The Role of Generalization Thinking in Mathematics Education and its Development

Through Technology

Dr. Muhammad Khalil* Dr. UmairKhalil** Dr. Zahoorulhaq***

Abstract

In the recent existing mathematics teaching in Pakistan there is need to focus on thinking rather than achievement. For which teachers' awareness about the mathematics teaching according to the documented requirements along with suitable technological tool is the need of the day. In this experimental research, the focus was to explore the awareness of the important aspect of mathematical thinking that is generalization and the way through which this aspect can be improved through GeoGebrain the teaching of analytic geometry at significance level after integrating it in the experimental group of the students. To uncover the effect of this dynamic geometry software on students' generalization thinking and their confidence, two groups (Experimental study. The study revealed that experimental students' generalization thinking and their confidence improved significantly with the use of Geogebra activities. Further, the teacher's conscious role in developing mathematical thinking is one of the most important factors.

Keywords: Confidence, Generalization, Geogebra, Mathematical thinking, Pattern investigation

Generalization

Explore the pattern, predict, and forecast the future. It is simply what the researcher does always. In solving problem researcher always struggles into in investigating the underline pattern that is essential for its solution. Similarly, the mathematical understanding of every single individual is the result of generalization process that are lying at variance in the age and stage of the particular learner (Orton, 2004). Students' always struggle in generalizing different pattern of different content structures that is necessary for developing mathematical

^{*}Lecturer FG Boy Inter College Mardan: <u>khalilmathematics1977@gmail.com</u>

^{**}Assistant Professor, Department of Statistics, AWKUM:<u>umairkhalil@awkum.edu.pk</u>

^{***}lecturer, Department of education, Bacha Khan University Charsadda KPK: zahoorulhaqnsr@gmail.com

thinking. The tasks and activities in generalization content mostly include are involving pattern, out of the box thinking and the extension of individual observation in larger context (Zaman, 2011; Lew, 2004 & Ma 'moon, 2005). Lew (2004) further described that in algebra content; most of the algebraic understanding is the result of additive generalization. Consider, all the linear function in analytic geometry in which the relation is the result of some constant addition that mostly relevant to arithmetic progression. Take the specific example of y-5x=0 or y= 5x. The input, output table value of this linear function is given below:

x	0	1	2	3	4	
У	0	5	10	15	20	

The above linear pattern is the result of constant addition in the successive terms. Exploring this five and then generalize the pattern for "n" or for general x always tough for students in understanding this phenomenon.

Moreover, process is always there to reach at the generalization. In addition, many steps that consist of actions along with algorithm are essential and pre-requisites for developing this thinking skill in students. Students' generalization thinking varies across different stages; some time students use their observation to look at ascending or descending pattern or searching for similarities or they use the rule of assimilation and accommodation to organize their cognitive disc. Tall (1991) also declared that generalization and abstraction processes results in understanding the concept in mathematics learning and mathematics content also expand as a result of those cognitive processes. Further he did not stick this generalization process to one aspect, but he described it for three different interconnected categories are:1) reconstructive; 2) expansive 3) and disjunctive generalizations. Students can be involved to investigate and understand the pattern across different levels for the sake of to improve their skill of generalization thinking (Dindyal, 2007). Generally, in generalization process, extension and expansion of reasoning involves in both way: within system and out of the system.

Generally, mathematics content evolves spirally and learner previous understanding always plays integral for new idea. They use their generalization thinking based on similarity in their existing cognitive unit. By Dorkoand Weber (2014), students used taxonomy of generalization process in different situations, so, in teaching process, instructions should be design to support this process. Furthermore, they divided and described the taxonomy of this generalization process into two distinct categories with different attributes: generalization action, generalization reflection.

Generalization in Analytic Geometry

Specifically, in teaching of the concept of equation in algebra, pattern recognizing and analyzing along with articulating its structure are very important (Geraniou, Mavrikis, Hoyles & Noss, 2008). On the same token, in the teaching of analytic geometry, every algebraic geometric concept in this subject has some definite structure that always form a specific pattern. Moreover, to know its complexity students must understand its different pattern. Further, the variables that are responsible for designing the geometric pattern in the coordinate plane, their understanding are very important to know the reality of this subject. That why, in this experimental research, the instructions were integrated with Geogebra tool to strengthen this important mathematical thinking ability. For assessing this ability, almost six items were constructed. Moreover, all the questions are underlying some patterns: slope pattern, intercept pattern, numeric pattern, process pattern or result generalized pattern.

