Nonlinear Vibration of Partially Cracked Plates Using Higher Order Perturbation Method

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Abstract - In this study the effect of crack in the aircraft panel structure modeled as an isotropic plates subjected to harmonic excitation is considered for obtaining the vibration response. The model is designed for analyzing the vibration response of aerospace and mechanical structures such as airplane wings, fuselage, tail, etc. The crack is in the form of continuous line located at the center and along the x-direction of the plate element. The system of nonlinear equation is solved by means of higher order perturbation method of multiple scales. The method of multiple scales is very popular method for solving nonlinear vibration problems competently. The result obtained from this study is then compared with the solution of the low order perturbation method of multiple scales.

Index Terms—**Plate, Crack, Duffing equation, Perturbation method, Method of multiple scales**

I. INTRODUCTION

Plates are used in a variety of structural applications. Many structures like aircraft wings, ships, helicopter rotor blades, etc., are modeled as isotropic plate elements. In this study, aircraft wing structure modeled as an isotropic plate having a part-through crack of length '2*a*' consists of continuous line located at the center and along the *x*-direction of the plate is discussed. The effects of rotary inertial and through thickness stresses are assumed to be negligible. It can be seen that the crack within an isotropic plate changes the vibrational response of the system. The configuration of the plate with crack subjected to point load at specified position is shown in Figure 1.

The governing equation of motion of nonlinear vibrations of isotropic plate as indicated in Fig. 1 is obtained by the classical plate theory [1]. The equation reproduce here is:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -\rho h \frac{\partial^2 w}{\partial t^2} + n_x \frac{\partial^2 w}{\partial x^2} + P_z + \frac{2aD\left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right)}{3\left(\frac{\alpha^o{}_{bt}}{6} + \alpha^o{}_{bb}\right)(3+\nu)(1-\nu)h + 2a} - \frac{2a}{(6\alpha^o{}_{tb} + \alpha^o{}_{tt})(1-\nu^2)h + 2a} n_{rs} \frac{\partial^2 w}{\partial y^2} \quad (1)$$

Galerkin's method [2] is then used to reformulate the governing equation into time dependent modal coordinates on the basis of different boundary conditions, i.e., CCFF (Clamped-Clamped-Free-Free), CCSS (Clamped-Clamped-Simply Supported-Simply Supported) and SSSS (all sides Simply Supported). Berger's formulation [3] is then used to express the in-plane forces that transform the governing equation into nonlinear system.The equation obtained is as follows [1,6]:

$$\ddot{\psi}_{mn}(t) + 2\mu\dot{\psi}_{mn}(t) + \omega^2_{mn}\psi_{mn}(t) + \beta_{mn}\psi^3_{mn}(t) = \frac{\lambda}{mD}F_0(t) \quad (2)$$

Eqn. (2) is called a Duffing equation that contains a cubic nonlinear term. The solution of nonlinear Eqn. (2) has already been carried out by Israr [1] by means of low order perturbation method of multiple scales [4].

The primary objective of this studiesare to propose an approximate vibration response of cracked plate subjected to transverse loading via the higher order perturbation method of multiple scales for different cases of boundary conditions. The results obtained from the higher order solution arethen compared with the low order perturbation results to see the effectiveness of the present method followed by some useful conclusions.

II. PERTURBATION METHOD OF MULTIPLE SCALES

Nonlinear problems are very difficult to solve, therefore, they can be solved by using approximate solution techniques. In this particular study, higher order perturbation method of multiple scales is adopted for obtaining a closed-form solution.The method of multiple scales [5] is used to formulate the Duffing's equation for the vibration analysis of a rectangular plate having a part-through crack at the center of the plate. It comprises the techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable to construct a global valid solution.
The method leads to an expression in terms of a power series in some small parameter.

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots + \varepsilon^n\psi_n \quad (3)$$

where the parameter ε is called the perturbation parameter.

