The Construction of Polynomial Spiral Segment Using Cubic Ball Basis Functions

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Abstract

B-splines, Bezier, Ball curves and NURBS (non-uniform rational B-splines) are commonly used in CAD and CAGD applications. Unfortunately their fairness is not guaranteed. Spiral segments help us in designing improved form of curves called fair curves. Such fair curves are useful in sophisticated applications such as design of routes of high ways and railways and mobile robot trajectories. In this paper we have developed the polynomial cubic Ball spiral segment with degree of freedom. The effect of shape parameters is also observed. In the end results are represented in graphical form.

Keywords: Curvature, Cubic Ball Basis Functions, Polynomial Spiral Segment, Fair Curves

1 Introduction

Curvature continuous curves with undesirable extrema are attributes to specific applications such as design of the trajectories of mobile robot and high ways or railway routes [1, 2]. These curves are called fair curves [3]. These curves are equally important in computer aided design CAD and computer aided geometric design CAGD applications such as [3, 4]. B-spline, Bezier, Ball curves and NURBS are also used in CAD and CAGD applications. But their fairness may not be guaranteed. To adhere this problem polynomial spirals are developed.

We study spiral in terms of a curved line segment with variation of signed curvature in a monotonic way. Spiral segment refers to any one of aspects: firstly Any segment of a curve between two consecutive curvature extrema an secondly between the first endpoint and first curvature extremum and third between the last endpoint and finally last curvature extremum. Enriched literature on Bezier spiral is found in research world. A planar cubic Bezier curve developed [5] has been generalised [6, 9] having point of zero curvature at one end point. Where as cubic Spiral segments without any point of zero curvature also exists [7] which is used for shape control [8].

In this manuscript we have proposed the polynomial spiral segment (cubic Ball spiral) using cubic Ball basis functions by incorporating both the options (zero curvature at one end or non zero curvature at both ends). The advantage of Ball basis functions can be broadly divided in two, when comparing it with Bezier basis functions.

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Firstly, a robust algorithm has been developed to evaluate the Ball curve which suits interactive design environment [13]. Secondly, generalized Ball basis suits much better in degree elevation and reduction which eases data portability and curve approximation in CAD systems [10, 11].

The following section describes the notation and conventions used in this work; it is followed by a section on cubic Ball curve. The development of the cubic Ball spiral is done in Section 4 which is followed by a section with graphical presentation and concluding remarks.

2 Notation and Conventions

The usual Cartesian coordinate system with $x-axis$ and $y-axis$ is supposed. Angles are measured anti clockwise direction. Vectors and points are represented as, for example $\overrightarrow{W}$. Points and vectors may also be indicated using the ordered pair notation, e.g, (x, y). In particular, the components of a vector $\overrightarrow{u}$ may be denoted as (u_x, u_y), or in the case of a subscripted vector, e.g, $\overrightarrow{W_0}$ as $(W_{0,x}, W_{0,y})$ or $(W_{b,0,x}, W_{b,0,y})$ for a doubly subscripted vector, $\overrightarrow{W}_{b,0}$. The dot product of two vectors, $\overrightarrow{u}$ and $\overrightarrow{v}$ is denoted as $\overrightarrow{u}.\overrightarrow{v}$. The norm or length of a vector $\overrightarrow{u}$ is denoted as $\|\overrightarrow{u}\| = \sqrt{\overrightarrow{u}.\overrightarrow{u}}$. The derivative of a function (scalar or vector valued) is denoted with a prime, e.g $\overrightarrow{P}'(\psi)$. A planar parametric curve is defined by the set of points $\overrightarrow{P}(\psi) = (X(\psi), Y(\psi))$ for real ψ. To aid concise writing of mathematical expressions, the symbol $\times$ is used to denote the signed z-component of the usual three-dimensional cross-product of two vectors in the xy plane, i.e, $\overrightarrow{u} \times \overrightarrow{v} = u_x v_y - u_y v_x$. The tangent vector of a plane parametric curve $\overrightarrow{P}'(\psi)$ is given by $\overrightarrow{P}'(\psi) = (X'(\psi), Y'(\psi))$. If $\overrightarrow{P}'(\psi) \neq \overrightarrow{0} = (0, 0)$, then the signed curvature of $\overrightarrow{P}(\psi)$ is defined as [3]

$$c(\psi) = \frac{\overrightarrow{P}'(\psi) \times \overrightarrow{P}''(\psi)}{\|\overrightarrow{P}'(\psi)\|^3}. \tag{2.1}$$

