# Computational Results for Lorenz Dynamical Model

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***Abstract*— This article deals with the Lorenz chaotic system, which is used in biological science, astronomical science and environmental science. Lorenz dynamical system is well known due to its sensitivity. This article presents the detailed computational study of Lorenz chaotic system and compensating the sensitivity which depends on the parameters and initial conditions. We use a convergent technique for nonlinear equation known as Successive-Over-Relaxation as well as use finite difference method and RK4 method. Discretized the Lorenz chaotic system through a convergent technique known as SOR, and it is applied to get equations for better accuracy and for comparable results. The overall comparisons of proposed and existing strategies are discussed through graphical measures, phase portrait and in the form of tables.**

***Index Terms*—Lorenz chaotic model; Successive-Over-Relaxation method; finite difference method; RK4 method**

## I. INTRODUCTION

n 1880s Poincaré was studying the 3-body problem and Idiscovered chaos but his discovery has no any major development in this regards. In 1963 Ed Lorenz was shown a simplest mathematical Ordinary differential equation which has skill to make chaos. This system became a model of chaos and associated with strange attractor, which shows the wings of butterfly. This time Lorenz was not sure to discover the chaos but he trying to find the solution of system which is complicated as compare to periodic. He was very surprise to see the system, which is highly sensitive to its initial conditions the "butterfly effect", but the importance was understood very fast [1]. Lorenz had shown that the long range weather prediction is challenging due to the error in the date set used which is dependent on initial conditions. Lorenz defines the complex structure of chaotic attractor and estimate the predictability [2].

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Atmospheric gases climates and the chaotic behavior of climate are amazing with the discovery of Lorenz indication, and producing the changes in weather as however unimagined also ice caps, the temperature of the oceans, and promoting changes in many other factors which affect the climate. Ed Lorenz achieved his system from thermal convection in a fluid Layer as reduced model and the due to its physical meaning, parameters are considered to be positive [3].Chaos has many features and one of them is that, it will occur in nonlinear, low dimensional and very simple equations. There are several autonomous ordinary differential equation with containing one or two quadratic nonlinearities and less than seven terms containing chaotic solution. Instead of equations describing the physical processes and which are of practical and mathematical interest, the simplicity lies in the algebraic expressions. A practical example of which are the chaotic electrical circuits used for the encryption and decryption for reliable communication [4]. Continuous system with three or more dimensions has a collective behavior of deterministic chaos [5]. In many engineering systems chaos has been found. Chaotic behavior is very complex and irregular, especially undesirable in mechanical system. Improved system performance or the escaping of tiredness requires controlling the system and hence the chaos is removed, providing the system stable and predictable behavior. Therefore the Lorenz chaotic system is used as a paradigm. In the area of nonlinear dynamics chaos of controlling or chaos of ordering is getting enlarge consideration for research because it captures several characteristic of chaotic dynamics. In 1990 Ott et al Grebogi and Yorke Ja showed that the chaotic attractor can be converted into large number of possible time periodic motions. By using a differential geometric method Fun and Tung controlled Lorenz chaos by a very effective approach.[6]. The system performance and response will decrease when the nonlinearity present in control input and also the chaotic system perturbs to unpredictable results due to its sensitivity to any system parameter. In control design the effect of nonlinearity cannot be ignored for the realization of chaotic system [7]. In several discipline like biomedical, human brain liquid mixing with low power consumption, engineering applications, through secret communication chaos is found to be a very great potential. It is a very interesting nonlinear phenomenon. Several new types of synchronization have appeared over the last decade such as phase synchronization, adaptive synchronization, lag synchronization, generalized synchronization and chaotic synchronization [8]. As time goes on, the importance of synchronization increased and people began to realize its important role in fast convergence speed

[9]. The study of different uncertainties is also important because it affect the synchronization even break it.

There are two types of chaos control one is create or enhance chaos when it is necessary and the second one is suppress which is dynamical behavior when it is harmful and known as anticontrol of chaos or called chaotification. Chaotification is very interesting theoretical subject, because it includes some very difficult but well-manage dynamical behaviors, which is generally contain bifurcations and fractals [10]. Chaotic systems also provide detailed mechanism for generation and signal design, and large applications in communications and signal processing due to chaotic signals that is normally unpredictable, noiselike and broadband. Chaotic signals can also be used in many context like in spread spectrum system used as a modulating wave forms and for masking information bearing waveforms [11]. If the chaotic system decomposes into two subsystems a transmitter (drive system) and receiver (stable response subsystem) it will self-synchronized.

