LQR and LMI Based Optimal Control Design for Aircraft

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Abstract—The maneuvering of an aircraft requires design and implementation of advance control techniques for both longitudinal and lateral motion. The main objective of this paper is to design an optimal controller for aircraft dynamics in terms of quadratic performance index. The problem formulation is carried out using different control techniques like linear quadratic regulator (LQR) and linear matrix inequalities (LMI). The controller design process begins with obtaining appropriate mathematical model for aircraft dynamics. The model is obtained using first principle approach and linearized using small perturbation theory. Open loop analysis is carried out for model using MATLAB and results are obtained in step responses. The model response is compared with LQR and LMI based results.The simulation results reveal the effectiveness of these control techniques applied and ensure stability of aircraft dynamics. Longitudinal model of aircraft is used for analysis and aircraft used as reference is Cessna 310.

Keywords— Aircraft, LMI, LQR, Longitudinal, Cessna 310.

I. INTRODUCTION

The development and advancements of aircraft of various design is still a research area of growing interest because of its civil and military applications. The challenges that any aircraft goes through during its mission assigned not only depend upon the aerodynamic design but also on system design. Aircraft of light weight low cost with efficient control techniques and strategies are of great interest. Effective control methods are applied during designing which can be helpful in case of failure of control surface, actuator etc and also provide with efficient performance during its flight [1].

Aircraft is a highly non-linear and complex system. Free flight motion for any aircraft is very complicated task [2].

The aircraft motion basically consists of three translational and three rotational motion along x-, y- and z-axes. The three translation motion includes horizontal motion along x-axes, vertical motion along z-axes and transverse motion along y-axes while three rotational motion includes roll, pitch and yaw motion. The main control surfaces designed for control of different types of motion are elevator, rudder and aileron. Aileron located on wing controls pitch motion, rudder on vertical tail affect yaw motion and elevator is responsible for roll motion [3]. The aircraft motion can be mainly categorized into longitudinal and lateral motion resulting in two main types of control i.e. longitudinal control and lateral control. In case of longitudinal control, elevator plays its role in controlling longitudinal motion and pitch while for lateral control, aileron and rudder perform their role [4].

The aircraft being highly coupled and non-linear dynamical system is difficult to modeled accurately. Therefore, certain assumptions are made to make derivation of model easy and retaining the model accuracy within desired limit. Aircraft is assumed to be a rigid body with constant mass and also some deviations exist in motion from equilibrium level flight [5]. The mathematical modeling of aircraft can be carried out using first principle approach or using system identification technique. The first principle method is a theoratical method uses Newton-Euler equations for description of dynamic characteristics of aircraft. This paper also considers this method for derivation of longitudinal motion equations which are linearized using small perturbation theory.

The controller design for aircraft dynamics which are coupled and highly non-linear is a difficult task and it becomes further difficult due to its sensitivity to disturbance. Research interest for control design techniques preferred to achieve desired performance without causing increase in complexity. Modern control methods exist for overcoming all short comings exist in flight control of aircraft but proportional integral derivative is still in use because of its ease in implementation and reduced cost [6]. For flight control system to maintain robustness along with maintaining desired performance uses a robust control technique named H_2/H_∞ [7,8]. The main idea of using this method is to maintain boundary limit between performance of aircraft and robustness. State output feedback (SOF) is another method for autopilot design of aircraft dynamics and its significance of this method is that it can work on signal information to be controlled. The SOF technique calculate static gain in order to meet desired requirement for system in closed loop [9]. Dynamic inversion is among other methods which is applied in control community and is mainly energized due to aerospace control significance [10- 13]. It helps in retaining non-linear characteristics along with flexibility and its simplicity. LQR is another controller design approach used to achieve stability and control of aircraft motion. It can maintain trim state of aircraft if uncertainties are generated due to loss of control surface. LQR method calculate feedback gain which stabilizes the motion and overcome disturbance uncertainties [14-15].