Differentiation of Eq. (2.1) yields

$$c'(\psi) = \frac{w(\psi)}{\|\overrightarrow{P}'(\psi)\|^5}. \tag{2.2}$$

where,

$$w(\psi) = \{\overrightarrow{P}'(\psi).\overrightarrow{P}'(\psi)\}\{\overrightarrow{P}'(\psi) \times \overrightarrow{P}'''(\psi)\} - 3\{\overrightarrow{P}'(\psi) \times \overrightarrow{P}''(\psi)\}\{\overrightarrow{P}'(\psi).\overrightarrow{P}''(\psi)\}. \tag{2.3}$$

3 Cubic Ball basis curve

The cubic Ball polynomial basis was first proposed by Ball [12] for CAD systems application. The Ball basis functions are defined as

$$\begin{aligned} &S_0(\psi) = (1-\psi)^2,\ S_1(\psi) = 2\psi(1-\psi)^2, \\ &S_2(\psi) = 2\psi^2(1-\psi),\ S_3(\psi) = \psi^2. \end{aligned} \tag{3.1}$$

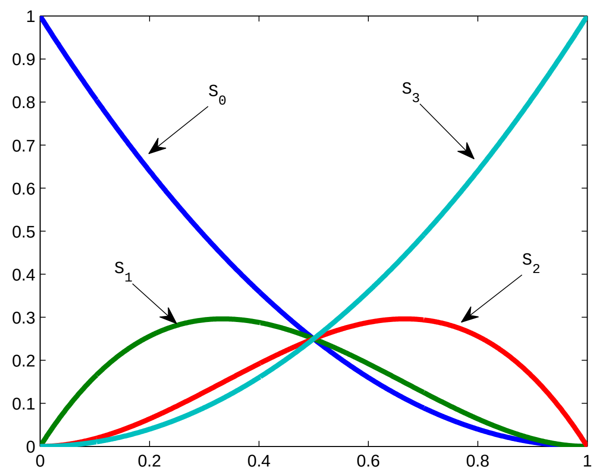


Figure 1: Ball basis functions

Figure 1 illustrates these functions against its parameter ψ. The cubic Ball curve $P(\psi)$ with control points $\overrightarrow{A_i}$ is defined as

$$P(\psi) = \overrightarrow{A}_0(1-\psi)^2 + 2\overrightarrow{A_1}(1-\psi)^2\psi + 2\overrightarrow{A_2}(1-\psi)\psi^2 + \overrightarrow{A_3}\psi^2, \ 0 \leq \psi \leq 1. \tag{3.2}$$

The Ball basis functions and curve obey all the properties of curve like linearly independent, non-negativity, symmetric, monotonicity, partition of unity, convex hull property and affine under linear transformation.

4 Cubic Ball Spiral

In Eq. (3.2) assume the control points $\overrightarrow{A}_0$, $\overrightarrow{A}_1$, $\overrightarrow{A}_2$ and $\overrightarrow{A}_3$ are distinct. Without loss of generality, translate, rotate, and if necessary, reflect the curve such that $\overrightarrow{A}_0$ is at the origin, $\overrightarrow{A}_1$ is on the positive x-axis and $\overrightarrow{A}_3$ is above the $x-axis$, Eq. (3.2) may now be written as