For the method of multiple scales, the solution of Eqn. (2) is approximated by a uniformly valid expansion of the form as indicated below:

$$\begin{aligned}\psi_{mn}(t,\epsilon) = \psi_{0mn}(T_0,T_1,T_2,T_3) &+ \epsilon\psi_{1mn}(T_0,T_1,T_2,T_3)\\ &+ \epsilon^2\psi_{2mn}(T_0,T_1,T_2,T_3)\\ &+ \epsilon^3\psi_{3mn}(T_0,T_1,T_2,T_3) + O(\epsilon^4)\end{aligned} \quad (4)$$

where $\psi_{0mn}(T_o,T_1,T_2,T_3)$, $\psi_{1mn}(T_o,T_1,T_2,T_3)$, $\psi_{2mn}(T_o,T_1,T_2,T_3)$ and $\psi_{3mn}(T_o,T_1,T_2,T_3)$ are functions yet to be determined.

The derivative perturbations rely on the notion that the real time 't' can be expressed in the form of a set of successively independent time scales 'T_n' and is given by:

$$T_n = \epsilon^n t \quad \text{for } n = 0,1,2,3,... \quad (5)$$

In Eqn. (5), T_o is nominally considered as fast time and T_1, T_2, T_3 as slow times, such that

$$D_0 = \frac{\partial}{\partial T_0},\ D_1 = \frac{\partial}{\partial T_1},\ D_2 = \frac{\partial}{\partial T_2},\ D_3 = \frac{\partial}{\partial T_3} \quad (6)$$

Before apply this method, it is necessary to order the cubic term, the damping and the excitation. To accomplish this, we choose to set the following [7]

$$\mu = \epsilon^3\mu;\ \beta_{mn} = \epsilon\beta_{mn}; \qquad p = \epsilon^3 p \quad (7)$$

For this particular study, third order perturbation is carried out so the above values depend on the higher order power of perturbation. Now find the derivatives with respect to time and solve the Duffing'sEqn. (4). Furthermore, a detuning parameterσ_{mn} is introduced, which quantitatively describes the nearness of Ω_{mn} toω_{mn}. Accordingly [7],

$$\Omega_{mn} = \omega_{mn} + \epsilon^3\sigma_{mn} \quad (8)$$

After substituting all the known values from Eqns.(7) & (8) into Eqn. (2), leads to

$$\begin{aligned}\ddot{\psi}_{mn}(t) + 2\epsilon^3\mu\dot{\psi}_{mn}(t) + \omega^2_{mn}\psi_{mn}(t) &+ \epsilon\beta_{mn}\psi^3_{mn}(t)\\ &= \epsilon^3\frac{\lambda_{mn}}{D}p\cos(\omega_{mn} + \epsilon^3\sigma_{mn})t\end{aligned} \quad (9)$$

This introduces damping and the excitation to third order perturbation. Substituting the expansion of power series Eqn. (3) and the three times derivative from Eqn. (6) into the Eqn. (9) and separate like order terms, we get,

For order ε^0:

$$D_0^2\psi_{0mn} + \omega^2_{mn}\psi_{0mn} = 0 \quad (10)$$

For order ε^1:

$$D_0^2\psi_{1mn} + \omega^2_{mn}\psi_{1mn} = -2D_0D_1\psi_{0mn} - \beta_{mn}\psi^3_{0mn} \quad (11)$$

For order ε^2:

$$\begin{aligned}D_0^2\psi_{2mn} + \omega^2_{mn}\psi_{2mn} = &-D_1^2\psi_{0mn} - 2D_0D_2\psi_{0mn}\\ &- 2D_0D_1\psi_{1mn} - \beta_{mn}\psi^3_{1mn}\end{aligned} \quad (12)$$

For order ε^3:

$$\begin{aligned}D_0^2\psi_{3mn} + \omega^2_{mn}\psi_{3mn} = &-2(D_0D_3 + D_1D_2)\psi_{0mn}\\ &-(D_1^2 + 2D_0D_2)\psi_{1mn}\\ &-2D_0D_1\psi_{2mn} - \beta_{mn}\psi^3_{mn}\\ &+\frac{\lambda_{mn}}{D}p\cos(\omega_{mn} + \epsilon^3\sigma_{mn})t\end{aligned} \quad (13)$$

The general solution of Eqn. (10) can be written as:

$$\psi_{0mn} = B(T_1,T_2,T_3)e^{i\omega_{mn}T_0} + \bar{B}(T_1,T_2,T_3)e^{-i\omega_{mn}T_0} \quad (14)$$

where$B$ is unknown complex amplitude, and $\bar{B}$ is the complex conjugate of B.

