Bifurcation Analysis of Heartbeat Model

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Abstract— **In this article bifurcation analysis for nonlinear chaotic Heartbeat Model is presented. Equilibrium points of the system are calculated and critical value of the bifurcation parameter is extracted. Hopf bifurcation is used for nonlinear analysis and to determine the possible existence of periodic orbits in the dynamical system. Interesting results related with nonlinearity of the system are presented through numerical simulations**.

Keywords: **Heartbeat Model, Hopf Bifurcation, Hopf Bifurcation Theorem.**

I. INTRODUCTION

HEART is the most important part of human body. It is also a complex system. There are two states in a cycle of the heart of a human heart beat namely *diastole* and *systole*. At *diastolic* state, the cycle of the heart starts whereas it is at systolic state when it is at contraction position. The top right atrium positioned at the top slot of the heart consists of a pacemaker and. The pacemaker causes a slow spreading electrochemical wave in the heart and consequently causing the muscle fibers to contract so that the blood is pushed into ventricles. Ventricles comprise of bottom portion of the heart. This very wave then rapidly spreads over the heart's lower portion, the lower chamber which actually are the ventricles. This wave causes the ventricles to get into systolic state which ultimately results the blood to run into the lung and arteries. The *systolic* state immediately causes the muscle fibers to get into relaxation state and the heart comes at *diastolic* position. This is how one cycle of the heart completes [1].

Different mathematical model have been represented so far which describe the working of human heart [2, 3, 4]. In [3], E.C Zeeman presented a mathematical model for heart beat which focuses three crucial qualities of the heart- steady state, the start of causing an action potential and finally coming back to steady state. The model is second order nonlinear. E.C Zeeman also discussed the third order nonlinear model in his work [2] which is also applicable to nerve impulse. Many recent studies are also dedicated to the study of biological heart beat systems by applying the concept of nonlinear control system theory [4], using fuzzy-genetic classifier to discuss the cardiac arrhythmias in order to get improved results of electrocardiogram [5]. In [6], flow curvature method, an analytical technique is applied to four dimensional heartbeat autonomous unforced heartbeat model and six dimensional non-autonomous forced heartbeat model. The introduction of forcing term in order to see additional features of the heart beat dynamics and the chaotic effects there also became part of recent research work [7]. There are other aspects of investigating the cardiac models which discuss the cardiac performance by comparing the exercise done [8]. All the models are investigated based on the mathematical frame work to have more accurate results. The nonlinearity remains key features of all type of dynamical models related to the cardiac study.

Bifurcation theory is an active and vast area of research. It occurs in many phenomena as spontaneous oscillations such as airfoil flutter [9] , wind induced oscillations –Tacoma-Narrows Bridge collapse [10], bifurcations in electrical circuits [11], periodic oscillations as discussed by the Van der Pol oscillator [12], the FitzHugh-Nagumo model which depicts periodic forcing of neurons in nervous systems [13], the Belousov-Zhabotinsky reaction as described by Brusselator [14], bifurcation phenomena in predator-prey models [15] and the epidemic models describing some type of disease [16].

In her recent work [17], Luca Guerrini analyzed Hopf bifurcation in a model presented by E.C.Zeeman [3]. Our present work primarily focuses on the detection of bifurcation parameter, its critical value and its bifurcation analysis along with its chaotic behavior detected numerically. The model is an extended form of [3] with two additional parameters p and q . The former is related to muscle fiber length when the heart is in systolic state while the latter is control parameter playing the role of cardiac pacemaker mechanism which directs the heart switch between both the states. It is analyzed that model has a unique equilibrium point. The Hopf bifurcation is analyzed analytically. It is proved for fixed parameters value at the equilibrium point; the nonlinear model exhibits the supercritical Hopf bifurcation.