The organization of paper is carried out as: The first part of paper discusses modeling of aircraft longitudinal motion using first principle approach with linearization of non-linear differential equations using small perturbation theory and obtaining state space form. The open loop analysis of model is carried out and simulation results are obtained for step responses. The next part describes control technique named linear quadratic regulator (LQR). The LQR calculated gain

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matrix stabilize the system dynamics and results obtained are compared with that of open loop. The second part of the third section deals with another control method named linear matrix inversion (LMI) which is a static feedback controller design, stabilizing the system as evident from simulation results. LQR and LMI simulation results are compared and conclusion is drawn in last section.

II. AIRCRAFT DYNAMICS

This section describes mathematical modeling of aircraft which makes easier to understand characteristics of aircraft. The different axes of aircraft are shown in figure 1 describing roll, pitch and yaw motion. The geometric and flight data regarding aircraft is shown in table I.

The different assumptions made for derivation of mathematical model of aircraft are; it is assumed to be a rigid body with constant mass and small disturbances exist in its motion. The aircraft equations of motion are derived using Newton-Euler equations and these equations are as follow [16-17];

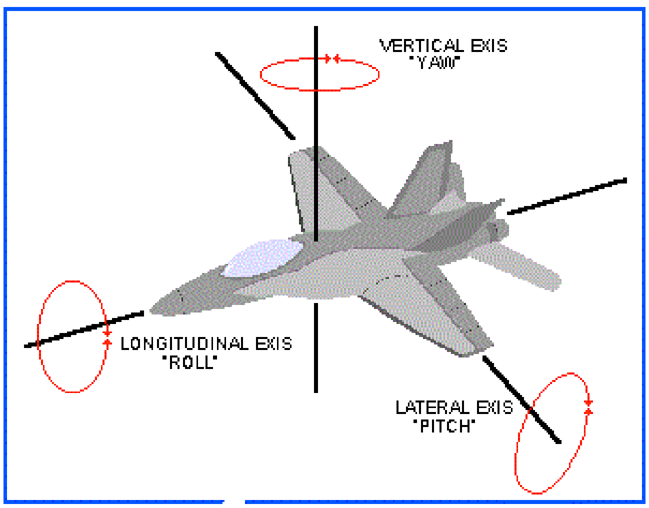

Fig. 1. Aircraft axis

Table I
Geometric and flight data

Mass (lbs)	4600
Mean aerodynamic chord (ft)	4.79
Wing Surface (ft^2)	175
Wing Span (ft)	36.9
Altitude (ft)	8000
Mach Number	0.288
Air Speed (ft/s)	312.5
Dynamic Pressure (lbs/ft^2)	91.2
Location of CG % of MAC	0.33
Moment of inertia x-axes ($slug - ft^2$)	8,884
Moment of inertia y-axes ($slug - ft^2$)	1,935
Moment of inertia z-axes ($slug - ft^2$)	11,001

$$\sum F = {d(mV)}/{dt} \quad (1)$$

$$\sum H = {dH}/{dt} \quad (2)$$

Disturbance act upon aircraft dynamics and its forces and moment equations get modified to following form:

$$\sum F = \sum F_0 + \sum \Delta F \quad (3)$$

$$\sum M = \sum M_0 + \sum \Delta M \quad (4)$$

Where $\sum F_0$ and $\sum M_0$ represent sum of steady state forces and moments while sum of disturbance forces and moments are described by $\sum \Delta F$ and $\sum \Delta M$ respectively. The main reason for disturbance are control surfaces deflection or atmospheric disturbances.

