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Integrity of Wheel Related Graphs

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Abstract.: If a network modeled by a graph, then there are various graph theoretical parameters used to express the vulnerability of communication networks. One of them is the concept of integrity. In this paper, we determine exact values for the integrity of wheel related graphs.

AMS (MOS) Subject Classification Codes: 05C40; 05C69; 90C35
Key Words: Integrity, wheel related graphs, communication network.

1. Introduction

In this paper, we consider simple, finite, undirected graphs. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. we refer to Harary [13] for notations and terminologies not defined here.

The stability of a communication network composed of processing nodes(vertices) and communication links(edges) are of primary importance to network designers. As the network begins losing links or nodes, eventually there will be a decrease in its effectiveness. Thus, communication networks constructed as stable as possible, not only with respect to the earliest damage, but also with respect to the possible reformation of the network. Integrity is the best parameter to measure the stability of network.

Barefoot et al. in [6] introduced the concept of integrity. The integrity of a graph G is defined as

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\},$$

where $m(G - S)$ denotes the order of the largest component of $G - S$. In [6], the authors have compared integrity, connectivity, toughness and binding number for several classes of graphs. In 1987, Barefoot et al. [7] have investigated the integrity of trees and powers of cycles. In 1988, Goddard et al. [11] have obtained integrity of the join, union, product and composition of two graphs. The authors in [2, 14] have studied the integrity of cubic graphs. Inspired by this, we obtain integrity of wheel related graphs. For more details on integrity of a graph refer to [3, 4, 5, 8, 10, 12].