The item given below is an actual test item for this research study.

G 1:	Cor	m plet	te the	table	for	the	linear	function	
	х	0	1	2	3	4	5	6	
	Y	-1	1	3	5	7	-	-	
A1s	othe	egene	eralize	e form	ı of	the	above	linear patte	rn will be
A)	у =	= x - 1		B)	х	= 2y	+ 1		
C)	y =	= 2x -	1	D)	у	= 3x	: - 1		

Here the main object of this item is to explore and to estimate students' pattern investigation ability and its generalized result. In the same way, the third item was like this:

 G_3 : Evaluate the line slope having the points are: (a, c) and (b, d). Students have a lot of difficulty when they confront with general numbers instead of numeric. Their lack of arithmetic generalization ability stops them toward the exact solution. The six questions in this experimental study are specific to the content of grade-12 students' analytic geometry. As, most of the mathematical thinking' aspects are highly correlated because of their interdependency. Similarly, the interconnected concept of generalization thinking aspects, specialization, induction, symbolism and generalization are highly interrelated and correlated. Because of this attribute, Zaman (2011) mixed two of them in single identity.

moon (2005) combined induction and deduction along with symbolism as a single aspect of mathematical thinking.

Generalization Process in Linear Equation

No doubt, pattern investigation always leads mathematical thinking. The linear equation in which the pattern is additive and students' understanding about this phenomenon is important and help them in solving linear problem (Geraniou, Mavrikis, Hoyles & Noss, 2009). As linear equations are the generalization of arithmetic and, we use symbols to compact and represent the huge number population. These symbols always create problem for algebraic struggling students. Therefore, to minimize this cognitive burden teacher must give them arithmetic scaffolding. Because of minimum diversity in different aspects of generalization thinking, in this research study the induction, symbolism and generalization thinking was taken as a one-unit disc. Although, students always struggle to use symbols or parameters to generalize the linear pattern that are essential for algebraic reasoning. In linear equation, teacher must have the knowledge of two unknown parameters and in parabolic pattern the three parameters (Ellis, 2011).

The Importance of Generalization in Teaching of Mathematics

Mathematics teaching is not just to solve the question or to follow the algorithmic procedure towards a specific solution without any conscious behavior. Rather the main aim of mathematics is to inculcate the problem solving behavior. Although students having this ability at fraction level. Because, so many other behaviors are the pre-requisites for this aspect, in which generalization or pattern investigation is of paramount. In generalization process, students have to analyze the particular case, then through systematic behavior organizing the data and in the last conjecturing and generalizing. In most of the country, the pattern investigation and replacing pattern through variables and symbols is consider as integral potential (Barbosa, Palhares & Vale 2007). Across the grade from kindergarten to 12 grades the essence of pattern tasks for school in learning mathematics is also highly recognized and stressed by National Council of Teachers of Mathematics (NCTM, 2000). So many skills may be developing through these types of tasks that involve pattern. In short, the tool for mathematics learning is the result of this particular aspect generalization (Barbosa, Vale & Palhares, 2009).

The Role of Geogebra Technology in Developing Mathematical Thinking

Analytic geometry is an abstract subject and most of students having lot of difficulties in understanding its various nested concepts. Although having the 4th stage of formal thinking of Piaget, students lack behind to think abstractly. Further, because of their limited thinking behaviors about the content of analytic geometry they struggle with most of the concepts. In such situation, the teacher role is to know about the desired thinking behavior and the application of relevant tools that can assist their unconscious behavior to conscious, and to make the content from concrete to abstract in their teaching. With reference to this context, the GeoGebra is the best tool that can visualize most of the concepts of the analytics geometry with ease and it keep every individual as an active-independent learner in mathematics learning process (Purniati & Sudihartinih, 2015; Safdar, Yousuf, Parveen & Behlol, 2011).