The equations of motion describing aircraft dynamics are as given as follows;

$$\begin{bmatrix} F_{ax} + F_{tx} \\ F_{ay} + F_{ty} \\ F_{az} + F_{tz} \end{bmatrix} = \begin{bmatrix} m\,(\dot u + qw - rv) - mg_x \\ m\,(\dot v - rv + pw) - mg_y \\ m\,(\dot w - uq + vp) - mg_z \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} L_{ax} + L_{tx} \\ M_{ay} + M_{ty} \\ N_{az} + N_{tz} \end{bmatrix} = \begin{bmatrix} I_{xx}\dot p - I_{xz}\,(\dot r + pq) + rq\,(I_{zz} - I_{yy}) \\ I_{yy}\dot q + pq\,(I_{xx} - I_{zz}) + I_{zz}\,(p^2 - r^2) \\ I_{zz}\dot r - I_{xz}\dot p + pq\,(I_{yy} - I_{xx}) + qr\,I_{xz} \end{bmatrix} \quad (6)$$

Where u, v and w are components of translational velocity, p, q and r are components of rotational velocity, m is mass of aircraft and g_x, g_y and g_z are components of gravity along $x-$, $y-$ and $z-$axes. Similarly,L, M and N are rolling, pitching and yawing moments along x-, y- and z-axes. The forces, moments, linear velocity and angular velocity components are shown in figure 2.

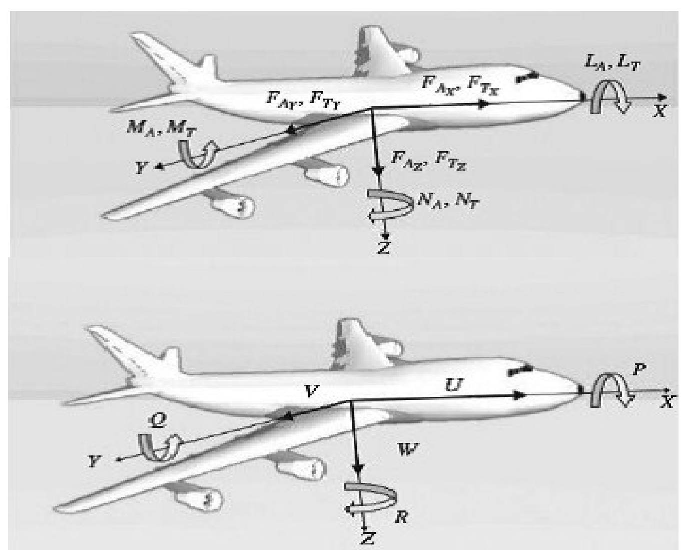

Fig. 2. Forces, Moments and velocities components

Euler angles describes the aircraft motion in inertial space and equations for $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$

$$\dot{\theta} = q cos\varphi - r sin\varphi \quad (7)$$
$$\dot{\phi} = p + q sin\varphi tan\theta + r cos\varphi tan\theta \quad (8)$$
$$\dot{\psi} = q\frac{sin\varphi}{cos\theta} + r\frac{cos\varphi}{cos\theta} \quad (9)$$

The linearization of non-linear equations is carried out using small perturbation theory and linearized equation are converted to state space representation. The generalized state model is given as;

$$\dot{x} = Ax + Bu \quad (10)$$
$$y = Cx + Du \quad (11)$$

Where A, B, C and D describes system matrix, control matrix, output matrix and feed forward matrix. In our case, the state vector for longitudinal model is given by $x = [u\ \alpha\ q\ \theta]^T$, Where u is longitudinal component of true airspeed, α is the angle of attack, q is pitch rate and θ is the pitch angle. The elevator control surface δ_e is used as input which affects the longitudinal motion. The numeric values obtained for the matrices A and B of the longitudinal model of the aircraft are:

$$A = \begin{bmatrix} -0.0422 & 15.28 & 0 & -32.17 \\ -0.0020 & -1.1877 & 0.9544 & 0 \\ 0.0064 & 3.5648 & -9.5413 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -0.2307 \\ 95.0425 \\ 0 \end{bmatrix}$$

The eigen values obtained for the longitudinal model are: -9.93e+000, -7.44e-001, -4.79+1.84e-002i and -4.79+1.84e-002i. The open loop response obtained for the model shows unsatisfactory performance and these simulations are shown in fig. 3-5.

