 2$. Therefore, the Bohr radius for the class $\tau S_p(k, \gamma)$ of k-uniformly starlike functions of order γ with negative coefficients is*

$$r^\star = \frac{2(k+1)}{(k+1)+|\gamma-1|+\sqrt{(k+1)^2+|\gamma-1|^2+6(k+1)(\gamma-1)}}.$$

If we also take $k = 1$ and $\gamma = 0$, then we get the class $\tau \cap S_p$, where S_p is the class of uniformly starlike functions defined in [19], *therefore for the class τS_p we have*

$$r^\star = \frac{4}{3+\sqrt{17}}.$$

Corollary 3.9. *For $\lambda = 1, q \longrightarrow 1^-$, we have the class $\tau \cap k - ST[\Lambda, \Theta]$, where the class $k - ST[\Lambda, \Theta]$ was investigated by Noor and Malik in* [15]. *So for the class $\tau k - ST[\Lambda, \Theta]$, we get $s = 2(k + 1), t = 2(\Theta + 1), v = (\Lambda + 1)$ and $w = \Theta - \Lambda$. Hence the Bohr radius for this class is*

$$r^\star = \frac{2[2(k+1)+|2\Theta-\Lambda+1|-|\Theta-\Lambda|]}{2(k+1)+|2\Theta-\Lambda+1|+\sqrt{[2(k+1)+|2\Theta-\Lambda+1|]^2+4|\Theta-\Lambda|[2(k+1)+|2\Theta-\Lambda+1|-|\Theta-\Lambda|]}}.$$

If we take $\Lambda = 1, \Theta = -1$ then we obtain

$$r^\star = \frac{2(k+1)}{(k+2)+\sqrt{k^2+8k+8}}.$$

Corollary 3.10. *For $\Lambda = 1 - 2\gamma, 0 \leq \gamma < 1, \Theta = -1, \lambda = 1, k = 0$ and $q \longrightarrow 1^-$, we obtain the class $\tau S^*(\gamma)$ of starlike functions of order γ with negative coefficients. So for this class we get*

$$r^* = \frac{2}{\gamma+\sqrt{\gamma^2-8\gamma+8}}.$$

Corollary 3.11. *For $\Lambda = 1, \Theta = -1, \lambda = 0, k = 0$ and $q \longrightarrow 1^-$, we get the class $\tau R = \tau \cap R$, where $R = \{g \in A : \Re(g'(z)) > 0\}$. For the class τR we obtain*

$$r^* = \frac{1}{1+\sqrt{2}}.$$

4. Application

As an application of our work, we have obtained the Bohr radius of class defined by using Bessel functions. Bessel functions are the special functions, used to study solutions of differential equations. The class of generalized Bessel functions of first kind of order u, consists of the functions

$$\upsilon_(u,c,d)(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n!\Gamma(u+n+\frac{c+1}{2})}(\frac{z}{2})^{2n+u}. \quad (4.\ 15)$$

Eq (4.15) is the particular solution of the following equation, see [6].

$$z^2 v''(z) + czv'(z) + [dz^2 - u^2 + (1-c)u]v(z) = 0,\ u, c, d \in \mathbb{C}.$$

The function $\varphi(u,c,d)(z)$ in terms of $\upsilon_(u,c,d)(z)$ can be defined as

$$\varphi(u,c,d)(z) = 2^u \Gamma(u + \tfrac{c+1}{2}) z^{1-\frac{u}{2}} \upsilon_(u,c,d)(\sqrt{z}).$$

By using the well known Pochhammer symbol defined as:

$$(\delta)_\varepsilon = \tfrac{\Gamma(\delta+\varepsilon)}{\Gamma(\varepsilon)} = \begin{cases} 1 & if \varepsilon = 0 \\ \delta(\delta+1)...(\delta+\varepsilon-1) & if \varepsilon \in \mathbb{N}. \end{cases}$$

We get the following series representation for $\varphi(u,c,d)(z)$

$$\varphi(u,c,d)(z) = z + \sum_{n=1}^{\infty} \frac{(\frac{-d}{4})^n}{(\kappa)_n n!} z^{n+1}, \quad (4.\ 16)$$

where $\kappa = u + \frac{c+1}{2} \notin \{0,-1,-2,...\}$. For our ease we write $\varphi(u,c,d)(z) = \varphi(\kappa, d)$. The operator $B_{(d,\kappa)} : A \longrightarrow A$ is defined by

$$B_{(d,\kappa)}(g(z)) = \varphi(\kappa, d) * g(z), \quad \forall g(z) \in A.$$

Ramachandran et al. [18] introduced a class $UB(\gamma, \eta, \beta, d)$ of analytic functions with negative coefficients, using the normalized form of generalized Bessel functions of first kind defined as:

Let $d > 1, 0 \leq \gamma < 1, \beta \geq 0, 0 \leq \eta < 1$ and $z \in \nabla$, then a function $g(z) \in \tau$ is said to be in the class $UB(\gamma, \eta, \beta, d)$, if and only if

$$\Re[\tfrac{zG''(z)}{G'(z)}] > \beta|\tfrac{zG''(z)}{G'(z)} - 1| + \eta,$$

where

$$G(z) = (1-\gamma)B_{(d,\kappa)}(g(z)) + \gamma(B_{(d,\kappa)}(g(z)))'. \quad (4.\ 17)$$

Now by using some results of [18] and the above technique of finding the Bohr radius we can show that the class $UB(\gamma, \eta, \beta, d)$ satisfies Bohr's phenomenon for the following Bohr radius

$$r^\star = \frac{(\eta + \beta + 2(2-\eta)\varsigma)}{\beta + 2(2-\eta)\varsigma + \sqrt{(\beta + 2(2-\eta)\varsigma)^2 + 4(1-\eta)(\eta + \beta + 2(2-\eta)\varsigma - 1)}}, \quad (4.\ 18)$$

with $\varsigma = (1+\gamma)(|\frac{-d}{4\kappa}|)$.

5. Conclusion

Keeping in sight the importance and applications of Bohr's phenomenon in many areas of research, we found the Bohr radius of different subclasses of analytic functions with negative coefficients. The derived results in the present investigation, continue to hold for a variety of subclasses of analytic functions. Moreover, the technique used in the paper can be utilized to investigate the Bohr radius of various other classes.

6. Acknowledgments

The authors are thankful to the rector of Comsats University Islamabad for providing an excellent research oriented environment.

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