Pak. J. Commer. Soc. Sci.
2011 Vol. 5 (1), 68-80

Cyclic Polygonal Designs with Block size 3 and Joint Distance 3

M. H. Tahir (Corresponding author)
Department of Statistics, The Islamia University of Bahawalpur, Pakistan
E-mail: mtahir.stat@gmail.com, mht@iub.edu.pk

Ijaz Iqbal
Department of Statistics, Bahuddin Zakariya University Multan, Pakistan
Email: drijaziqbal@yahoo.com

Jamal Abdul Nasir
Department of Statistics, The Islamia University of Bahawalpur, Pakistan
E-mails: njamal76@yahoo.com

Abstract

Polygonal designs, a class of partially balanced incomplete block designs with regular polygons, are useful in survey sampling in terms of balanced sampling plans excluding contiguous units (BSECs) and balanced sampling plans excluding adjacent units (BSAs) when neighboring units in a population provide similar information. In this paper, the method of cyclic shifts is used and cyclic polygonal designs (CPDs) are constructed with block size $k = 3$ and $\lambda = 1, 2, 3, 4, 6, 12$ for joint distance $\alpha = 3$ and $v \in \{21, 22, \ldots, 100\}$ treatments

Keywords: BIBD; cyclic BSA; cyclic polygonal design; cyclic shifts; distance between the units, joint distance; PBIBD; polygonal design.

MSC (2000): 05B05; 62K10; 62D05.

1. Introduction

Polygonal designs (PDs), a class of partially balanced incomplete block designs (PBIBDs) with regular polygons, are useful in survey sampling in terms of balanced sampling plans excluding contiguous units (BSECs) and balanced sampling plans excluding adjacent units (BSAs) when neighboring units in a population provide similar information. The use of PDs or balanced sampling plans is essential in situations where the units in a population (or in an experimental region) are found physically close (as neighbors) to each other. Studies in ecological and environmental sciences are often conducted to investigate the abundance and diversities of species, where a balanced sampling plan serves the purpose of generating samples from the population by avoiding the selection of neighboring (contiguous or adjacent) units which essentially provide redundant information. In other words, these neighboring units are deliberately prevented (or excluded) from being selected under the situation that they provide little new information to the sampling effort. These plans attempt to provide ways of sampling the units from geographical region when a spatial pattern in the response is expected (Christman, 1997; and See et al. 2000). Cochran (1977) also pointed out that neighboring units are often more alike than the units that are far apart, thus giving a poor contribution to the sampling information.

Now consider an example which highlights the usefulness of cyclic polygonal designs (CPDs) or balanced sampling plans in practical situations.

Example 1. A study was conducted to investigate the species abundance, diversity and richness, of certain insects in a forest. For the selected plots, a net was placed under the trees, the trees were fogged with the insecticide and the insects of the species of interest that landed on the net were counted. This is an expensive and time-consuming procedure that can be applied to only a relatively small number of plots with small areas. In this study, it is expected that counts from neighboring plots would be very similar, and that "fogging" one plot could alter the responses in neighboring plots. There were certain regions in the forest where a low insect count was expected from all trees due to recent fires, and other regions, near a creek, where a relatively high count was expected. Therefore, a sampling plan that avoids the simultaneous selection of neighboring plots within a region was utilized (See and Song, 2002).

Let a population consists of circular ordered units labeled as $0,1,\ldots,v-1$ then i and $i+1$ are said to be contiguous for all i such that $0 \le i \le v-2$, as are $v-1$ and 0. Let $\delta_{(i,j)}$ be the distance between the sampling units (or design points) i and j in this circular population such that $\delta_{(i,j)} = \min(|i-j|, v-|i-j|)$ and maximum distance between any pair of units cannot exceed $[v/2]$. For simplicity, denote the distance between sampling units (or design points) i and j as $\delta_{(i,j)} = 1,2,\ldots,[v/2]$. The notation cyclic BSEC(v,k,λ) is used to denote a cyclic polygonal design CPD($v,k,\lambda;1$) Similarly, the notation cyclic BSA ($v,k,\lambda;\alpha$) is used to denote a cyclic polygonal design CPD($v,k,\lambda;\alpha$). A simple cyclic polygonal design with minimal distance $\alpha = 1$ is denoted by CPD($v,k,\lambda;1$). A CPD($v,k,\lambda;\alpha$) is actually a PBIBD ($v,b,r,k,\lambda;\alpha$) for which association relations between the treatments are defined through the distance (see Frank and O'Shughnessy, 1974; Wei, 2002; Stufken et al. 1999).