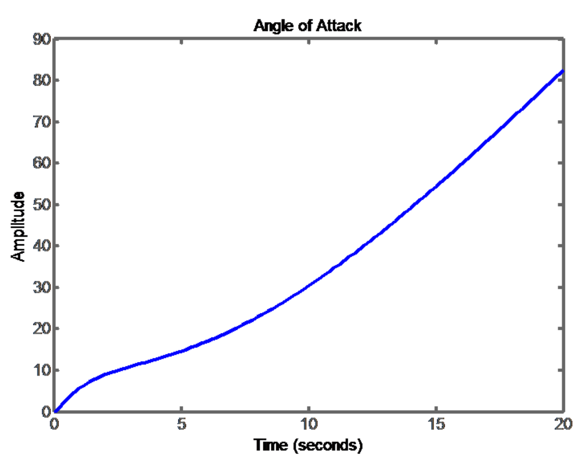

Fig. 3. Response for angle of attack

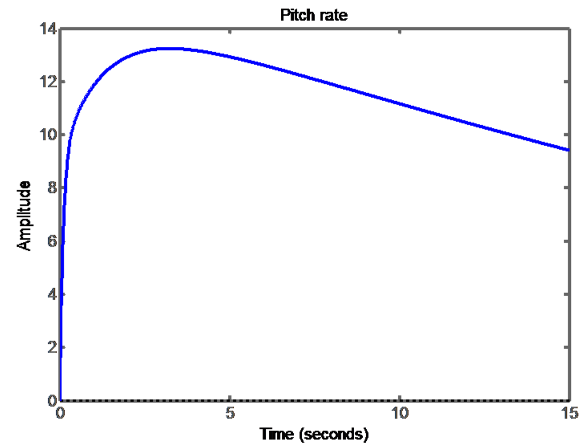

Fig. 4. Response for pitch rate

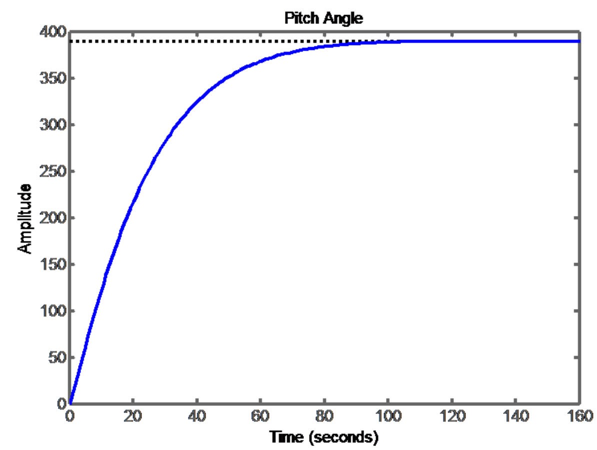

Fig. 5. Response for pitch angle

The simulation results in fig. 3-5 shows quite unsatisfactory behavior for the mentioned state. The angle of attack, pitch rate and pitch angle should obtain zero as their steady-state value according to trim conditions.
Similarly the frequency response of the open loop longitudinal model can be studied from fig. 6.

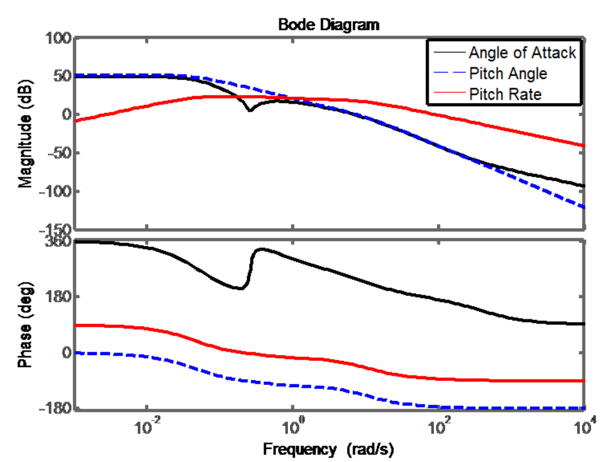

Fig. 6. Longitudinal model frequency response

These unwanted characteristics should be eliminated anyhow and it became necessary to design controller(s) so that desired results can be achieved. These different control strategies designed and implemented are discussed in the next section.

