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Some Inclusion Theorems related to Asymptotically Deferred Statistical Equivalent Measurable Functions

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Abstract.: The main purpose of this article is to present the new concepts of asymptotically deferred statistical equivalent and strongly asymptotically deferred statistical equivalence by considering two nonnegative real-valued Lebesgue measurable functions on $(1, \infty)$. Additionally, we examine some inclusion theorems that are not presented in the literature before.

AMS (MOS) Subject Classification Codes: 40H05; 40C05
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1. Introduction and Motivation

One of the widespread research areas of summability theory is the notion of statistical convergence that was presented by Fast [6] in 1951. After the work of Fridy [7], extensive collection of literature has appeared and many other applications that are related to this concept were studied by [4], [5], [15]. On the other hand, in 1980 Pobyvancts [14] presented the idea of asymptotically regular matrices which preserve the asymptotic equivalence of two non-negative sequences. Afterward, Patterson [12] introduced the concept of asymptotically statistical equivalent sequences by combining the notion of asymptotically equivalent introduced by Marouf [10] and the concept of statistical convergence [7] as follows:

Definition 1.1. *Two nonnegative sequences $y = (y_r)$ and $z = (z_r)$ are said to be asymptotically statistically equivalent of multiple ξ if for every $\varepsilon > 0$,*

$$\lim_{w\to\infty} \frac{1}{w}\left| \left\{ r \leq w : \left| \frac{y_r}{z_r} - \xi \right| \geq \varepsilon \right\} \right| = 0.$$

If this condition is met, it is denoted by $y \overset{S_\xi}{\sim} z$.

Recently, the concept of asymptotically statistical equivalent sequences has been studied in many different aspects of summability theory (see [2], [8], [13]). In other direction, Agnew [1] presented the deferred Cesáro mean by

$$(D_{\tau,\upsilon}y)_m := \frac{1}{\upsilon(m)-\tau(m)} \sum_{r=\tau(m)+1}^{\upsilon(m)} y_r,\ m=1,2,3,...,$$

where $\{\tau(m)\}$ and $\{\upsilon(m)\}$ are sequences of positive natural numbers satisfying $\tau(m) < \upsilon(m)$ and $\lim_{m\to\infty} \upsilon(m) = \infty$.

Following Agnew's results, Küçükaslan and Yılmaztürk [9] introduced the notions of deferred density as follows:

Definition 1.2. *Let E be a subset of $\mathbb{N}$ and let the set $\{\zeta : \tau(m) < \zeta \leq \upsilon(m),\ \zeta \in E\}$ denote by $E_d(m)$. Deferred density of E is defined by*

$$\delta_d(E) = \lim_{m\to\infty} \frac{1}{\upsilon(m)-\tau(m)} |E_d(m)|$$

if the limit exists. The vertical bars show the cardinality of the set $E_d(m)$.

Also, strongly summable single valued functions were introduced by Borwein in [3] in the following:

Definition 1.3. *A nonnegative real-valued Lebesgue measurable function $g(\varsigma)$ on interval $(1,\infty)$ is said to be strongly summable to ξ if,*

$$\lim_{m\to\infty} \frac{1}{m} \int_1^m |g(\varsigma) - \xi|\, d\zeta = 0.$$

Afterward, Nuray [11] introduced the notion of $\lambda-$statistically convergent functions. Following his results, Savas and Ozturk [16] presented $\lambda-$statistical asymptotically equivalent of the measurable functions. The primary focus of this article is to study the ideas of asymptotically deferred statistically equivalent functions and strongly asymptotically deferred statistical equivalent functions by using $g(\varsigma)$ and $h(\varsigma)$. Additionally, we examine some main properties of these concepts. Note that $g(\varsigma)$ and $h(\varsigma)$ shall be two measurable real valued functions on $(1,\infty)$ thoughout this paper.