The organization of presented work is as follows. In section 2 the mathematical model of heartbeat is presented and unique equilibrium point is explored. Section 3 comprises of the analytical Hopf bifurcation theorem which is applied to the presented nonlinear model. Also the classification of bifurcation is detected applying the theorem mentioned therein. Section 4 deals with the nonlinear behavior of the presented nonlinear model using numerical technique for different parameter values through phase portraits and time history. Finally section 5 concludes the analysis of the presented model by showing the importance of Hopf bifurcation for the given model and importance of the critical parameter values.

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II. MATHEMATICAL ANALYSIS

Our proposed mathematical model is taken from [4], which is modified form of the one presented originally in [3] b y E.C Zeeman. The model is as follows;

$$\dot{x} = \frac{1}{\varepsilon}(x^3 - Tx + y), T > 0$$

$$\dot{y} = (x - r) + (x - p)q$$

$$x(0) = x_0, y(0) = y_0$$

(1)

where $x(t)$ is related to length of muscle fiber, $y(t)$ represents electrochemical activity, ε represents small positive constant, T represents the tension in the muscle fiber, r is a scalar quantity representing a typical length of muscle fiber in the *diastolic* state, p is parameter related to length of muscle fiber in *systolic* state, and the parameter q is the control parameter related to pacemaker of the heart. It symbolizes as control mechanism directing the heart to be in the diastolic and the systolic state. The parameter q having either of two values 0 or 1, which actually represent the respective diastolic or systolic states of the heart. The unique fixed point of system (1), given by

$$P_1(q(p-r)+q, T(q(p-r)+r)+(q(p-r)+r)^3)$$

Hopf Bifurcation Theorem: Consider

$$\dot{x} = P_\alpha(x, y)$$

$$\dot{y} = Q_\alpha(x, y)$$

where α is representative of parametric value. Let it is having equilibrium point which can be considered to situated at $(x, y) = (0,0)$. Assuming the eigenvalues about equilibrium points for the linearized system are given as $\lambda(\alpha), \bar{\lambda}(\alpha) = \mu(\alpha) \pm i\rho(\alpha)$ Assume further that for a particular value of μ, the underlined conditions are fulfilled:

1. Non-hyperbolicity condition:

conjugate pair of eigenvalues

$$\alpha(0) = 0, \rho(0) = \omega \neq 0$$

where $\mathrm{sgn}(\omega) = \mathrm{sgn}[(\frac{\partial Q_\alpha}{\partial x})]_{|\alpha=0}(0,0)$

2. **Transversality condition**

It states that the eigenvalues of the linearized system cross the imaginary axis with non-zero speed.

$$\left.\frac{d\mu(\alpha)}{d\alpha}\right|_{\alpha=0} = \eta \neq 0$$

3. Genericity condition

$a \neq 0$ where

$$a = \frac{1}{16}(P_{xxx} + P_{xyy} + Q_{xxy} + Q_{yyy}) + \frac{1}{16\omega}(P_{xy}(P_{xx} + P_{yy}) - Q_{xx}(Q_{xx} + Q_{yy}) - P_{xx}Q_{xx} + P_{yy}Q_{yy})$$

with $P_{xy} = \left.(\frac{\partial^2 P_\alpha}{\partial x \partial y})\right|_{\alpha=0} (0,0)$

Then the periodic solutions are separated from origin into the region for $\alpha > 0$ if $a\eta < 0$ or $\alpha < 0$ if $a\eta < 0$ by a unique curve. The origin is a stable equilibrium point when $\alpha > 0$ and an unstable fixed point when $\alpha < 0$ provided $\eta < 0$, similarly the origin is a stable equilibrium point for $\alpha < 0$ and an unstable fixed point for $\alpha > 0$ when $\eta > 0$ .The are stable periodic solutions if the origin is unstable about $\alpha = 0$ where the periodic solutions exist and there exist unstable periodic solutions when the origin is stable about $\alpha = 0$ where the periodic solutions exist. The periodic trajectories grow like $\sqrt{|\alpha|}$ in amplitude whereas their periods incline to 2π/ω as $|\alpha|$ approaches zero [18].