Substituting the value of ψ_{0mn}in Eqn. (11), we get,

$$\begin{aligned}D_0^2\psi_{1mn} + \omega^2_{mn}\psi_{1mn} = [&-2iD_1B\omega_{mn}\\ &-3\beta_{mn}B^2\bar{B}]e^{i\omega_{mn}T_0} + cc\end{aligned} \quad (15)$$

where cc denotes the complex conjugate of the preceding terms.

This amplitude is determined by eliminating the secular terms from ψ_{1mn}. To eliminate the secular term from the Eq. (15), we must put,

$$-2iD_1B\omega_{mn} - 3\beta_{mn}B^2\bar{B} = 0 \quad (16)$$

It is convenient to write B in the polar form as,

$$B = \frac{1}{2}ae^{i\alpha} \quad (17)$$

where $B = B(T_1,T_2,T_3)$, the physical reasoning behind that is to keep amplitude in a steady state or nearly so. It also has to impose the conditions that real amplitude $b = b(T_1,T_2,T_3)$ and phase, $\alpha = \alpha(T_1,T_2,T_3)$.

Substituting all the known values in Eqn. (16), we get,

$$b = b(T_2), \quad \alpha = \frac{3\beta_{mn}}{8\omega_{mn}}b^2T_1 + \alpha_0(T_2) \quad (18)$$

The solution of Eqn. (15) is then obtained as

$$\psi_{1mn} = -\frac{\beta_{mn}}{8\omega^2_{mn}} B^3 e^{3i\omega_{mn}T_0} + cc = -\frac{\beta_{mn}}{32\omega^2_{mn}} b^3\cos(3\omega_{mn}T_0+3\alpha) \tag{19}$$

Similarly, from Eqn. (12), we get,

$$-2D_2Bi\omega_{mn} = \frac{15\beta^2_{mn}}{8\omega^2_{mn}} B^3\bar{B}^2 \tag{20}$$

The solution of order three is obtained as

$$\psi_{2mn} = -\frac{3\beta^2_{mn}}{208\omega^4_{mn}} B^5 e^{5i\omega_{mn}T_0} + cc \tag{21}$$

Substitute the values of ψ_{0mn}, ψ_{1mn} and ψ_{2mn} into the Eqn. (13) and eliminate the secular terms

$$-2iD_3B\omega_{mn} - 2D_1D_2B + \frac{\lambda_{mn}}{2D} p e^{i\sigma_{mn}T_3} = 0 \tag{22}$$

Now, substituting the values from Eqn. (16) and Eqn. (18) into Eqn. (22), we get,

$$-2iD_3B\omega_{mn} - \frac{45\beta^3_{mn}}{16\omega^4_{mn}} B^4\bar{B}^3 + \frac{\lambda_{mn}}{2D} p e^{i\sigma_{mn}T_3} = 0 \tag{23}$$

Eqn. (23) is much simpler and can directly be solved using the conventional particular integral method. To do this substituting the value of Eqn. (17) into Eqn. (23) and differentiate, we obtain,

$$\begin{aligned} -\omega_{mn} b\alpha' - i\omega_{mn} b' - i\omega_{mn}\mu b + \frac{45\beta^3_{mn}}{2048\omega^4_{mn}} b^7 \\ + \frac{\lambda_{mn}}{2D} p[\cos(\sigma_{mn}T_3 - \alpha) \\ + i\sin(\sigma_{mn}T_3 - \alpha)] = 0 \end{aligned} \tag{24}$$

where prime denotes the derivative with respect to T_3.
Now, separating the result into real and imaginary parts, we get,

Real Part:

$$b\alpha' = -\frac{45\beta^3_{mn}}{2048\omega^4_{mn}} b^7 - \frac{\lambda_{mn}}{2\omega_{mn}D} p\cos(\sigma_{mn}T_3 - \alpha) \tag{25}$$

Imaginary Part:

$$b' = -\mu b + \frac{\lambda_{mn}}{2\omega_{mn}D} p\sin(\sigma_{mn}T_3 - \alpha) \tag{26}$$