To solve a Lorenz system, at first step is to discretize, and then first order derivatives are replaced by the finite difference and, then get the system of nonlinear equations to investigate the dynamics of the Lorenz system and proved that the exact solution of the system is difficult to achieve from the said techniques. Now to achieve the approximate solution we utilized the well-known technique Successive-Over-Relaxation for the discretized equations to get the approximate solution of the Lorenz system. In this article we use Successive-Over-Relaxation (SOR) technique for Finite Difference to approximate the Lorenz chaotic model. Successive-Over-Relaxation (SOR) is an iterative technique which solves the matrix equations.

The Successive-Over-Relaxation method is the version of Gauss-Seidel method for the solution of linear system of equations. Gauss-Seidel is also an iterative method based on Jacobi's method which is based on back substitution. The method is useful to increase the rate of convergence. The Successive-Over-Relaxation (SOR) method is the extrapolation of Gauss-Seidel [12]. The SOR technique is used to minimize error or to minimize the residual.

The present and the previous iterations give the weighted average value for the extrapolation. The present Gauss-Seidel iteration successively for each component is,

$$y_i^{(k+1)} = \omega \bar{y}_i^k + (1-\omega) y_i^k \qquad [13]$$

Where $\overline{y_i^k}$ represents the $i^{th}$ iteration by Gauss Seidel Method and '$\omega$' is the relaxation parameter. The successive over-relaxation is working as catalyst and hence improves the results in aforesaid techniques. The formulation of the Successive-Over-Relaxation (SOR) method depends on a relaxation parameter. Let us denote the iteration matrix of the SOR matrix as $Y_\omega$, then the speed of its convergence is computed by the spectral radius $\rho Y_\omega$ which is absolute value of the largest eigenvalue in magnitude of $Y_\omega$. The optimal value for $\omega$ that is used to minimize the spectral radius$\rho Y_\omega$, is required to include the Successive-Over-Relaxation (SOR) method in comparisons between the iterative methods. The relaxation parameter for the optimal value is given by,

$$\omega_{opt} = \frac{2}{1+\sin(\pi h)} \tag{2}$$

The Eq. (2) is depending on the Taylor series expansion [14]. The SOR technique is competent numerical tool and has vast range application like to predict the structure behavior of plate, to investigate the chaotic and sensitive system. Also, this technique converges rapidly and reduces the computational time [15]. After discretization of Lorenz model results into a nonlinear equations that are very difficult to approximate[16]. Newton's method is well-known method to approximate the solution of such type of nonlinear equations. Convergence technique of order third and order fourth have been proposed and analyzed to obtain effective results.

Section-2 of this paper provides the detailed implementation of Lorenz system and algorithm of Successive-Over-Relaxation (SOR) technique. Section-3, numerical scheme are discussed by the help of graphs and tables.

## II. Mathematical Algorithm

Now in this section we define the model and also the short introduction of finite difference with SOR technique. Consider the following nonlinear system of equations,

$$\dot{\vec{Y}} = \vec{F}(y_1, y_2, y_3, ..., y_n) \tag{3}$$

where

$$\dot{\vec{Y}} = \begin{bmatrix} \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dots & \dot{y}_n \end{bmatrix}^t$$

and

$$\vec{F}1 = f_1(y_1, y_2, y_3, \dots, y_n) f_2(y_1, y_2, y_3, \dots, y_n) f_3(y_1, y_2, y_3, \dots, y_n) = [f_1(y_1, y_2, y_3, \dots, y_n) f_2(y_1, y_2, y_3, \dots, y_n) \dots f_n(y_1, y_2, \dots, y_n)]$$