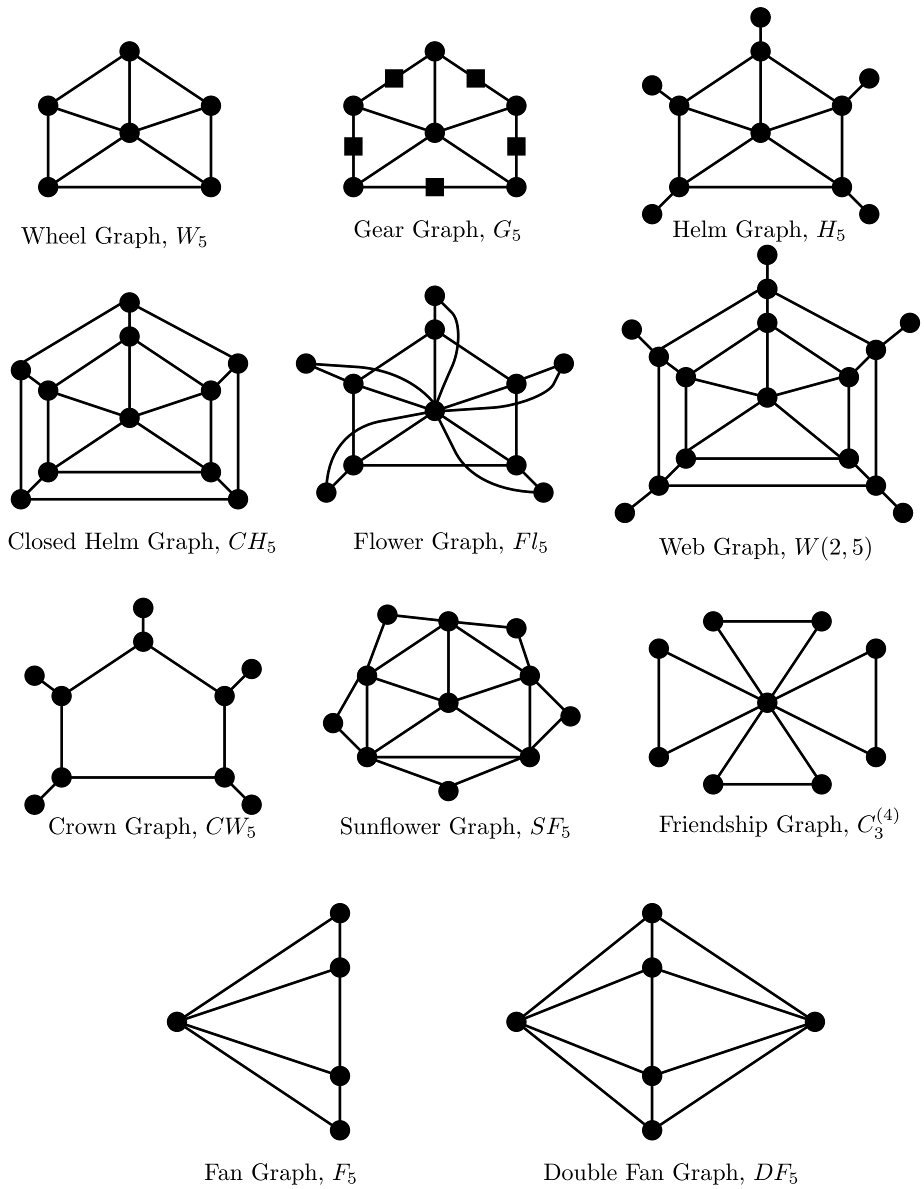

FIGURE 1. Some wheel related graphs

2. Preliminaries

We list some of the basic results.

Theorem 2.1. [5] *The integrity*
(i) complete graph K_n, $I(K_n) = n$,
(ii) null graph $\overline{K_n}$, $I(\overline{K_n}) = 1$,
(iii) star $K_{1,n}$, $I(K_{1,n}) = 2$,
(iv) path P_n, $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$,
(v) cycle C_n, $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$,
(vi) complete bipartite graph $K_{a,b}$, $I(K_{a,b}) = 1 + min\{a, b\}$,
(vii) wheel W_n, $I(W_n) = \lceil 2\sqrt{n-1} \rceil$.

Theorem 2.2. [5] *For any graphs* G *and* H,

$$I(G + H) = min\{I(G) + |H|, I(H) + |G|\}.$$

3. Integrity of wheel related graphs

In this section, we obtain integrity of wheel related graphs.

Definition 3.1. [9] *The graph* $W_n = K_1 + C_n$ *is called a wheel graph. In wheel graph, the vertex* c *of degree* n *is called the central vertex while the vertices on the cycle* C_n *are called rim vertices.*

Definition 3.2. [9] *The gear graph* G_n *is obtained from a wheel graph by inserting a vertex between each pair of adjacent vertices of the outer cycle.*

Theorem 3.3. *For a gear graph* G_n *of order* $2n + 1$,

$$I(G_n) = \begin{cases} n + 1, & if\ 3 \leq n \leq 7, \\ \lceil 2\sqrt{2n}\, \rceil, & otherwise. \end{cases}$$

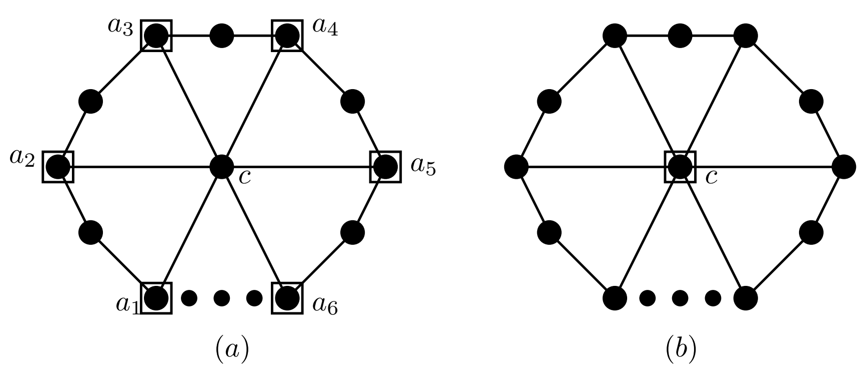

FIGURE 2. Choosing S in gear graph.

Proof. Let S be the collection of some rim vertices and central vertex c. Let $S \subset V(G_n)$. To choose S, we have two cases.

Case 1. If $3 \leq n \leq 7$, then choose set $S = \{a_i : i = 1, 2, ..., n\}$ in such a way that it is containing all vertices adjacent to central vertex of G_n (as shown in Figure 2(a)). It is clear to write $|S| = n$. $G_n - S$ results in a graph with $n + 1$ components of order 1. Hence, $m(G_n - S) = 1$. $|S| + m(G_n - S)$ is minimum for the above defined set S. Thus, $I(G_n) = n + 1$.
Case 2. Suppose $n \geq 8$. Then choose $S = \{c\}$, where c is central vertex of gear graph (as shown in Figure 2(b)). Then, $G_n - S$ will be the cycle of order $2n$.
Therefore, $I(G_n) = |S| + I(C_{2n})$. By Theorem 2.1(v), we get the required result.