III. HOPF BIFURCATION ANALYSIS

The linearized form of system (1) in its Jacobian matrix at $P_1(q(p-r)+q, T(q(p-r)+r)+(q(p-r)+r)^3)$ is given by

$$J_{(x,y)} = \begin{pmatrix} \frac{-1}{\varepsilon}(3x^2 - T) & \frac{-1}{\varepsilon} \\ 1 & 0 \end{pmatrix}$$

$$J|_{P_1} = \begin{pmatrix} \frac{-1}{\varepsilon}(3(q(p-r)+r)^2 - T) & \frac{-1}{\varepsilon} \\ 1 & 0 \end{pmatrix}$$

The eigenvalues are roots of the characteristic equation $\lambda^2 - \tau\lambda + \sigma = 0$ given by

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\sigma}}{2}$$

Here we have $\tau = \frac{-1}{\varepsilon}(3(q(p-r)+r)^2 - T)$ and $\sigma = \frac{1}{\varepsilon}$ which gives

$$\lambda_{1,2} = \frac{\frac{-1}{\varepsilon}(3(q(p-r)+r)^2 - T) \pm \sqrt{\frac{-1}{\varepsilon}(3(q(p-r)+r)^2 - T) - 4(\frac{1}{\varepsilon})}}{2}$$

With $\mu(p,q,r,T) = \mathrm{Re}(\lambda_{1,2}) = \dfrac{\frac{-1}{\varepsilon}(3(q(p-r)+r)^2 - T)}{2}$

and

$$\rho(p,q,r,T)=\mathrm{Im}(\lambda_{1,2})\big|_{T_0}=\frac{\sqrt{4(\frac{1}{\varepsilon})+((\frac{-1}{\varepsilon})(3(q(p-r)+r)^2-T_0))^2}}{2}=\frac{1}{\sqrt{\varepsilon}}$$

Choosing T as bifurcation parameter and applying the Hopf bifurcation theorem

C1) $\mu(p,q,r,T)=\mathrm{Re}(\lambda_{1,2})=0$ which gives

$$\frac{\frac{-1}{\varepsilon}(3(q(p-r)+r)^2-T)}{2}=0$$

Implying the bifurcation point for parameter is,

$$T_0=3(q(p-r)+r)^2$$

$$\omega=\rho(p,q,r,T)=\mathrm{Im}(\lambda_{1,2})\big|_{T_0}=\frac{\sqrt{4(\frac{1}{\varepsilon})+((\frac{-1}{\varepsilon})(3(q(p-r)+r)^2-T_0))^2}}{2}$$

$$=\frac{1}{\sqrt{\varepsilon}}$$

Thus the non-hyperbolicity condition is satisfied.

C2) $\eta=\frac{d\mu(p,q,r,T)}{dT}\big|_{T_0}=\frac{1}{\varepsilon}\neq 0$ satisfying transversality condition.

C3) At equilibrium point P_1 with $T_0=3(q(p-r)+r)^2$ we have

$$P_{xy}=0, P_{xx}=\frac{-6(q(p-r)+r)}{\varepsilon}, P_{xxx}=\frac{-6}{\varepsilon}, P_{xyy}=0,$$

$$P_{yy}=0$$

and

$$Q_{xy}=0, Q_{xx}=0, Q_{yy}=0, P_{xyy}=0, P_{yy}=0, Q_{xxy}=0, Q_{yyy}=0$$

$$P_{xy}=0, P_{xx}=\frac{-6(q(p-r)+r)}{\varepsilon}, P_{xxx}=\frac{-6}{\varepsilon}, P_{xyy}=0, P_{yy}=0$$