Now by using the Taylor series expansion w.r.t't' the first order finite difference approximation is given as,

$$\frac{df}{dt} = \frac{f_{i+1} - f_i}{h} + O(h) \tag{4}$$

where $h$ represents the step size. Now by using Eq (4), into Eq (3), we get the following equations,

$$\begin{aligned} y_{1(i+1)} &= y_{1(i)} + hf_1(y_1, y_2, y_3, \dots, y_{n-1}) \\ y_{2(i+1)} &= y_{2(i)} + hf_2(y_1, y_2, y_3, \dots, y_{n-1} \\ y_{3(i+1)} &= y_{3(i)} + hf_3(y_1, y_2, y_3, \dots, y_{n-1}) \\ &\vdots \\ y_{n(i+1)} &= y_{n(i)} + hf_n(y_1, y_2, y_3, \dots, y_{n-1}) \end{aligned} \tag{5}$$

Aforesaid is nonlinear system of equations, using the SOR technique to approximate the above system and also the mathematical form of SOR-Newton method for system of nonlinear equation is given below:

$$y_i^{k+1} = y_i^k - \omega \frac{H_i(Y^{k,i-1})}{H_{ii}(Y^{k,i-1})}; \quad i = 1, 2, \ldots, n \text{ and } k = 0, 1, 2\ldots$$

where

$$H_{ii} = \frac{\partial H_i}{\partial Y_i}$$

$H$ represents the vector of system of nonlinear equations, and

$$Y^{(k,0)} = (y_1^{(k,0)}, y_2^{(k,0)}, \ldots, y_n^{(k,0)})^t = Y^{(k)}$$
$$Y^{(k,l)} = (y_1^{(k,l)}, y_2^{(k,l)}, \ldots, y_n^{(k,l)})^t$$
$$= (y_1^{(k+1)}, y_2^{(k+1)}, \ldots, y_j^{(k+1)}, y_{j+1}^{(k)}, \ldots, y_n^{(k+1,l)})^t$$

for $j = 1,2, \ldots, n-1$. It gives us,

$$y_l^{(k,l)} = y_l^{(k,j-1)}, l \neq j, l = 1, 2, \ldots, n.$$

Now to apply the SOR-Newton technique on the system (5), we get,

$$x_{1(i+1)} = (1-\omega)x_{1(i)} + \omega\left\{x_{1(i)} + hf_1(x_1, x_2, x_3, \ldots, x_{n-1})\right\},$$
$$x_{2(i+1)} = (1-\omega)x_{2(i)} + \omega\left\{x_{2(i)} + hf_2(x_1, x_2, x_3, \ldots, x_{n-1})\right\},$$
$$x_{3(i+1)} = (1-\omega)x_{3(i)} + \omega\left\{x_{3(i)} + hf_3(x_1, x_2, x_3, \ldots, x_{n-1})\right\}, \quad (6)$$
$$\vdots$$
$$x_{n(i+1)} = (1-\omega)x_{n(i)} + \omega\left\{x_{n-1(i)} + hf_1(x_1, x_2, x_3, \ldots, x_{n-1})\right\}$$

$\omega$ works as a relaxation parameter in SOR method. Aforesaid are the general formulation and we can apply on every nonlinear system to solve it by the help of finite difference SOR technique. Now we can apply the above formulation on Lorenz system,

$$\dot{Y}_1 = \alpha(y_2 - y_1)$$
$$\dot{Y}_2 = -y_1 y_3 - \gamma y_2 \quad (7)$$
$$\dot{Y}_3 = y_1 y_2 - \frac{\beta}{\alpha} - \gamma y_3$$

In the above system $y_1, y_2$ and $y_3$ are represents x, y, and z. here $x$ is the intensity of fluid motion, $y\ and\ z$ are the variables related to the vertical and horizontal direction temperature variation. Here are three parameters, which Lorenz uses as $\alpha, \beta\ and\ \gamma$. The parameters $\alpha = 0.3, \beta = 0.35$ depend on the geometrical and material property of the fluid layer.[17].

Initial conditions for the Lorenz system $y_1(0) = 0.0645, y_2(0) = 0, y_3(0) = -0.9645$ and $h = 0.005$ is step size. Eq. (7) is well-known simplified climatic model of Lorenz system, which has been widely studied in the past 50 years. Now by using the first order finite difference method the Lorenz system become,

$$y_{1(i+1)} = y_{1(i)} + h\{\alpha(y_2 - y_1)\}$$
$$y_{2(i+1)} = y_{2(i)} + h\{-y_1 y_3 - \gamma y_2\}$$
$$y_{3(i+1)} = y_{3(i)} + h\left\{y_1 y_2 - \frac{\beta}{\alpha} - \beta y_3\right\}.$$