$$\overrightarrow{P}(\psi) = (X(\psi), Y(\psi)) \tag{4.1}$$

where,

$$\begin{aligned} X(\psi) &= 2ag\psi(1-\psi)^2 + 2(g + kcos(\theta) - bkcos(\theta))\psi^2(1-\psi) + (g + kcos(\theta))\psi^2, \\ Y(\psi) &= 2(ksin(\theta) - bksin(\theta))\psi^2(1-\psi) + (ksin(\theta))\psi^2. \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\|\overrightarrow{A}_1 - \overrightarrow{A}_0\| > 0, \\ &\|\overrightarrow{A}_2 - \overrightarrow{A}_1\| > 0, \\ &\|\overrightarrow{A}_3 - \overrightarrow{A}_2\| > 0, \end{aligned} \tag{4.3}$$

where θ is a angle between $\overrightarrow{A}_3 - \overrightarrow{A}_2$ and $x-axis$.
The curvature at $\psi = 0$ and $\psi = 1$ using Eq. (2.1) is

$$c(0) = \frac{(3-2b)ksin(\theta)}{2a^2g^2},$$

and

$$c(1) = \frac{(3-2a)gsin(\theta)}{2b^2k^2}.$$

Theorem 1. *The curvature of* $\overrightarrow{P}(\psi)$ *is 0 at 0 iff* $b = 3/2$.

Now by using the above theorem, Eq. (4.2) becomes

$$X(\psi) = 2ag\psi(1-\psi)^2 + 2(g - 1/2kcos(\theta))\psi^2(1-\psi) + (g + kcos(\theta))\psi^2, \quad (4.4)$$
$$Y(\psi) = -ksin(\theta))\psi^2(1-\psi) + ksin(\theta))\psi^2. \quad (4.5)$$

and curvature becomes

$$c(0) = 0,$$
$$c(1) = \frac{2(3-2a)gsin(\theta)}{9k^2}.$$

The derivatives of Eq. (4.4) are

$$\begin{aligned} X'(\psi) &= 2ag(1-\psi)^2 - 4ag(1-\psi)\psi + 4(1-\psi)\psi(g - 1/2kcos(\theta))... \\ &\quad -2\psi^2(g - 1/2kcos(\theta)) + 2\psi(g + kcos(\theta)), \\ X''(\psi) &= -8ag(1-\psi) + 4ag\psi + 4(1-\psi)(g - 1/2kcos(\theta)) - 8\psi(g - 1/2kcos(\theta)) + 2(g + kcos(\theta)) \\ X'''(\psi) &= 12ag - 12(g - 1/2kcos(\theta)) \end{aligned} \quad (4.6)$$

$$\begin{aligned} Y'(\psi) &= 2k\psi sin(\theta) - 2k(1-\psi)\psi sin(\theta) + k\psi^2 sin(\theta) \\ Y''(\psi) &= 2ksin(\theta) - 2k(1-\psi)sin(\theta) + 4k\psi sin(\theta) \\ Y'''(\psi) &= 6ksin(\theta) \end{aligned} \quad (4.7)$$

Using Eq. (4.4), Eq. (4.6) and Eq. (4.7) we can write

$$\begin{aligned} \overrightarrow{P}'(\psi).\overrightarrow{P}'(\psi) &= (9k^2)\psi^4 + (+36gkcos(\theta) - 8agkcos(\theta))\psi^3(1-\psi)... \\ &\quad +(36g^2 - 32ag^2 + 12agkcos(\theta))\psi^2(1-\psi)^2 + ... \\ &\quad (24ag^2 - 16a^2g^2)\psi(1-\psi)^3 + 4a^2g^2(1-\psi)^4 \\ \overrightarrow{P}'(\psi).\overrightarrow{P}''(\psi) &= (-18gkcos(\theta) + 18k^2 + 12agkcos(\theta))\psi^3 + (+46gkcos(\theta)... \\ &\quad -4k^2 + 32ag^2 - 40g^2 - 24agkcos(\theta))\psi^2(1-\psi) + (32g^2 - 84ag^2 - 8gkcos(\theta)... \\ &\quad -4k^2 + 40a^2g^2 + 12agkCos(\theta))\psi(1-\psi)^2 + (12ag^2 - 16a^2g^2)(1-\psi)^3 \\ \overrightarrow{P}'(\psi) \times \overrightarrow{P}''(\psi) &= (18gksin(\theta) - 8agksin(\theta))\psi^2 - 4agk\psi^3 Sin(\theta)... \\ &\quad +8agk\psi(1-\psi)Sin(\theta) + 4agk(1-\psi)^2 Sin(\theta) - 4agk(1-\psi)^3 Sin(\theta) \quad (4.8) \\ \overrightarrow{P}'(\psi) \times \overrightarrow{P}'''(\psi) &= (36gksin(\theta) - 24agksin(\theta))\psi - 12agk\psi^2 sin(\theta) + 12agk(1-\psi)^2 sin(\theta) \end{aligned}$$