$$a=\frac{1}{16}(\frac{-6}{\varepsilon})+\frac{1}{16\omega}(0)$$

So that the value of a at unique equilibrium point P_1 at parameter value T_0,

$$a=\frac{-3}{8\varepsilon} \qquad (2)$$

In order to detect critical value of bifurcation parameter value, we fix values of the other parameters, let $\varepsilon=0.09, r=1.9, p=0.05, q=1$, the value of bifurcation point is $T_0=0.075$ while $\omega=1.054$ and $a=-0.4167$. Here $q=1$ implies that the bifurcation point is at a point when the heart is at systolic state. When $q=0$ with all the remaining parameters having same fix value defined above, the bifurcation point in the diastolic state of the heart will be having critical parameter value $T_0=10.83$. It can be verified by the Hopf bifurcation theorem that the Hopf bifurcation exists for the fixed values of the parameters. We have $\eta>0$ and consequently $a\eta<0$. The fixed point $P_1(q(p-r)+q, T(q(p-r)+r)+(q(p-r)+r)^3)$ is stable for $T<3(q(p-r)+r)^2$ and unstable for $T>3(q(p-r)+r)^2$. The Hopf bifurcation theorem justifies that a unique curve of periodic solutions bifurcates from P_1 into the region for $T>3(q(p-r)+r)^2$ as $a\eta<0$.

Classification of Hopf Bifurcation

In the following we present the Hopf bifurcation theorem which will define the type of bifurcation in our model.

Theorem: Assume that for $l=f_\eta(m), l\in\Re^n, \eta\in\Re$ has an equilibrium (l_0,η_0) fulfilling the properties given below [17]:

(A1) $D_l f_{\eta_0}(l_0)$ is the only pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

A2) There will be a unique center manifold which passes through $(l_0(\eta_0),\eta_0)$ in $\Re^n\times\Re$ and a system of coordinates system. For $a\neq 0$, the center manifold will have a surface having periodic solutions such that it has quadratic tangency having eigen-space of $\lambda(\eta_0), \bar{\lambda}(\eta_0)$ satisfying parabolic equation $\eta=-(\frac{a}{d})(l^2+m^2)$. The periodic solution will yield a stable limit cycle for $a<0$ and bifurcation will be supercritical, while for $a>0$ the repelling periodic solutions give subcritical bifurcation.

It is easy to verify from (2) that the value of a remains negative for all values of the parameter ε, supporting the fact the Hopf bifurcation is of supercritical type.

IV. NUMERICAL SIMULATIONS

This section is dedicated to the numerical results of the system (1) showing its nonlinear behavior. The phase portraits and the time history graphs of system (1) are shown for both states of the heart. The initial conditions are taken as $x(0)=0.1, y(0)=0.1$ for both the states.

Fig. 1 shows the phase portrait of the model in systolic state which apparently depicts the nonlinear behavior of the system with single cubic nonlinearity. While Fig. 2 represents the time history of variable $x(t)$ representing the muscle fiber length, depicts the nonlinear behavior.

For diastolic state, Fig. 3 represents the phase diagram of the model for the diastolic state, while time history for variable is shown in Fig. 4.

It can be observed from phase portraits and the time history for both the states of the heartbeat that the model represents the oscillatory nonlinear behavior.