III. FLIGHT CONTROLLERS DESIGN

The simulation results obtained for the longitudinal model of the aircraft show diverging behavior. These states should be stable and required to obtain their desired steady state value within the required time. In order to achieve desired performance the different control techniques implemented using Matlab/Simulink are Linear Quadratic Regulator (LQR) and Linear Matrix Inequality (LMI). These control methods are described in the following sections.

A. Linear Quadratic Regulator (LQR)

Optimal control design approach for multi-input multi-output (MIMO) dynamical systems is one of the important control method which work effectively for systems. LQR is a control strategy which when design properly provide with desired performance of a system which can be stated in terms of performance index. It provides optimal approach to calculate feedback gain and for that all states of systems should be available for controller. The block diagram for LQR is shown in figure 7.

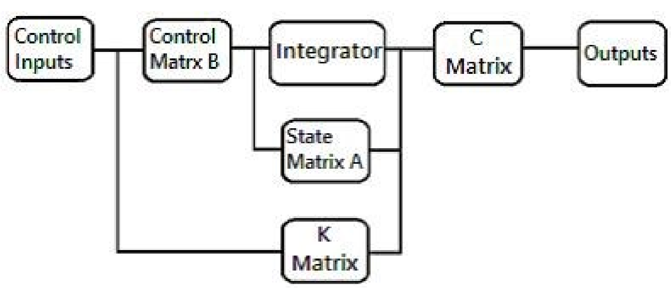

Fig.7. LQR Block diagram

The feedback control law implemented for the state space form of the aircraft dynamics in (10) is given as [18-19];

$$u = -K(x - x_{desired}) \tag{12}$$

After obtaining value for gain matrix K, the longitudinal model dynamics modifies to following form:

$$\dot{x} = (A - BK)x + BKx_{desired} \tag{13}$$

$x_{desired}$ is input to external closed loop system and representing desired state vector. The modified system matrix for plant dynamics after K calculation is A-BK. The performance index J which need to be minimized and gives cost function is sum of terminal cost and integral along way is given by;

$$J = \Psi(x(t), t) + \int_{t_o}^{t} L(x(t), u(t), t)dt \tag{14}$$

Where terminal cost is described by $\Psi(x(t), t)$ and $L(x(t), u(t), t)$ is non-negative. Also $\Psi(x(t), t) = 0$ for linear quadratic regulator and L is given as;

$$L = \frac{1}{2}x^T(t)Q(x) + \frac{1}{2}u^T Ru \tag{15}$$

The feedback control law obtained on the basis of linear model assumptions is given as;

$$u(t) = -R^{-1}B^T Px(t) \tag{16}$$

The value of gain matrix K obtained for longitudinal model is given as;

$$K = [-0.9840 \ -4.1368 \ 0.9660 \ 10.8945]$$

The LQR controller stabilize the plant model very effectively. After introducing K into the model dynamics the simulation results obtained for longitudinal model are shown in fig. 8-10.