Now using the SOR technique we get,

$$y_{1(i+1)} = (1-\omega)y_{1(i)} + \omega\{h(\alpha(y_2 - y_1))\}$$
$$y_{2(i+1)} = (1-\omega)y_{2(i)} + \omega\left\{h(-y_1 y_3 - \gamma y_2)\right\}$$
$$y_{3(i+1)} = (1-\omega)y_{3(i)} + \omega\left\{h(y_1 y_2 - \frac{\beta}{\alpha} - \beta y_3)\right\}. \quad (9)$$

To control the convergence of the Newton's method we will find the best value of $\omega$ by approximating the above Newton's method. In the next section we will presents the numerical results.

## III. Numerical Discussion and conclusion

The nonlinear system is developed from the discretization of chaotic model by utilizing the SOR-Newton's method as we discussed earlier. Now to compare the above technique to investigate the results with a familiar method RK4 also solves the system of ordinary differential equations. Finite difference method is also including in the comparison and they do not contain the relaxation parameters. Now the number of iterations and time elapsed are given below in the Table 1.

Table I: Time elapsed by three numerical methods in seconds

| **No. of iterations x $10^5$** | **Time Elapsed-RK4** | **Time Elapsed-FD** | **Time Elapsed-FD-SOR** |
|---|---|---|---|
| 13 | 40.8992 | 0.3909 | 0.0287 |
| 4 | 26.6208 | 0.3928 | 0.0319 |
| 2 | 8.4802 | 0.3900 | 0.0283 |
| 0.5 | 1.3081 | 0.3904 | 0.0284 |
| 0.25 | 0.7913 | 0.3781 | 0.0297 |

Now the Table I provides the time elapsed by using three numerical schemes such as SOR, RK4, and finite difference methods to find the minimum number of iterations. For such purpose we use $y_1(0) = 0.0654, y_2(0) = 0$ and $y_3(0) =$

$-0.9645$ as an initial conditions and take the step size $h = 0.0005$. As we can see in the Table I, the number of iterations obtained from numerical scheme, Successive-Over-Relaxation (SOR) takes minimum time to compute the results as compare to its competitor like RK4, and finite difference method. Now to investigate the number of iterations for finite difference method and SOR with relaxation parameter we understand the effect of tolerance on the number of iterations.

Table II: Tolerance with number of iterations

| **Tolerance** | **No. of Iterations** | |
|---|---|---|
| | **Finite Difference** | **SOR-Method** |
| $1x10^{-9}$ | 25773 | 1415 |
| $1x10^{-8}$ | 1631 | 1414 |
| $1x10^{-7}$ | 1616 | 1403 |
| $1x10^{-6}$ | 1416 | 1287 |

From Table II, we can observe that SOR method provides minimum number of iterations for defined value of tolerance and also converges very faster as compare to its competitor finite difference method. So it is clear that we can find the required results with short interval of time without using an extra time as well as extra memory.

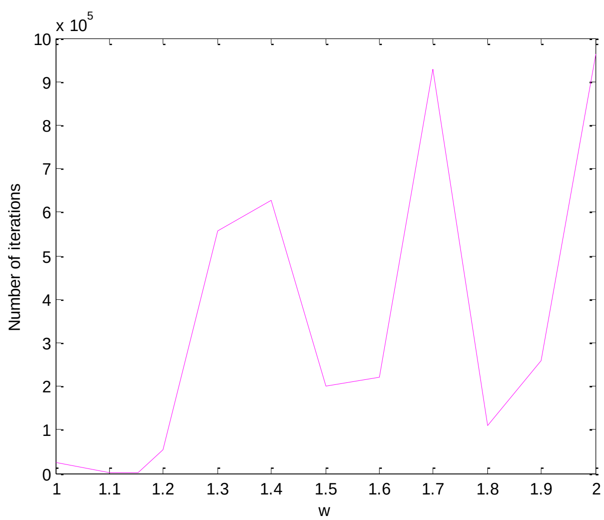


Figure 1: Variation of iterations with Relaxation parameter

The relaxation parameter plays an important role in SOR method. Better the value of relaxation parameter will give us the minimum number of iterations and the convergence will also depends on the value of relaxation parameter. In figure-1, for converged solution we required the number of iterations for different values of relaxation parameter$\omega$. Form the fig. 1 we notice that after 1415 iterations against the tolerance $10^{-9}$ will give converged solution as$\omega = 1.1536$.