Substitution of Eq. (4.8) into Eq. (2.3), followed by some algebraic maipulation yields $w(\psi) = 12gkf(\psi)sin(\theta)$ where

$$f(\psi) = \sum_{i=0}^{6} f_i(1-\psi)^{6-i}\psi^i, \quad 0 \le \psi \le 1, \quad (4.9)$$

with

$$f_0 = 4a^3g^2 \tag{4.10}$$

$$f_1 = 24a^3g^2 \tag{4.11}$$

$$f_2 = -2a((21 + 2a(-94 + 25a))g^2 - 6k^2 + 12(-1 + a)gkCos(\theta)) \tag{4.12}$$

$$f_3 = 2(-2(9 + a(-87 + 14a))g^2 + 3(3 + 2a)k^2 + 2(9 + 10(-3 + a)a)gkCos(\theta)) \tag{4.13}$$

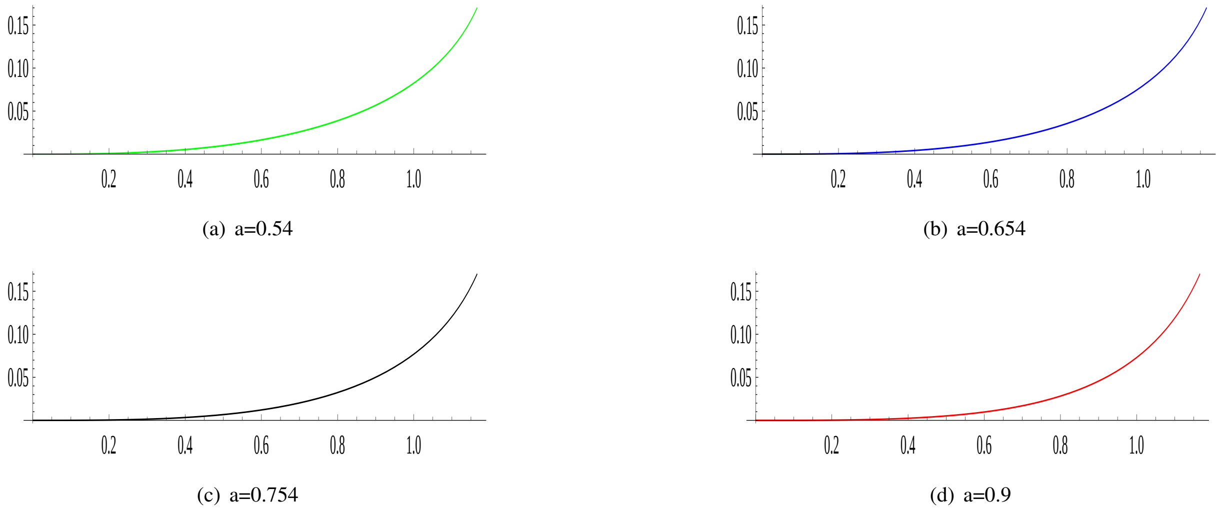
$$f_4 = 6(24 + a(21 + 4a(-14 + 5a)))g^2 + 3(12 - 19a)k^2 + (-63 + 4(6 - 5a)a)gkCos(\theta) \tag{4.14}$$

$$f_5 = -6(-2(-3 + 2a)(-5 + 4a)g^2 + (6 + 5a)k^2 + (3 + 2a(-5 + 4a))gkCos(\theta)) \tag{4.15}$$