2. Main Definitions

Before we present the main results of this study, we will introduce the following defnitions:

Definition 2.1. *Let $\{\tau(m)\}$ and $\{\upsilon(m)\}$ be two sequences, then two functions $g(\varsigma)$ and $h(\varsigma)$ are asymptotically deferred statistically equivalent of multiple ξ if for each $\varepsilon > 0$,*

$$\lim_{m\to\infty} \frac{1}{\upsilon(m)-\tau(m)} \left| \left\{ \tau(m) < \varsigma \leq \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geq \varepsilon \right\} \right| = 0.$$

We symbolize this equivalence by

$$g(\varsigma) \overset{S_D^\xi}{\sim} h(\varsigma)$$

and if $\xi = 1$, *then we call as asymptotically deferred statistically equivalent. If* $\upsilon(m) = m$ *and* $\tau(m) = 0$, *then the notion coincides the concept of asymptotically statistically equivalent functions of multiple* ξ *which was denoted by* $g(\varsigma) \overset{S^{\xi}}{\sim} h(\varsigma)$. *Also, if* $h(\varsigma) = 1$ *and for every* $\varepsilon > 0$, *then* $g(\varsigma)$ *is said to be deferred statistically convergent function to* ξ.

Definition 2.2. *Let* $\{\tau(m)\}$ *and* $\{\upsilon(m)\}$ *be two sequences, then two functions* $g(\varsigma)$ *and* $h(\varsigma)$ *are strongly deferred asymptotically equivalent of multiple* ξ *if for every* $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\upsilon(m) - \tau(m)} \int\limits_{\zeta=\tau(m)+1}^{\upsilon(m)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma = 0.$$

We symbolize it by

$$g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$$

and if $\xi = 1$ *then we call it a simply strongly deferred asymptotically equivalent. If* $\upsilon(m) = m$ *and* $\tau(m) = 0$, *then the concept coincides the notion of strongly asymptotically equivalent functions of multiple* ξ *which was denoted by* $g(\varsigma) \overset{V^{\xi}}{\sim} h(\varsigma)$, *and also if* $h(\varsigma) = 1$, *for every* $\varepsilon > 0$, *then* $g(\varsigma)$ *is said to be stronlgy* $D_f^{\tau,\upsilon}-$*convergent to* ξ.

Definition 2.3. *Let the index sequences* $\varpi(m)$ *is strictly increasing single sequences of positive integers and* $g(\varsigma)$ *and* $h(\varsigma)$ *are measurable functions.* $g(\varsigma)$ *and* $h(\varsigma)$ *are said to be* $C_{\varpi}-$*asymptotically equivalent of multiple* ξ *if*

$$\lim_{m\to\infty} \frac{1}{\varpi(m)} \int\limits_{\zeta=1}^{\varpi(m)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma = 0.$$

We symbolize it by $g(\varsigma) \overset{C_{\varpi}^{\xi}}{\sim} h(\varsigma)$.

If we take $\tau(m) = \varpi(m-1)$ and $\upsilon(m) = \varpi(m)$, then we denote $g(\varsigma) \overset{V_{D_{\varpi}}^{\xi}}{\sim} h(\varsigma)$ as $g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$.

3. Inclusion Theorems

In this section we present some interesting results.

Theorem 3.1. *Let* $\{\tau(m)\}, \{\upsilon(m)\}, \{\tau^*(m)\}$ *and* $\{\upsilon^*(m)\}$ *be sequences of non-negative integers and let*

$$\tau(m) \le \tau^*(m) < \upsilon^*(m) \le \upsilon(m) \text{ for all } m \in \mathbb{N}$$

such that the sets $\{\varsigma : \tau(m) < \varsigma \le \tau^*(m)\}$ *and* $\{\varsigma : \upsilon^*(m) < \varsigma \le \upsilon(m)\}$ *are finite. If*

$$\lim_{m\to\infty} \frac{1}{(\upsilon^*(m) - \tau^*(m))} \left| \left\{ \tau^*(m) < \varsigma \le \upsilon^*(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \ge \varepsilon \right\} \right| = 0$$

which is denoted by $g(\varsigma) \overset{S_{D^*}^{\xi}}{\sim} h(\varsigma)$, *then* $g(\varsigma) \overset{S_{D^*}^{\xi}}{\sim} h(\varsigma)$ *implies* $g(\varsigma) \overset{S_D^{\xi}}{\sim} h(\varsigma)$.