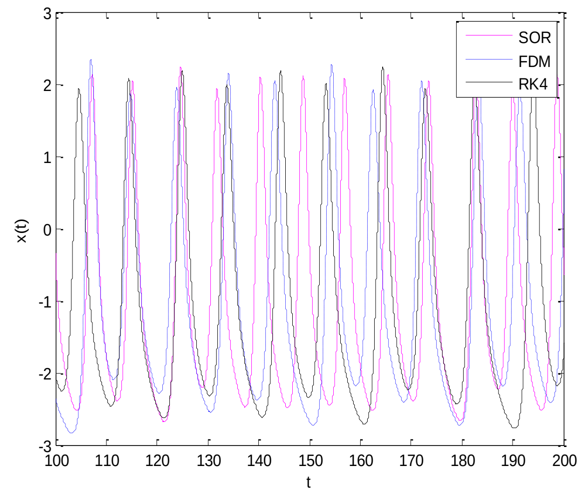


Figure 2: Time history of x(t) through three schemes

Now we compare the above three methods in term of time history show in fig. 2, and analyzed that the above figure confirm the nonlinearity of the Lorenz chaotic system. As we can clearly observe that other numerical techniques (FD and RK4) achieve their converged value later than SOR. For large value of $t$, the SOR scheme can takes its position much earlier than FDM and RK4. The time history plot validates the numerical schemes and also the selection of relaxation parameter.

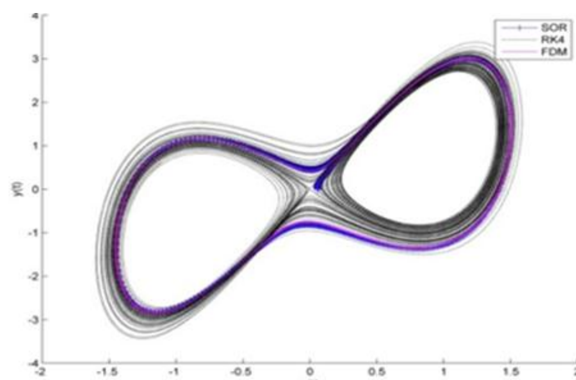


Figure 3: Phase portrait of xy-plane through three numerical schemes

Now we can create the phase portrait to notice the chaotic behavior of the Lorenz model. Also the above fig. 3 provides the strange attractor of the Lorenz model which is sketch through RK4, finite difference method and SOR method. The graphical comparison of these methods is shown over here,

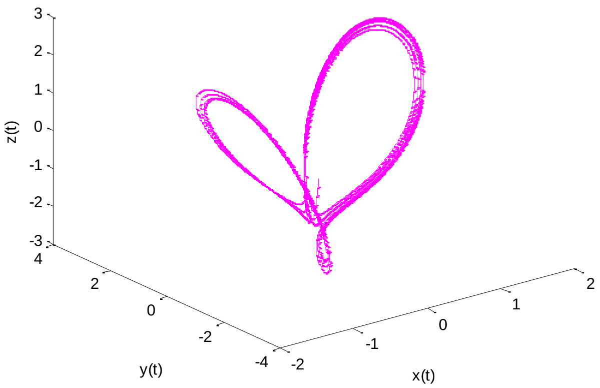


Figure 4: Stranger attractor of chaotic model in xyz-space

Fig. 4 is three dimensional sketch of Lorenz chaotic system through SOR method. Finally all results and discussion shows that the SOR method is fast convergence with guaranty to obtain the approximate solution of Lorenz chaotic model.

## IV. CONCLUSION

In present work, different computational aspects are compared using different techniques like SOR, RK4 Finite Difference technique. To check the time-efficiency Table-1 is calculated and found that SOR give the desired results in minimum time as compare to other two techniques. From Table-II we can observe that SOR get the required results in small number of iterations, table also show that comparison of number of iterations and the selection of tolerance over iterative algorithm. Successive-Over-Relaxation (SOR) technique shows the principal role in time history map over other two techniques. From all these results we can conclude that Lorenz model and other similar model can be approximated accurately using SOR technique and also get the reasonable results with respect to memory space and computational time.

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