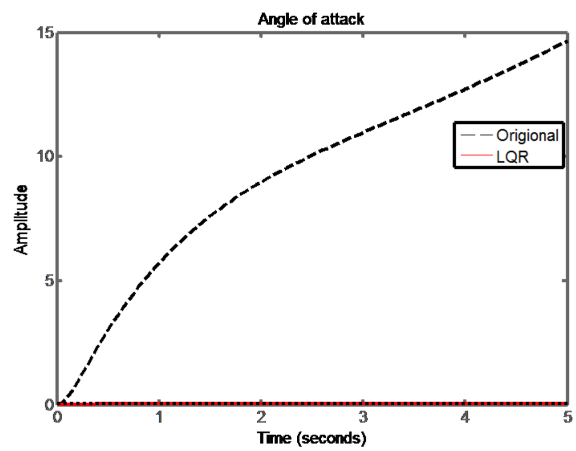

Fig. 8. LQR response for angle of attack

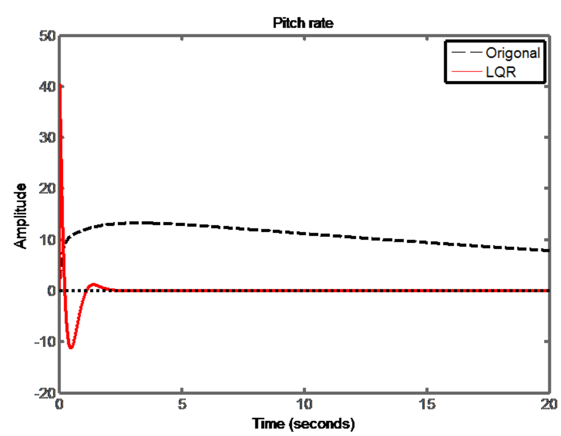

Fig. 9. LQR response for pitch rate

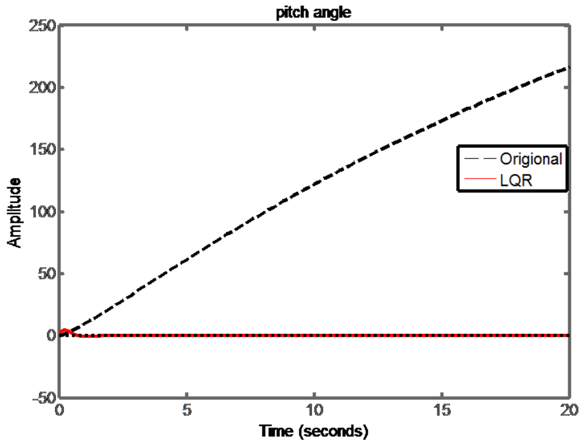

Fig. 10. LQR response for pitch angle

The simulation results obtained in fig. 8-10 using LQR control technique shows significant performance in improving the response and characteristics of the longitudinal model. The angle of attack response in fig. 8 has zero steady state value according to requirement without any delay of time. Similarly, for pitch rate and pitch angle, their desired steady state value is achieved efficiently. There is slight variation from their trim value at the beginning but after short time of few second, the desired final value is attained.

B. Linear Matrix Inequality (LMI)

For an affine function with decision vector $x = [x_1 \ x_2 \ x_3 \cdots x_j]^T$ and with real symmetric matrices F_0, F_1, $F_2 \cdots, F_j$, the matrix inequality is given as[20-22];

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_j F_j > 0 \qquad (17)$$

is known as Linear Matrix Inequality (LMI).

The disturbance showing impact on longitudinal motion is given by $[u_d \ \alpha_d \ q_d \ \theta_d]^T$ hence affecting horizontal velocity, angle of attack, pitch rate and pitch angle. The main objective is to design control law of form: $u = -Kx$ which guarantees the system stability and its performance index J is given by;

$$J = \int_0^\infty x(t)^T (Q + K^T RK) x(t) dt \qquad (18)$$

The Lyapunov stability matrix P obtained is given as;

$$P = \begin{bmatrix} 0.0025 & 1.3100 & 0.4181 & 0.0000 \\ 1.3100 & 702.0943 & 228.3084 & 19.3461 \\ 0.4181 & 228.3084 & 498.1163 & 18.2680 \\ 0.0000 & 19.3461 & 18.2680 & 169.2893 \end{bmatrix}$$

The gain matrix K calculated for state feedback is given as;

$$K = 1.0e + 003[0.0039 \ \ 2.1437 \ \ 4.7390 \ \ 0.1732]$$

The system performance shows significant improvement using LMI based dynamic model. The effect of disturbance is accounted effectively. The performance shows high degree efficiency of LMI based control approach. The simulation results obtained are shown in fig. 11-13.