Proof. Let us presume that the sets $\{\varsigma : \tau(m) < \varsigma \leq \tau^*(m)\}$ and $\{\varsigma : \upsilon^*(m) < \varsigma \leq \upsilon(m)\}$ are finite and $g(\varsigma) \overset{S_{D^*}^{\xi}}{\sim} h(\varsigma)$. Then for any $\varepsilon > 0$, we are granted the following:

$$\begin{aligned}
\left\{\varsigma : \tau(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\} \quad = \quad & \left\{\varsigma : \tau(m) < \varsigma \leq \tau^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\} \\
& \cup \left\{\varsigma : \tau^*(m) < \varsigma \leq \upsilon^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\} \\
& \cup \left\{\varsigma : \upsilon^*(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}
\end{aligned}$$

and so

$$\begin{aligned}
& \frac{1}{\upsilon(m) - \tau(m)} \left|\left\{\varsigma : \tau(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right| \\
\leq \quad & \frac{1}{\upsilon^*(m) - \tau^*(m)} \left|\left\{\varsigma : \tau(m) < \varsigma \leq \tau^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right| \\
& + \frac{1}{\upsilon^*(m) - \tau^*(m)} \left|\left\{\varsigma : \tau^*(m) < \varsigma \leq \upsilon^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right| \\
& + \frac{1}{\upsilon^*(m) - \tau^*(m)} \left|\left\{\varsigma : \upsilon^*(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

By taking limit as $m \to \infty$, we have $g(\varsigma) \overset{S_{D}^{\xi}}{\sim} h(\varsigma)$. $\square$

Theorem 3.2. *Let* $\{\tau(m)\}, \{\upsilon(m)\}, \{\tau^*(m)\}$ *and* $\{\upsilon^*(m)\}$ *be sequences of non-negative integers and let*

$$\tau(m) \leq \tau^*(m) < \upsilon^*(m) \leq \upsilon(m) \text{ for all } m \in \mathbb{N}$$

such that

$$\lim \left(\frac{\upsilon(m) - \tau(m)}{\upsilon^*(m) - \tau^*(m)}\right) > 0,$$

then $g(\varsigma) \overset{S_{D}^{\xi}}{\sim} h(\varsigma)$ *implies* $g(\varsigma) \overset{S_{D^*}^{\xi}}{\sim} h(\varsigma)$.

Proof. It is clear to see the following inclusion

$$\left\{\varsigma : \tau^*(m) < \varsigma \leq \upsilon^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\} \subset \left\{\varsigma : \tau(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}$$

holds and so the following inequalities as well

$$\left|\left\{\varsigma : \tau^*(m) < \varsigma \leq \upsilon^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right| \leq \left|\left\{\varsigma : \tau(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right|.$$

Hence, we obtain the following

$$\begin{aligned}
& \frac{1}{\upsilon^*(m) - \tau^*(m)} \left|\left\{\varsigma : \tau^*(m) < \varsigma \leq \upsilon^*(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right| \\
\leq \quad & \left(\frac{\upsilon(m) - \tau(m)}{\upsilon^*(m) - \tau^*(m)}\right) \frac{1}{\upsilon(m) - \tau(m)} \left|\left\{\varsigma : \tau(m) < \varsigma \leq \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

By taking limit as $m \to \infty$, we have $g(\varsigma) \overset{S_{D^*}^{\xi}}{\sim} h(\varsigma)$. □

Theorem 3.3. *Let $\{\tau(m)\}, \{\upsilon(m)\}, \{\tau^*(m)\}$ and $\{\upsilon^*(m)\}$ be sequences of non-negative integers satisfying*

$$\tau(m) \leq \tau^*(m) < \upsilon^*(m) \leq \upsilon(m) \text{ for all } m \in \mathbb{N}$$

such that the sets $\{\varsigma : \tau(m) < \varsigma \leq \tau^*(m)\}$ and $\{\varsigma : \upsilon^*(m) < \varsigma \leq \upsilon(m)\}$ are finite. If $g(\varsigma)$ and $h(\varsigma)$ are bounded then $g(\varsigma) \overset{V_{D^*}^{\xi}}{\sim} h(\varsigma)$ implies $g(\varsigma) \overset{V_{D}^{\xi}}{\sim} h(\varsigma)$.