□

Definition 3.4. [9] *The helm H_n is a graph obtained from a wheel W_n with central vertex c, by attaching a pendant edge to each rim vertex of W_n.*

Theorem 3.5. *For a helm graph H_n of order $2n + 1$,*

$$I(H_n) = \begin{cases} \lceil \frac{n}{2} \rceil + 3, & if\ 3 \leq n \leq 10, \\ \lceil \frac{n}{3} \rceil + 5, & otherwise. \end{cases}$$

Proof. Let S be the collection of some rim vertices and central vertex c. Let $S \subset V(H_n)$. The set S can be chosen in two ways.

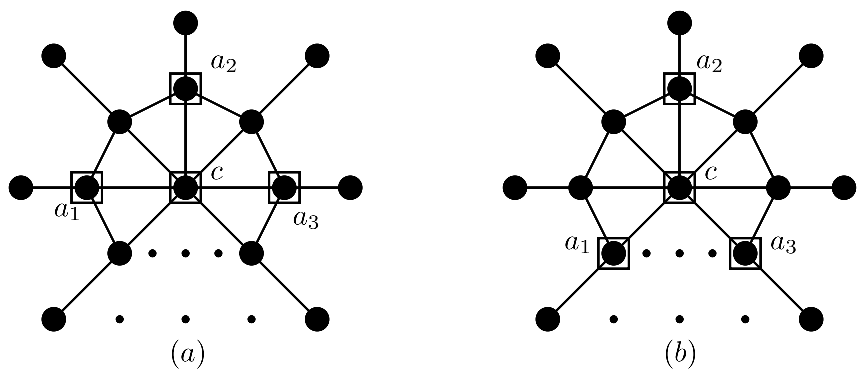

FIGURE 3. Choosing S in helm graph.

case 1. Suppose $3 \leq n \leq 10$. Then $S = \{a_i : i = 1, 2, ..., \lceil \frac{n}{2} \rceil\} \cup \{c\}$(as shown in Figure 3(a)) such that $d(a_1, a_{\lceil \frac{n}{2} \rceil}) = 1$ or 2 and $d(a_i, a_j) = 2$, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{2} \rceil$. Clearly, $|S| = \lceil \frac{n}{2} \rceil + 1$. $H_n - S$ is a graph with components K_1 and K_2.
Hence, $m(H_n - S) = 2$. The set S gives least value of $|S| + m(H_n - S)$ which gives the value of integrity. Thus, $I(H_n) = \lceil \frac{n}{2} \rceil + 3$.
Case 2. If $n \geq 11$, then $S = \{a_i : i = 1, 2, ..., \lceil \frac{n}{3} \rceil\} \cup \{c\}$(as shown in Figure 3(b)) such that $d(a_1, a_{\lceil \frac{n}{3} \rceil}) = 1$ or 2 or 3 and $d(a_i, a_j) = 3$, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{3} \rceil$. Clearly, $|S| = \lceil \frac{n}{3} \rceil + 1$. $H_n - S$ results in a graph with components of order 1, 2 and 4. Hence, $m(H_n - S) = 4$. $|S| + m(H_n - S)$ is least for the above defined set S . Thus, $I(H_n) = \lceil \frac{n}{3} \rceil + 5$. □

Definition 3.6. [9] *The flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex c of the helm.*

Theorem 3.7. *For a flower graph Fl_n of order $2n+1 (n \geq 4)$,*

$$I(Fl_n) = \begin{cases} \lceil \frac{n}{2} \rceil + 3, & if\ 4 \leq n \leq 10, \\ \lceil \frac{n}{3} \rceil + 5, & otherwise. \end{cases}$$

Proof. The proof is similar to that of Theorem 3.5.

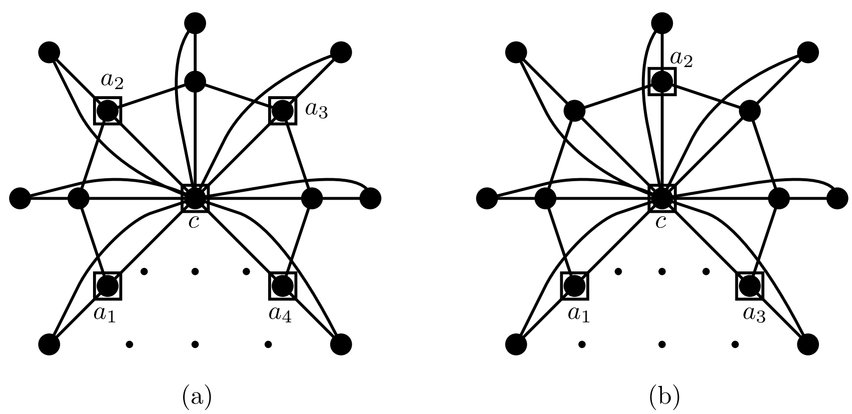

FIGURE 4. Choosing S in flower graph.