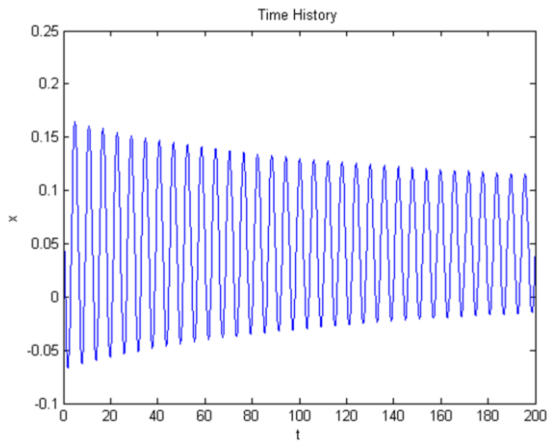

Fig. 2 Time History map for heartbeat model at systolic state

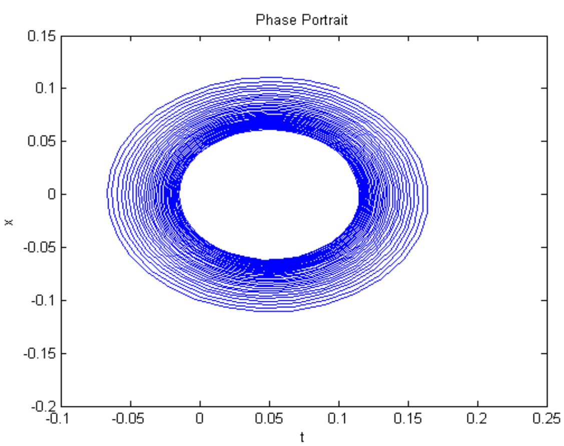

Fig. 1 Phase portrait of heartbeat model at systolic state when $T_0 = 0.0075$

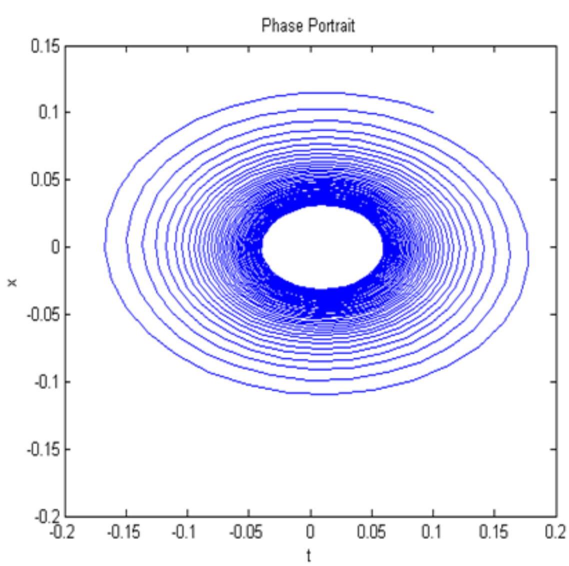

Fig. 3 Phase portrait of heartbeat model at diastolic state when $T_0 = 0.0003$

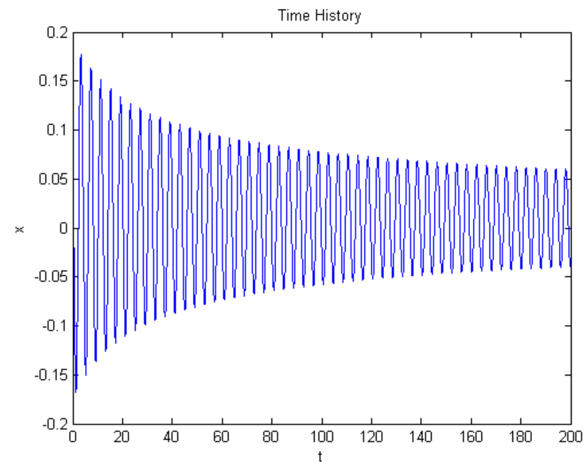

Fig. 4 Time History map for heartbeat model at diastolic state

V. CONCLUSION

The nonlinear heartbeat model with cubic nonlinearty is explored by finding the unique equilibrium point. Hopf bifurcation is analysed using Hopf bifurcation theorem. The bifurcation parameter T related to tension in muscle fiber is explored for its critical value, both for *systolic* and *diastolic* states. It is interesting to note that as many other applications of bifurcation descirbed above, the bifurcation phenomena aslo makes its presence in a simple heartbeat model. The oscillatory behavior of the system is analysed by detecting the Hopf bifurcation and then verifying the results numerically. It is shown that the supercritical Hopf bifurcation exists in the presented model.

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