Proof. Let $g(\varsigma)$ and $h(\varsigma)$ are bounded functions, then there is a positive real number K such that $\left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \leq K$. Then we obtain

$$\begin{aligned}
\frac{1}{\tau(m) - \upsilon(m)} \int\limits_{\tau(m)+1}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| &= \frac{1}{\tau(m) - \upsilon(m)} \int\limits_{\tau(m)+1}^{\tau^*(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \\
&\quad + \frac{1}{\tau(m) - \upsilon(m)} \int\limits_{\tau^*(m)+1}^{\upsilon^*(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \\
&\quad + \frac{1}{\tau(m) - \upsilon(m)} \int\limits_{\upsilon^*(m)+1}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \\
&\leq \frac{2}{\upsilon^*(m) - \tau^*(m)} K O(1) \\
&\quad + \frac{1}{\upsilon^*(m) - \tau^*(m)} \int\limits_{\upsilon^*(m)+1}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right|.
\end{aligned}$$

Consequently, $g(\varsigma) \overset{V_{D}^{\xi}}{\sim} h(\varsigma)$. □

Theorem 3.4. *Let $\varpi(m)$ is strictly increasing single sequences of positive integers and $\varpi(0) = 0$. If $g(\varsigma) \overset{V_{D_{\varpi}}^{\xi}}{\sim} h(\varsigma)$ then $g(\varsigma) \overset{C_{\varpi}^{\xi}}{\sim} h(\varsigma)$.*

Proof. Presume that $g(\varsigma) \overset{V^{\xi}_{D_{\varpi}}}{\sim} h(\varsigma)$ and for any $m \in \mathbb{N}$, we are granted

$$\frac{1}{\varpi(m)} \int_{\zeta=1}^{\varpi(m)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma$$

$$= \int_{i=0}^{m-1} \left(\frac{1}{\varpi(m)} \int_{\zeta=\varpi(i)+1}^{\varpi(i+1)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma \right)$$

$$= \int_{i=0}^{m-1} \left(\frac{1}{\varpi(m)} \int_{\zeta=\varpi(i)+1}^{\varpi(i+1)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma \right)$$

$$= \int_{i=0}^{m-1} \frac{\delta\varpi(i)}{\varpi(m)\,\delta\varpi(i)} \int_{\zeta=\varpi(i)+1}^{\varpi(i+1)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma$$

$$= \int_{i=0}^{m-1} a_{m,i} \frac{1}{\delta\varpi(i)} \int_{\zeta=\varpi(i)+1}^{\varpi(i+1)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d$$

where $\delta\varpi(i) = \varpi(i+1) - \varpi(i)$ and

$$a_{m,i} = \begin{cases} \frac{\delta\varpi(i)}{\varpi(m)}, i = 1, 2, ...m-1; \\ 0, \text{ otherwise.} \end{cases}$$

Since the matrix $a_{m,i}$ is regular, then

$$\lim_{m\to\infty} \frac{1}{\varpi(m)} \int_{\zeta=1}^{\varpi(m)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma = 0$$

It means $g(\varsigma) \overset{C^{\xi}_{\varpi}}{\sim} h(\varsigma)$. □

4. Conclusion

Recently, increasing interest in summability methods has increased. However, the concepts of asymptotically deferred statistically equivalent functions and strongly asymptotically deferred statistical equivalent functions have not been studied so far. Therefore, in this paper, we presented new methods by considering $g(\varsigma)$ and $h(\varsigma)$ that are two measurable real valued functions on $(1, \infty)$ so that the present paper is filled up a big gap in the existing literature.

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