□

Remark 3.8. *From the Theorems 3.5 and 3.7, we observe that the integrities of flower graph and helm graph are equal for $n \geq 4$.*

Definition 3.9. [9] *The crown (or sun) CW_n is a corona of form $C_n \circ K_1$ where $n \geq 3$. That is crown is a helm without a central vertex.*

Theorem 3.10. *For a crown graph CW_n order $2n (n \geq 3)$,*

$$I(CW_n) = \begin{cases} \lceil \frac{n}{2} \rceil + 2, & if\ 4 \leq n \leq 10, \\ \lceil \frac{n}{3} \rceil + 4, & otherwise. \end{cases}$$

Proof. Let S be the collection of some rim vertices and central vertex c. Let $S \subset V(CW_n)$. The set S can be chosen in two ways.

case 1. If $3 \leq n \leq 10$, $S = \{a_i : i = 1, 2, ..., \lceil \frac{n}{2} \rceil\}$(as shown in Figure 5(a)) such that $d(a_1, a_{\lceil \frac{n}{2} \rceil}) = 1$ or 2 and $d(a_i, a_j) = 2$, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{2} \rceil$. Clearly, $|S| = \lceil \frac{n}{2} \rceil$. $CW_n - S$ is a graph with components K_1 and K_2. Hence, $m(CW_n - S) = 2$. The set S gives least value of $|S| + m(CW_n - S)$ which gives the value of integrity. Thus, $I(CW_n) = \lceil \frac{n}{2} \rceil + 2$.

Case 2. If $n \geq 11$, $S = \{a_i : i = 1, 2, ..., \lceil \frac{n}{3} \rceil\}$(as shown in Figure 5(b)) such that $d(a_1, a_{\lceil \frac{n}{3} \rceil}) = 1$ or 2 or 3 and $d(a_i, a_j) = 3$, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{3} \rceil$. Clearly, $|S| = \lceil \frac{n}{3} \rceil$. $CW_n - S$ results in a graph with components of order 1, 2 and 4. Hence,

$m(CW_n - S) = 4$. $|S| + m(CW_n - S)$ is least for the above defined set S . Thus, $I(CW_n) = \lceil \frac{n}{3} \rceil + 4$. □

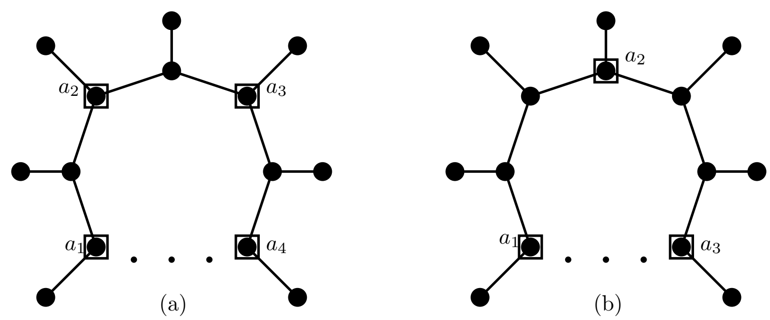

FIGURE 5. Choosing S in crown graph.

Corollary 3.11. *For a crown graph CW_n order $2n(n \geq 3)$,*

$$I(CW_n) = I(H_n) - 1.$$

Definition 3.12. [9] *The closed helm CH_n is the graph with central vertex c, obtained from a helm by joining each pendant vertex to form a cycle.*

Theorem 3.13. *For a closed helm graph CH_n of order $2n + 1$,*

$$I(CH_n) = \begin{cases} 2\lceil \frac{n}{2} \rceil + 2, & if\ 3 \leq n \leq 10, \\ 2\lceil \frac{n}{4} \rceil + 7, & otherwise. \end{cases}$$

Proof. We have to choose S_1 from rim vertices of inner cycle and S_2 from rim vertices of outer cycle such that $|S_1| = |S_2|$. Let c be a central vertex. The set S can be chosen in two ways.