Challenges in Teaching the Mathematical Thinking and GeogebraImplementation

Mathematical thinking is a process, which involves and invokes mental operation on math related operation (McDougall & Karadag, 2008). It is not limited to a single factor or single aspect but it is the cluster of different aspects and the performance of student in one aspect is highly correlated to the overall mathematical thinking and that effects achievement (Zaman, 2011 & Mubarak, 2005). This important element (mathematical thinking) necessary for learning mathematics (Stacy, 2006) but in our tradition approach the learning of mathematics is always done by memorizing facts and figure, and even the solution method of the problem. In such situation, student's role is just to memorize the written material by repetition action without understanding the underlining thinking behaviors. In spite of this reality, it works in our system because of the focus on one thing that is achievement (marks). It is because of the mostly common voice of parents, teachers, higher authorities and even students. Therefore, in such situation it is tough to change this concept with integration of GeoGebra environment and begin focus on understanding rather than memorization (Olsson, 2017).

Mathematical Confidence in Learning Mathematics

Indeed, students' positive attitude towards mathematics is positive indicator and vice versa. So, while teaching mathematics teacher should also keep this aspect in his/her mind along with skill and knowledge. It has a main and important role in conceptualizing the content of this subject. Students' affective domain should be in parallel dimension with the cognitive dimension. Out of different affective behaviors, students' mathematical confidence is one of the key indicators of his/her success (Grootenboer, & Hemmings, 2008).

To become a good problem solver in mathematics teacher as well as student attitude both are important and must have positive on measuring scale. Teacher' belief about teaching mathematics should be according to the demand of mathematic need. Their traditional concepts cause deficiency in learning mathematics of their students. So, prospective teacher program and mathematics teaching course training need to integrate this affective domain aspect in their program (Blanco, Guerrero & Caballero, 2013).

Methods and Procedure

Objectives of the Study

- 1) To explore the role of generalization thinking in teaching learning mathematics.
- 2) The development of generalization thinking through GeoGebra software.
- 3) To investigate students' confidence after experiment.

Delimitation

The current study was delimit to FG Boys Inter College Mardan Cantt.

Hypothesis of the Study

- H₀₁: The generalization thinking of the students do not effect at significance level on the integration of GeoGebra aided instructions.
- H₀₂: There is no difference in the mean and standard deviation of high and low achievers of the experimental and control group.
- H₀₃: Experimental group student gain more confidence in mathematics than control group after using Geogebra.

Population

The population was considered and included all govt as well as private higher secondary boys students. There numerical figure was almost, 384207 (EMIS, 2013-2014).

Sample participants and group construction

The participants (40 grade-12 students, age 16-18) of this study were belonged to a Federal Institute FG Boys Inter College Mardan Cantt. They were assigned to experimental and control group through pair random sampling based on their previous grade-12 mathematics

achievement. Further on the basis of same criteria they were identified as either higher or low achievers.

Content of the Experiment

Two chapters for this study were delimited:

- 1. Plane analytic geometry- straight line (chapter-6)
- 2. Conics-1 (circle portion, chapter-7)

(KPK Text Book for 2nd year)

Experiment Intervention

Activity and visualization are the main essence of understanding mathematics. For this study, 22 lessons were developed for the purpose of developing mathematical performance. The focus was to investigate the significance difference in the generalization thinking for the two treatments and to compare the mathematical confidence of the two groups after the treatment.

Both the group was taught by the same teacher under the assistance of supervisor in a well control environment. The experimental group was taught in a well-equipped computer lab through Geogebra software. They were kept as an active learner and freely involve in the exploring and discussing the concept while using Geogebra aided activity. The lessons were designed in such a way so that the students can validate and visualize the concept of the analytic geometry content.

In this study, the focus is on pattern and generalization exploration. As, every concept in mathematics underlying some specific pattern either: numeric, figural or pictorial. Therefore, the main purpose of these activities for the students was to look at the underlying pattern dynamically and observe the process in both ways i.e. algebraically and geometrically (Khalil, Sultana & Khalil, 2017).