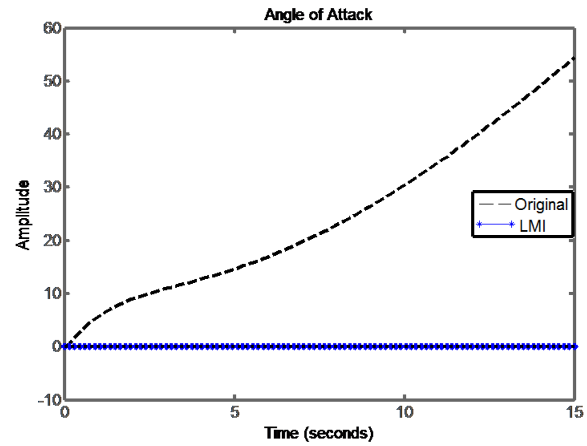

Fig. 11. LMI simulation for angle of attack

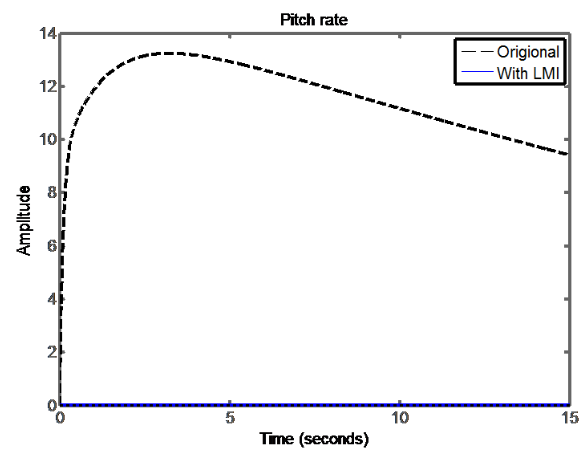

Fig. 12. LMI simulation for pitch rate

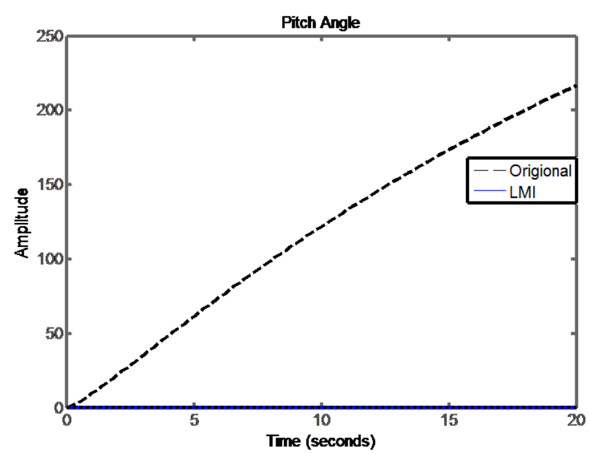

Fig. 13. LMI simulation for pitch angle

The simulation results in fig. 11-13, obtained for the longitudinal model of the aircraft using LMI control strategy have significantly improved its performance. The desired steady-state results are achieved effectively and system behavior modified efficiently.

IV. LQR and LMI PERFORMANCE COMPARISON

The Linear Quadratic Regulator (LQR) and Linear Matrix Inequality (LMI) control strategies designed for the longitudinal model of the aircraft have brought significant improvement in the performance of the model obtained. The unwanted behavior of the open loop model is eliminated efficiently and effectively. The performance of both LQR and LMI are compared in terms of their time domain responses to reach their desired steady state values. Their comparison is shown in fig. 14-16.