Case 1. If $3 \leq n \leq 10$, choose $S_1 = \{a_i : i = 1, 2, ..., \lceil \frac{n}{2} \rceil\}$(as shown in Figure 6(a)) such that $d(a_1, a_{\lceil \frac{n}{2} \rceil}) = 1$ or 2 and $d(a_i, a_j) = 2$. $S_2 = \{b_i : i = 1, 2, ..., \lceil \frac{n}{2} \rceil\}$ such that $d(b_1, b_{\lceil \frac{n}{2} \rceil}) = 1$ or 2 and $d(b_i, b_j)$ =2, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{2} \rceil$. a_1 must not be adjacent to b_1. $S = S_1 \cup S_2 \cup \{c\}$. So, $|S| = 2\lceil \frac{n}{4} \rceil + 1$. $CH_n - S$ results in a disconnected graph with components of order 1. Hence, $m(CH_n - S) = 1$. The set S defined above gives minimum value of $|S| + m(CH_n - S)$ which gives the value of integrity. Thus, $I(CH_n) = 2\lceil \frac{n}{2} \rceil + 2$.

Case 2. If $n \geq 11$, choose $S_1 = \{a_i : i = 1, 2, ..., \lceil \frac{n}{4} \rceil\}$(as shown in Figure 6(b)) such that $d(a_1, a_{\lceil \frac{n}{4} \rceil}) = 1$ or 2 or 3 or 4 and $d(a_i, a_j) = 4$. $S_2 = \{b_i : i = 1, 2, ..., \lceil \frac{n}{4} \rceil\}$ such that $d(b_1, b_{\lceil \frac{n}{4} \rceil}) = 1$ or 2 or 3 or 4 and $d(b_i, b_j) = 4$. where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{4} \rceil$. a_1 must be adjacent to b_1. $S = S_1 \cup S_2 \cup \{c\}$. So, $|S| = 2\lceil \frac{n}{4} \rceil + 1$. $CH_n - S$ results in a disconnected graph with components of order 2 or 4 or 6. Hence, $m(CH_n - S) = 6$. $|S| + m(CH_n - S)$ is least for above set S, the value of integrity. Thus, $I(CH_n) = 2\lceil \frac{n}{4} \rceil + 7$. □

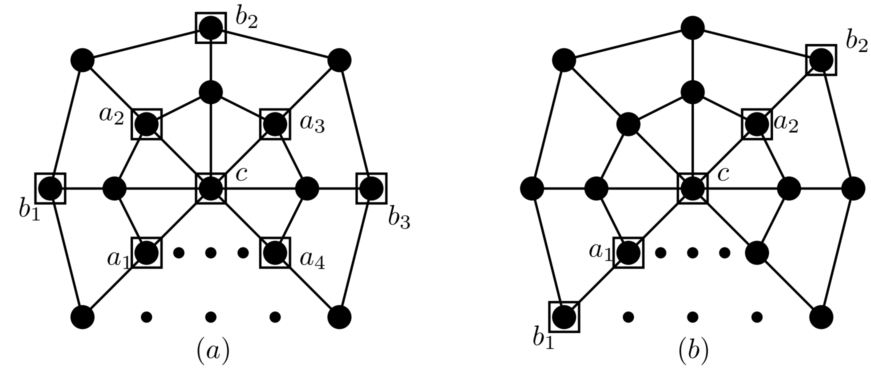

FIGURE 6. Choosing S in closed helm graph.

Definition 3.14. [9] *A web graph is the graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm.* $W(t, n)$ *is the generalized web with* t *cycles each of order* n.

Theorem 3.15. *Let* $W(t, n)$ *be a web graph of order* $3n + 1 (n \geq 3)$. *Then,*

$$I(W(t, n)) = n + 3.$$

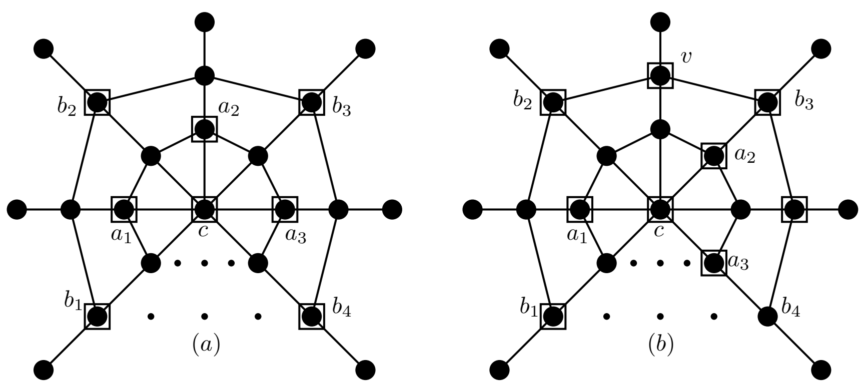

FIGURE 7. Choosing S in web graph.