The below Geogebra activity is one of the 22 lessons for experimental group. The main objectives of the lesson were:

- Generalization thinking (geometrically and algebraically)
- Students had to observe the pattern of the quadratic function as well as linear pattern.
- How to symbolize the linear and quadratic pattern through spread sheet in Geogebra.
- How to fit the numeric pattern into either linear or quadratic functions?



Figure 1: Pattern symbolizing activity

Data Tool

A test consist of 6 questions was developed to investigate the significance difference between two matched group after experiment.

Results

The t-test was used to know about the significance of difference between experimental and control group after experiment.

 Table 1: Significance of difference between the means scores of the two groups on generalization thinking scores

Groups	Ν	Mean	SD	df	t-test for equality of means				
Experimental	20	20.8	2.71		t	Sig. (2-tailed)	Mean Diff		
Control	20	17.6	4.4	38	2.77	0.009	3.2		

In the mean column of Table 1 indicates that the average score of experimental in comparison to control group is high. It shows that the experimental group performed well in comparison to control group. In addition, the low standard deviation of the experimental group shows that as whole the Geogebra aided instructions affected the performance this group well.

Further, for significance of difference the t-test result, that is t-value ($t_{cal} = 2.77$) and p-value= .009<.05. This shows that the difference is significant. So, we reject the null hypothesis.

Group		G1	G2	G3	G4	G5	G6
Experimental	Mean	3.95	3.80	3.70	3.70	2.70	2.95
N=20	Std. Dev	.22	.41	.66	.66	1.08	1.05
Control	Mean	3.45	2.90	3.40	2.70	2.80	2.35
N=20	Std. Dev	.89	.97	.99	1.08	1.10	1.18
Total	Mean	3.70	3.35	3.55	3.20	2.75	2.65

Table 2: Descriptive statistics of experimental and control group for six questions of generalization thinking

Table 2 shows the overall performance of the both group in each single question of the generalization thinking. It indicates that both the group performed well in these aspects because of the activity and thinking aided instructions. The mean cell of each of the group for six questions revealed that the experimental group mean's score for each question is greater than control group except Q_5 . Similarly, the low standard deviation for each question of generalization thinking except Q_5 shows that the Geogebra assisted instructions affected collectively the performance of experimental group.

 Table 3: Descriptive statistics of experimental and control group for six questions of generalization thinking

Diverse Grou	р	G1	G2	G3	G4	G5	G6
Exp.High	Mean	4	3.9	3.9	4	3.1	3.4
N=12	Std. Dev	.00	.29	.29	.00	.99	.9
Cont. High	Mean	3.8	3.4	3.8	3.2	3.3	2.7
N=12	Std. Dev	.57	.90	.62	.93	.89	.98
Exp.Low	Mean	3.9	3.62	3.38	3.25	2.12	2.25
N=8	Std. Dev	.35	.52	.91	.88	.99	.88
Cont. Low	Mean	2.9	2.1	2.9	2	2	1.9
N=8	Std. Dev	.99	.35	1.2	.92	.92	1.3
Total N=40	Mean	3.70	3.35	3.55	3.20	2.75	2.65

Table 3, indicates that the generalization thinking of the groups were improved statistically. But it vividly affected the performance of the experimental low achievers with greater difference. It means that Geogebra is a good influential tool for low achiever students.

		Experimental Group				Contr	ol Gro	up			
	Statement	Α	SA	DA	SDA	U	Α	SA	DA	SDA	U
1	I am sure that I can learn math										
	quickly	2	18	0	0	0	6	14	0	0	0
2	I can get good marks in math										
	Subject	6	12	2	0	0	7	10	2	1	0
3	I want to take part in math										
	Competition	8	10	1	1	0	12	6	0	2	0
4	I think in math, to understand										
	difficult concept is possible	8	11	1	0	0	10	8	0	2	0
	Cumulative response	24	51	4	1	0	25	38	2	5	0
	Percentage Cumulative										
	response	30	63.75	5	1.25	0	31.25	47.5	2.5	6.25	0

Table 4: Experimental and control group mathematical confidence responses statistics

Table 4 shows the responses of both the group after experiment. If look at the table statistics and last percentage row, it can be concluded that Geogebra increased the confidence of the experimental group more than the control group. The greater percentage of the respondents of the Geogebra group (30% agree and 63.75% strongly agree) in comparison with control group (31.25% and 47.5%) felt that the Geogebra is highly beneficial in favor of students.