$$f_6 = 9k(3(-2 + a)k + (3 - 2a)^2gCos(\theta)) \tag{4.16}$$

5 Results and Discussion

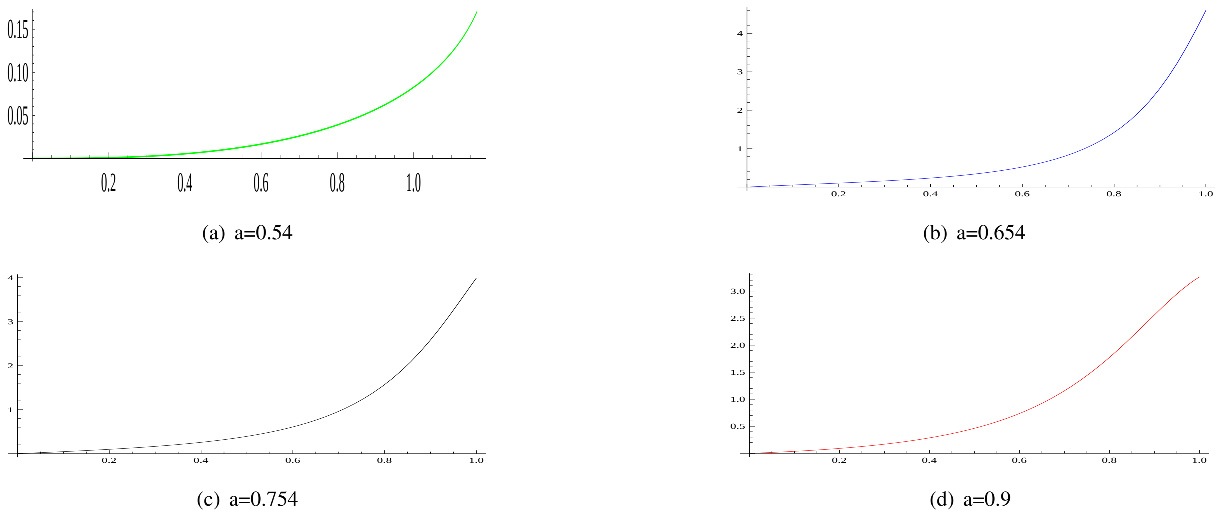
This section illustrates the different possibilities of cubic Ball spiral curve using proposed Cubic Ball spiral. As the proposed curve have 5 degree of freedoms namely $a,\ b,\ g,\ k,$ and θ. In Theorem 1 we have fixed $b = 3/2$ to get the zero curvature at 0. We have constructed the different possible Ball spiral curves. In Figure 2 we fixed the $\theta = \pi/4,\ g = 0.996$ and $k = 0.24$ randomly and change the free parameter a. Figure 3 shows that curvature of Ball spiral curves are monotone. Similarly Figure 4 and Figure 5 shows the different curves with their monotone curvature graphs. Figures 6 and 7 represent the comparison of different Ball spiral curves and curvature keeping $g,\ \theta,$ and k constant and changing the value of a. Figure 8 presents the different Ball spiral curves with different angles. In last Figure 9(a) is a cubic Ball curve and Figure 9(b) is the curvature of Ball curve. From Figure 9 we can conclude that simple cubic Ball curve looks smooth but it is not fair. Because the curvature graph is not monotone.



(a) a=0.54 (b) a=0.654

(c) a=0.754 (d) a=0.9

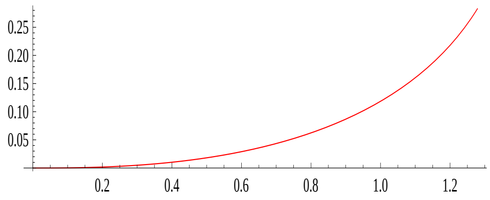
Figure 2: Cubic Ball spiral with g=0.996,k=0.24