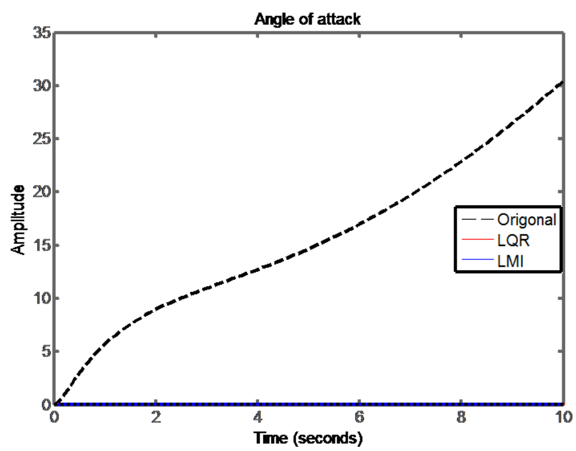

Fig. 14. LQR-LMI comparison for angle of attack

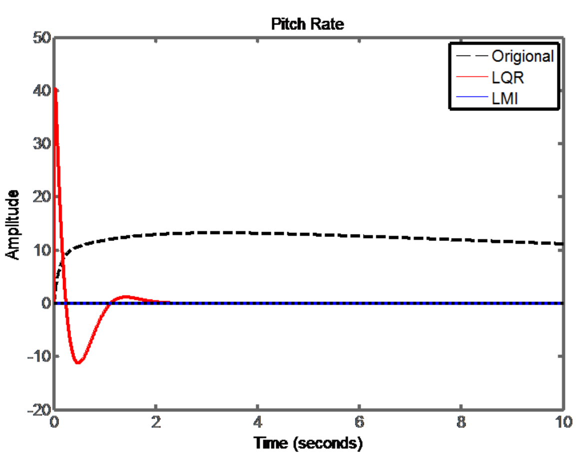

Fig. 15. LQR-LMI comparison for pitch rate

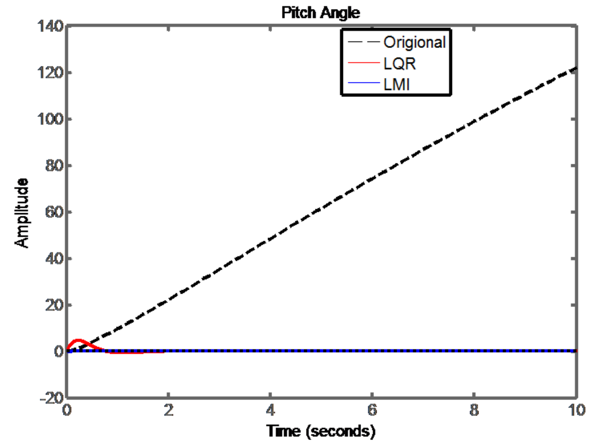

Fig. 16. LQR-LMI comparison for pitch angle

From fig. 14-16, it is evident that the LMI control strategy designed effectively and efficiently attains the desired trim value for each state of the model. The LQR controller performance for pitch rate and pitch angle undergoes through slight variation from the steady-state value and attain it after a short interval of time. Overall, the performance of the LMI is much effective than the LQR.

V. CONCLUSION

This paper has considered feedback stabilization for longitudinal motion of aircraft through LQR and LMI approach. The 6 degree-of-freedom model of an aircraft is derived using first principle approach and these non-linear equations are linearized using small perturbation theory and converted to state space model. The stability analysis is carried out for open loop system and simulation results are obtained for step responses. The control design for technique developed are linear quadratic regulator and linear matrix inversion. The solution of the problem with both LQR and LMI approach is demonstrated and results obtained shows that the performance and handling qualities are satisfactory with the reference model over wide range of nonlinear flight regime.

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