Proof. Let S_1={a_i : $i = 1, 2, ..., \frac{n}{2}.$} be the maximal independent set of vertices of outer cycle and S_2={b_i : $i = 1, 2, ..., \frac{n}{2}.$} be the maximal independent set of vertices of inner cycle, where a_i is not adjacent to b_i. Let c be a central vertex. We have two cases to choose the set S.

Case 1. If n is even, $S = S_1 \cup S_2 \cup \{c\}$ (as shown in Figure 7(a)). Clearly, $|S| = 2\beta_0(C_n) + 1$, where β_0 is point independence number. Thus, $|S| = n + 1$.

Case 2. If n is odd, choose vertex v from an outer cycle (or inner cycle) of $W(t, n)$ such

that v is adjacent to vertices of S_1(or S_2). So, $S = S_1 \cup S_2 \cup \{v\} \cup \{c\}$ (as shown in Figure 7(b)). So, $|S| = 2\beta_0(C_n) + 2$, where β_0 is point independence number. Therefore $|S| = n + 1$.
From the above two cases $|S| = n + 1$. $W(t, n) - S$ results in a disconnected graph with components K_1 and K_2. Hence, $m(W(t, n) - S) = 2$. The set S defined above gives least value of $|S| + m(W(t, n) - S)$. Thus, $I(W(t, n)) = n + 3$. □

Definition 3.16. [9] *The sunflower graph SF_n is a graph obtained from a wheel with central vertex c, n-cycle $v_0, v_1, ..., v_{n-1}$ and additional n vertices $w_0, w_1, ..., w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, ..., n - 1$ where $i + 1$ is taken modulo n.*

Theorem 3.17. *Let SF_n be a sunflower graph of order $2n + 1$. Then,*

$$I(SF_n) = \begin{cases} n+1, & if\ 3 \leq n \leq 5, \\ \lceil \frac{n}{2} \rceil + 4, & if\ 6 \leq n \leq 8, \\ \lceil \frac{n}{3} \rceil + 6, & otherwise. \end{cases}$$

Proof. Let $S \subset V(SF_n)$. The set S can be chosen in two ways.
Case 1. If $3 \leq n \leq 5$, then choose set S containing all vertices adjacent to central vertex of SF_n. It is clear to write $|S| = n$. $SF_n - S$ results in a graph with $n + 1$ components of order 1. Hence, $m(SF_n - S) = 1$. The set S gives minimum value of $|S| + m(SF_n - S)$. Thus, $I(SF_n) = n + 1$.
Case 2. Suppose $6 \leq n \leq 8$. Then choose $S_1 = \{a_i : i = 1, 2, ..., \lceil \frac{n}{2} \rceil\}$(as shown in Figure 8(a)) such that $d(a_1, a_{\lceil \frac{n}{2} \rceil}) = 1$ or 2 and $d(a_i, a_j) = 2$, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{2} \rceil$. Let $S = S_1 \cup \{c\}$. So, $|S| = \lceil \frac{n}{2} \rceil + 1$. $SF_n - S$ results in a disconnected graph with components of order 1 or 3. Hence, $m(SF_n - S) = 3$. Therefore, $I(SF_n) = \lceil \frac{n}{2} \rceil + 4$.
Case 3. If $n \geq 9$, choose $S_2 = \{b_i : i = 1, 2, ..., \lceil \frac{n}{3} \rceil\}$(as shown in Figure 8(b)) such that $d(b_1, b_{\lceil \frac{n}{3} \rceil}) = 2$ and $d(b_i, b_j) = 2$, where $i \neq j, j = 1, 2, ..., \lceil \frac{n}{3} \rceil$. Let $S = S_2 \cup \{c\}$. So, $|S| = \lceil \frac{n}{3} \rceil + 1$. $SF_n - S$ results in a disconnected graph with components of order 3 or 5. Hence, $m(SF_n - S) = 5$. The set S gives minimum value of $|S| + m(SF_n - S)$. Thus, $I(SF_n) = \lceil \frac{n}{3} \rceil + 6$. □