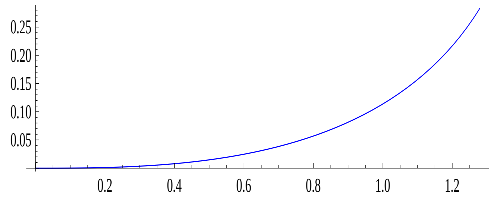
(a) a=0.54 (b) a=0.654

(c) a=0.754 (d) a=0.9

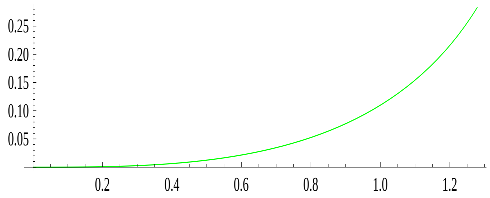
Figure 3: Monotone curvature with g=0.996,k=0.24



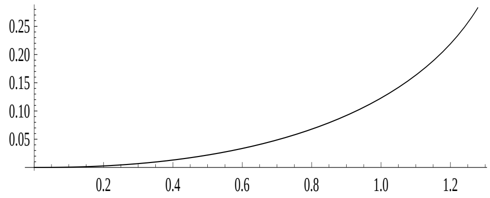
(a) a=0.459



(b) a=0.59

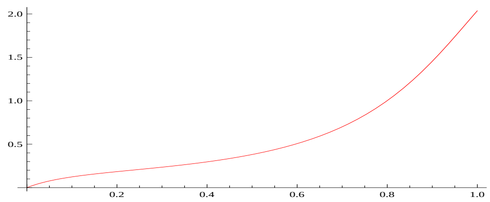


(c) a=0.69

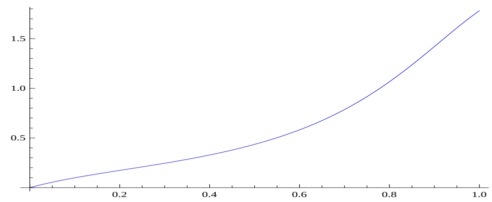


(d) a=0.312

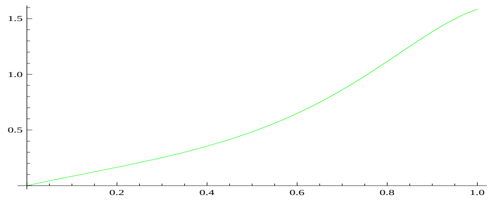
Figure 4: Cubic Ball spiral with g=0.996,k=0.4