Eqns. (25) & (26) contain $\acute{\alpha}$, where this is the slowly varying phase angle and $\acute{b}$, the slowly varying amplitude. These Eqns. i.e., (26) & (27) can be transformed into an autonomous system i.e. one in which T_3 does not appear explicitly, by letting

$$K = \sigma_{mn}T_3 - \alpha \tag{27}$$

In the case of steady state motion $\acute{b} = \acute{K} = 0$ and this corresponds to the singular points of Eqns.(26) & (27); that is,

$$\sigma_{mn} b + \frac{45\beta^3_{mn}}{2048\omega^4_{mn}} b^7 = -\frac{\lambda_{mn}}{2\omega_{mn}D} p\cos K \tag{28}$$

$$\mu b = \frac{\lambda_{mn}}{2\omega_{mn}D} p\sin K \tag{29}$$

Squaring and adding Eqns. (28) & (29), we finally obtain,

$$\sigma_{mn} = -\frac{45\beta^3_{mn}}{2048\omega^4_{mn}} b^6 \pm \sqrt{\frac{\lambda^2_{mn}}{4\omega^2_{mn}b^2D^2}p^2 - \mu^2} \tag{30}$$

This is the frequency response equation which gives the modal amplitude response 'b' is the function of detuning parameter 'σ_{mn}' and the amplitude of the excitation 'p' this being a measure of deviation from the perfect force resonance condition.

III. RESULTS& DISCUSSIONS

Eqn. (30) represents the frequency response equation for the 3rd order perturbation parameter which shows a considerably decreasing value as compared to the results obtained in first order perturbation by Israr [1] representing here in Eqn. (31) and the 2nd order perturbation results are shownin Eqn. (32).

Order first frequency response

$$\sigma_{mn} = -\frac{3\beta_{mn}}{8\omega_{mn}} b^2 \pm \sqrt{\frac{\lambda^2_{mn}}{4\omega^2_{mn}b^2D^2}p^2 - \mu^2} \tag{31}$$

Order second frequency response

$$\sigma_{mn} = -\frac{15\beta^2_{mn}}{256\omega^3_{mn}} b^4 \pm \sqrt{\frac{\lambda^2_{mn}}{4\omega^2_{mn}a^2D^2}p^2 - \mu^2} \tag{32}$$

The total excitation frequency response of the part through cracked rectangular plate of order 3rd is

$$\Omega = \omega_{mn}\left[\begin{aligned} &1 + \epsilon\left(-\frac{3\beta_{mn}}{8\omega_{mn}} b^2 \pm \sqrt{\frac{\lambda^2_{mn}}{4\omega^2_{mn}b^2D^2}p^2 - \mu^2}\right) \\ &+\epsilon^2\left(-\frac{15\beta^2_{mn}}{256\omega^3_{mn}} b^4 \pm \sqrt{\frac{\lambda^2_{mn}}{4\omega^2_{mn}b^2D^2}p^2 - \mu^2}\right) \\ &+\epsilon^3\left(-\frac{45\beta^3_{mn}}{2048\omega^4_{mn}} b^6 \pm \sqrt{\frac{\lambda^2_{mn}}{4\omega^2_{mn}b^2D^2}p^2 - \mu^2}\right) \end{aligned}\right]$$

IV. CONCLUSIONS

The nonlinear vibration response of the cracked plate is investigated when it is subjected to transverse harmonic excitations, using the method of multiple scales. In this study it is shown conclusively by using 3rd order multiple scales approximation that the nonlinear characteristics of the steady state responses are encoded within the non-autonomous modulation equations. The introduction of 2nd and 3rd order perturbation terms in the solution of cracked plate problems with arbitrary boundary conditions leads to a more refined vibrational response. It is seen from Eqn. (33) that the excitation frequency of cracked plate obtained by using the higher order perturbation method of Multiple Scales has less error as compared to the resonance frequency obtained by low order perturbation results [1]. Further work will examine numerically the comparison of resonance frequency obtained by the present method for the chosen boundary conditions.

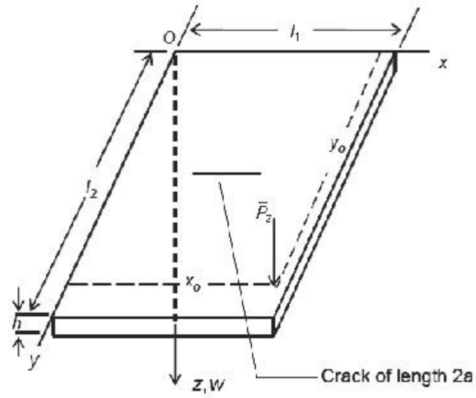

Fig. 1: A crack of length 2*a* at the center of an Isotropic plate [1]

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