Conclusions

In our education system, teachers always use memorization of solution techniques in their teaching and in the output, they expect from their students, conceptual understanding and problem solving behavior. Generalization thinking is an important, pre-requisite aspect, in teaching learning of mathematics. This thinking behavior can be integrated in the content of analytic geometry by introducing numeric, geometric, symbolism and process pattern. In such a case, the role of teacher and the use of specific technology are integral. The result of this study revealed that Geogebra significantly affected experimental group students' generalization thinking in comparison with control group. Further, it also increased the confidence of experimental group in learning mathematics. It is to be added that teachers' consciousness about the generalization thinking and expertise in the TPCK (technological pedagogical content knowledge) are necessary for its development.

Recommendations

As mathematics need understanding and thinking behavior, so, it is necessary that the mathematics curriculum should be reviewed in respect to mathematical thinking behavior. Its objectives should be assessed on the basis of its nature. To achieve its objectives the dynamic geometry software is a better choice and should be integrated in its process of implementation. Secondly, the Effect of GeoGebra is depends on the activity and its implementation. So, in this regards it is suggested for Higher Education Commission Pakistan (HEC) to launch a special degree program for mathematics teacher and integrate the courses mathematical thinking and Geogebra in its content which is still not acknowledged until now. Likewise, in mathematics teaching training program these aspects should also be considered. Generally, it can be done to develop standardized activities for specific thinking purpose to conserve the teaching learning process.

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Appendix

Generalization Thinking Questions

Name	G	Group	
College Name	FG Boys Inter College Mardan Ca	antt	Code No-

This test was designed to measure your mathematical thinking ability. The test consists of 6 questions.

- 1- Write your name, code no Exp or Cont and Group.
- 2- Read each question carefully before attempt.
- 3- Justify and give reason, how you arrive at the solution.
- 4- In front of each question or on extra sheet you can write your justification.
- 5- Multiple-choice questions have only one correct option you have to encircle the one option according to your thinking along with best justification.
- 6- This test is only for the research of this study along with confidential.Don't fill the below table, it's for evaluator.

Q. No	G ₁	G ₂	G ₃	G ₄	G ₅	G ₆	Total
Marks							

Questions related to Generalizatio

01	a 1.	- 1		0	. 1		C	
GI:	Complete	the	table	tor	the	linear	function	

Х	0	1	2	3	4	5	6
Y	-1	1	3	5	7		

Also, the generalize form of the above linear pattern will be

A)
$$y = x - 1$$

C) $y = 2x - 1$
B) $x = 2y + 1$
D) $y = 3x - 1$

D)
$$y = 3x - 1$$

G2: Find the missing numbers of quadratic pattern

Х	1	2	3	4	5	 n
Y	4	7	12			

Also, the generalize form for "n" the above pattern will be

A)
$$y = 5x^2 - 1$$

C) $y = 3x^2 + 1$
B) $y = 4x^2$
D) $y = x^2 + 3$

G3: The slope of a line passing through points (c, d) and (a, b) is

-	-	-	-		
A)	b-c			D)	b-d
A)	d = a			Б)	a-c
	","				"
\sim	a-b			D)	a-p
()				D)	
	a-c				c-d

Investigate the intercept pattern of the given parallel lines. G4:



The equation of L_4 is y = x - 2 L_3 is And the equation of y = x - 1Similarly the equation of L_2 is y = x

What will the equation of L1 and L5 if we construct it in the same way? Encircle the correct option below +

Equation for L ₁	Equation for L5
A) $y = x + 1$	A) $y = x + 1$
B) $y = x + 2$	B) $y = x - 2$
C) $y = x + 3$	C) $y = x - 3$
D) $y = x - 1$	D) $y = x - 1$