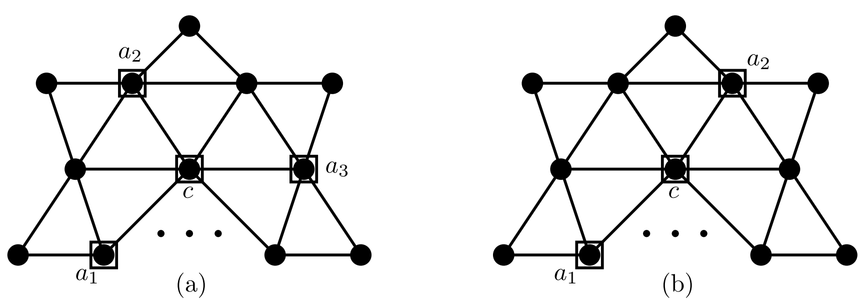

FIGURE 8. Choosing S in sunflower graph.

Definition 3.18. [9] *The friendship graph $C_3^{(t)}$ is a collection of t-triangles with a common vertex. Friendship graph can also be obtained from a wheel W_{2n} with cycle C_{2n} by deleting alternate edges of the cycle. That is $f_n = K_1 + nK_2$.*

Theorem 3.19. *For a Friendship graph $C_3^{(t)}$ of order $2t+1$,*

$$I(C_3^{(t)}) = 3.$$

Proof. Choose $S = \{c\}$, where c is a central vertex. Clearly, $|S| = 1$. Then, the graph $C_3^{(t)} - S \cong tK_2$. Hence, $m(C_3^{(t)} - S) = 2$. The set S defined above gives least value of $|S| + m(C_3^{(t)} - S)$, the value of integrity. Thus, $I(C_3^{(t)}) = 3$. □

Example 3.20. *Consider a graph $C_3^{(4)}$. Let $S = \{c\} \subset V(C_3^{(4)})$ (see Figure 9) such that $|S| = 1$ and $m(C_3^{(4)} - S) = 2$. Therefore, $I(C_3^{(4)}) = 3$.*

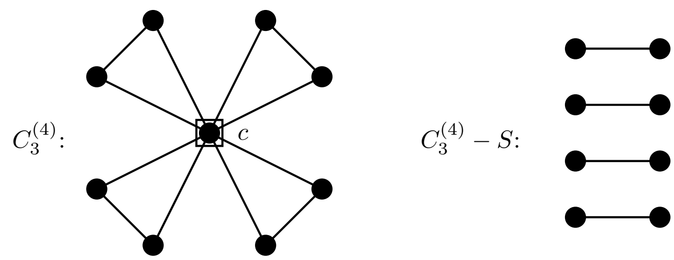

FIGURE 9. Graph $C_3^{(4)}$ and $C_3^{(4)} - S$.

Definition 3.21. [9] *The fan graph $F_n(n \geq 3)$ is defined as the graph $K_1 + P_{n-1}$, where K_1 is singleton graph and P_{n-1} is the path on $n-1$ vertices.*

Theorem 3.22. *Let F_n be a fan graph of order $n \geq 3$. Then*

$$I(F_n) = \lceil 2\sqrt{n} \rceil - 1.$$

Proof. The fan graph is join of K_1 and P_{n-1}. So, $I(F_n) = I(K_1 + P_{n-1})$. Thus, by Theorems 2.1(iv) and 2.2, we get the desired result. □

Definition 3.23. [9] *The double fan graph $df_n(n \geq 3)$ is defined as the graph $2K_1 + P_{n-1}$.*

Theorem 3.24. *For a double fan DF_n of order $n+1(n \geq 3)$,*

$$I(DF_n) = \lceil 2\sqrt{n} \rceil.$$

Proof. The double fan graph is join of $2K_1$ and P_{n-1}. So, $I(DF_n) = I(2K_1 + P_{n-1})$. From Theorems 2.1(iv) and 2.2, we get the desired result. □

4. CONCLUSION

In this paper, we have concentrated on integrity, a measure of network vulnerability. We have computed the integrity of wheel related graphs. Also, we have established the relationship between some wheel related graphs. Wheel related graphs are taken to model the network system and the integrity values of them reveal that how network can be made more stable than earlier. These results can help the network designers to choose a suitable topology for the network. This study can be very useful in the investigation of complex network robustness.

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