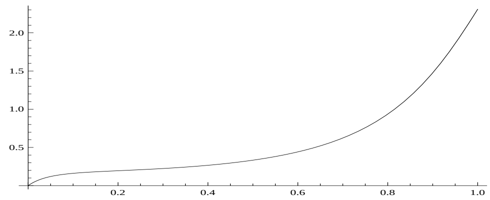
(a) a=0.45



(b) a=0.59



(c) a=0.69



(d) a=0.312

Figure 5: Monotone curvature with g=0.996,k=0.4

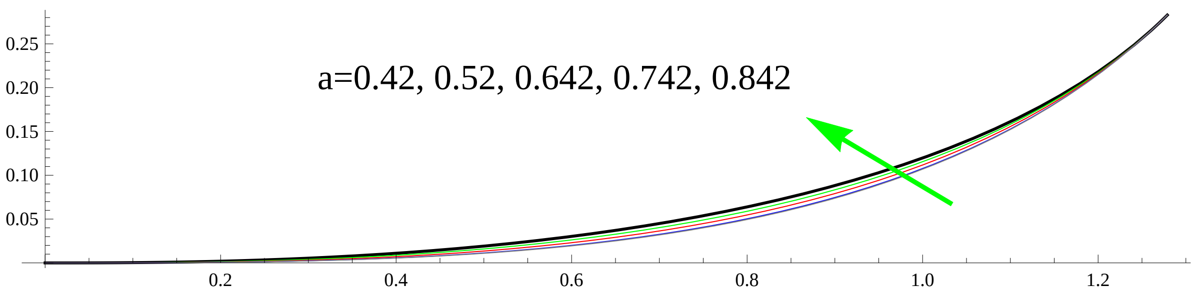


Figure 6: Ball spiral curves with different Values of a

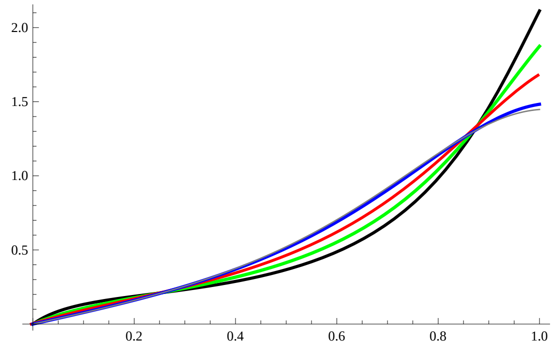


Figure 7: Curvature graph with different values of a

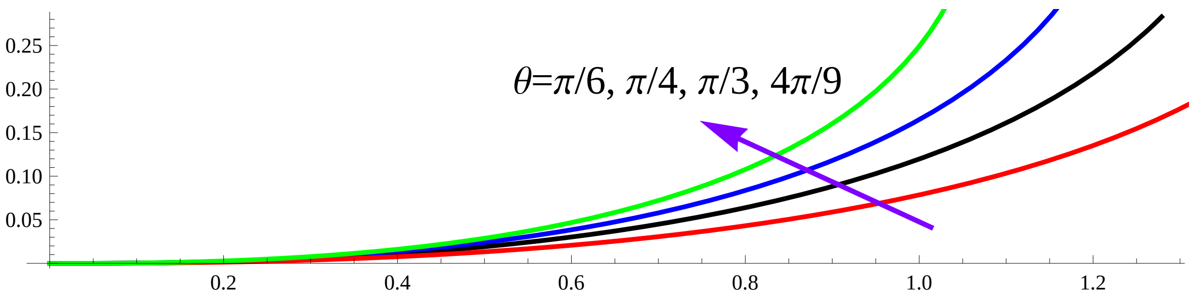
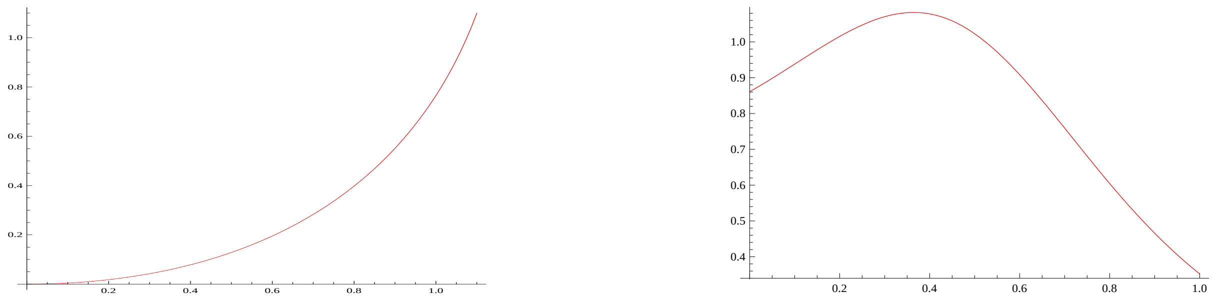


Figure 8: Ball spiral curves with different angles



(a) Cubic Ball Curve

(b) Cubic Ball Curvature

Figure 9: Cubic Ball curve with curvature graph

6 Conclusion

In this manuscript we have developed a cubic Ball spiral segment with the free parameters and the curvature of spiral is always monotone. The point of zero curvature introduces flat spots where they may not be desirable. Using proposed cubic Ball spiral, such flat spots can be avoided. We compared the cubic Ball curve with cubic Ball spiral curve. Although the cubic Ball curve may or may not be fair, in contrast the cubic Ball spiral is always fair curve. In respect to monotone the cubic Ball curve may or may not be monotone nevertheless the curvature of cubic Ball spiral is always monotone.

7 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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