HERMITIAN OPERATORS AND ITS APPLICATION TO UNCERTAINTY PRINCIPLE

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Abstract

Some properties and applications of Hermitian operators composed of any integral operator and its adjoint and its application to uncertainty principle are studied. It includes the mathematical definition, properties of Hermitian operators and its relevance in quantum mechanics. Uncertainty principle is verified by applying it to a problem of a particle in a box.

Key words: *Hermitian operators, quantum mechanics, momentum, standard deviation*

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INTRODUCTION

An operator is an operation which transforms a given function into another function (Levine, 1991) on application. For example, if P^is a momentum operator and could be operate on a given function f(x), it will transform the given function into another function. If we let f(x)

= $2x^2 + 4$, then P^f(x)= 4x

Moreover, let A^ be an operator and f(x) be a function on which A^ can act. If A^ operating on f(x) returns the same function f(x) multiplied by a constant k, that is:

$$\hat{A}f(x) = kf(x); \quad (1)$$

then f(x) is an eigenfunction of A^ with eigenvalue k[1]. Next, let A^ be an operator and let α and β represent any two arbitrary, well behaved functions. If the relation

$$\int \alpha (\hat{A} \beta)^* dx = \int \beta^* (\hat{A} \alpha) dx \quad (2)$$

is satisfied, where dx is over one dimension space, then A^ is a Hermitian operator (Atkins and Paula, 2002). The two basic properties of Hermitian operators form the basis of quantum mechanics. i.e. (1) Hermitian operator eigenvalues are always real, (2) Hermitian operators' eigenfunctions are orthogonal to each other. These properties of Hermitian operator are described in detail somewhere else (Levine, 1991, Atkins and Paula, 2002).

RESULTS AND DISCUSSION

In quantum mechanics, the concept of wavefunction often denoted as ψ is well understood. It provides all of the dynamical information of a system (Autschbach, 2007). This information can be extracted from the wavefunction through the use of operators. Each physical property, or observable, of interest has a corresponding operator which operates on the wavefunction. If the wavefunction is an eigenfunction of that particular operator, then its eigenvalue is the value of that observable (Autschbach, 2007). A good example of this is lies in the Schrodinger equation:

$$\hat{H} \psi = E \psi \quad (3)$$

where H^ is the Hamiltonian operator $-h^2/m\Delta^2 + V$, of which is an eigenfunction, and E is the is the corresponding eigenvalue of total energy of the system. As observables are

measurable physical quantities, therefore the obtained eigenvalues of the system is also real.
Furthermore, the operators corresponding to observables must be Hermitian in order to guarantee that the eigenvalues are real. If a wavefunction is not an eigenfunction of a Hermitian operator corresponding to an observable, the average (or expectation) value of that property can still be calculated. The expectation value of a property with operator $A^\wedge$ can be denoted as $<A>$ and is described as:

$$<A> = \int \Psi^* A^\wedge \Psi \, dx \qquad (4)$$

where Ψ is the normalized wavefuntion and dx is over one dimension space (Atkins and Paula, 2002).
The wavefunctions with different eigenvalues for a particular observable operator must be orthogonal to satisfied the second afore-mentioned property of Hermitian operators.
The application of Hermitian operators in quantum mechanics is well understood by considering the example of a particle in box. In this example, a particle of mass m is assumed to be confined in a box. The potential V, inside the box is taken zero while it is taken ∞ outside of the box. The particle is confined to a box from $0 < a < L$ (Kittel, 2005), where L is the length of the box, it is therefore the wavefunction ψ is zero outside of the box. The potential $V_{(a)}$ is zero, then the time independent schrodinger equation becomes:

$$|-h^2/m\Delta^2 \psi = E \Psi \qquad (5)$$

Where E is the eigenvalue of the eigenfunction. This is a second order differential equation which has a solution:

$$\Psi = A\sin(\alpha a) + B\cos(\alpha a) \qquad (6)$$

By using the boundary condition i.e. Ψ=0 at a=0, gives the constant B zero. By using the second boundary condition Ψ=0 at x=a, $\alpha = n\pi/L$ where n is any positive integer. By using the normalization condition:

$$\int \Psi^* \Psi \, dx = 1 \qquad (7)$$

gives the constant A value i.e. $\sqrt{2/a}$.
By putting the value of Ψ back in the schrodinger equation gives the value of eigenenergy.

$$-h^2/m\Delta^2 (A\sin(n\pi a/L)) = E\, A\sin(n\pi a/L) \qquad (8)$$

From equation (8) it can be shown that the value of eigenenergy is $n^2h^2/8mL^2$. This shows that the energy levels of the system is quantized since n is a positive integer.
By using the hermitian operator, we can derive the expectation values and uncertainty in position and momentum and further by using these uncertainties values, uncertainty principle can be verified for a particle in a box. By using equation (4), the expectation value of position x in one dimension is given as:

$$<X> = \int \Psi^* X^\wedge \Psi \, dx \qquad (9)$$

By putting vales of Ψ^* and Ψ in equation (9) gives:

$$<X> = {}_0\int^a X A^2 \sin^2(n\pi a/L)\, dx \qquad (10)$$

By solving the above integral:

$$<X> = a/2 \qquad (11)$$

This result shows that the particle is lying in the middle of the box. By using the same method the vale $<x^2>$ is given as:

$$<X^2> = a^2/3 \qquad (12)$$

The standard deviation or uncertainty in position is given by:

$$\Delta X = \sqrt{<X^2><X>^2} \qquad (13)$$
$$\Delta X = \sqrt{a^2/12} \qquad (14)$$

The expectation value of momentum $P^\wedge$ is given as:

$$<P> = \int \Psi^* P^\wedge \Psi \, dx \qquad (15)$$

$\langle P\rangle = {}_0\int^a A^2 \sin(n\pi a/L)\ \hbar/i\ \partial/\partial x\ \sin(n\pi a/L)\ dx$ (16)

$\langle P\rangle = 0$ (17)

The expectation value of P^2 is given as:

$\langle P^2\rangle = {}_0\int^a A^2 \sin(n\pi a/L)(\hbar^2)\ \partial^2/\partial x^2 \sin(n\pi a/L)\ dx$ (18)

$\langle P^2\rangle = n^2\pi^2\ \hbar^2/a^2$ (19)

The standard deviation or uncertainty in momentum is given by:

$\Delta P = \sqrt{\langle P^2\rangle \langle P\rangle^2}$ (20)

$\Delta P = n\pi\ \hbar/a$

The Heisenberg uncertainty principle $\Delta X\Delta P\ \hbar$ that states a fundamental limit on the accuracy with which certain pairs of physical properties of a particle, such as position and momentum, cannot be simultaneously known can be verified by using the above calculated values of standard deviation in ΔX and ΔP as give:

$\Delta X\Delta P = \sqrt{a^2/12}\ \ n\pi\ \hbar/a$ (21)

$\Delta X\Delta P = n\pi\ \hbar/\sqrt{12}$ (22)

Which gives:

$\Delta X\Delta P\ \hbar$ (23)

This result shows that uncertainty in position and momentum of a particle in box leads to verification of Heisenberg uncertainty principle.

CONCLUSIONS

Hermitan are discussed in detail. Two basics properties of Hermitian operator make it importance in quantum mechanics. First, its eigenvalues are always real, second their eigenfunctions are orthogonal to each other. The Hermitian operators are applied in the case a particle in a box. The Heisenberg uncertainty principle $\Delta X\Delta P\ \hbar$ is verified by using the Hermitian operators in this regard.

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