Hedayat, Rao and Stufken (1988a, b) introduced balanced sampling plans excluding contiguous units (BSECs) in which the contiguous units do not appear together in a sample whereas all other pairs of units appear equally often. The first- and second-order inclusion probabilities are $\pi_i = \frac{n}{N}$; $i = 1,2,\ldots,N$, $\pi_{ij} = \frac{n(n-1)}{N(N-3)}$ $\forall i \ne j = 1,2,\ldots,N$, $|i-j|>1$, and $\pi_{ij} = 0$ when $|i-j| \le 1$, respectively. Hedayat et al. (1988a) showed that the variance of the Horvitz-Thompson estimator of the population mean $\mu = \sum Y_i$ is given by $\left\{1-\frac{(n-1\_(1-\rho_1)}{N-3}\right\}\frac{\sigma^2}{n}$ where σ^2 denotes the population variance and ρ_1 is the first order serial correlation between the units. Hedayat et al. (1988a) proved that their sampling plan is more efficient than the simple random sampling without replacement provided that $\rho_1 > \frac{-1}{N-1}$.

Ultimately, the objective of BSEC plans is to provide more representative sample and to provide more efficient estimator of the population mean, when neighboring (or contiguous) units are expected to provide similar responses. Simply in a sampling plan, the entire sample of units is selected such that no two neighboring units are included.

Stufken (1993) generalized the concept of BSECs to balanced sampling plans excluding adjacent units (BSAs) where all those adjacent pairs of units are excluded whose distance is less than or equal to α. The first- and second-order inclusion probabilities are $\pi_i = n/N$; $i = 1, 2, \ldots, N$, and $\pi_{ij} = [n(n-1)]/[N(N-2\alpha-1)]$ $\forall i \neq j = 1, 2, \ldots, N$, $|i-j| > \alpha$, and $\pi_{ij} = 0$ when $|i-j| \leq \alpha$, respectively. Two units i and j are called adjacent when their distance is less than a specified number α whose choice depends on the surveyor or experimenter (Mandal et al., 2009). It is obvious that BSEC(v, k, λ) is equivalent to BSA($v, k, \lambda; 1$) or BSA(1) (Stufken, 1993; Wei, 2002). For more comparison of balanced sampling plans with other sampling schemes, see See et al. (2000, 2007a, b) and Wright (2005).

A PD is a PBIBD with 2-class associate scheme (i.e. with λ_1 and λ_2) such that the two units which are ith associate of each other occur together in λ_i (distinct) blocks. Thus a PBIBD($v, b, r, k, \lambda_1 = 0, \lambda_2 = \lambda$), satisfying the necessary condition of PD is having first of the $\lambda's$. We will add one more parameter α in PBIBD if represented in terms of PD as PBIBD ($v, b, r, k, \lambda_1 = 0, \lambda_2 = \lambda; \alpha$), where α denotes the distance between the units.

To establish the existence and construction of CPD($v, 3, \lambda; 3$)'s, we first give the definition.

Definition 1. A CPD($v, k, \lambda; \alpha$) for $\{0, 1, \ldots, v-1\}$ treatments in b blocks of size k each ($k < v$) and some λ is a binary block design in which a pair of treatments (i, j) do not appears in any block if $\delta_{(i,j)} \leq \alpha$ and the pair of treatments (i, j) appears together in λ blocks if $\delta_{(i,j)} > \alpha$, for all $\forall (i, j) = \{0, 1, \ldots, v-1\}$.

For block size $k = 3$ and joint distance $\alpha \geq 2$, the existence and construction of CPDs for joint distance $\alpha = 2$ first appeared in Hedayat et al. (1998b). Stufken (1993) considered the existence and construction of CPDs for $\alpha \geq 2$. Wei (2002) suggested the existence of CPDs by using the Langford sequence with $k = 3$ and $\lambda = 1, 2$ for arbitrary α. Zhang and Chang (2005) used Langford sequence and extended Langford sequence, and constructed CPDs with $k = 3$ and λ =1,2,3,4,6,12 for $\alpha = 3$ and for some v. Mandal et al. (2008) used symmetrically repeated differences and integer linear programming approach and constructed CPDs with $k = 3$ and λ =1,2,3,4,6,12 for $\alpha = 3$ and for some v.

In this paper, the method of cyclic shifts is used to extend the existence and construction of CPDs for block size $k = 3$ and joint distance $\alpha = 3$. The interesting feature, in addition to the simplicity, of the proposed method is that the properties of a CPD from the sets of shifts (used in a CPD) can easily be obtained without constructing the actual blocks of a CPD. The pattern of off-diagonal zero elements (in bold form) from the main-diagonal in a concurrence matrix (or first row of the concurrence matrix) is helpful in the identification of the distance α in a CPD. Further, the off-diagonal elements in a concurrence matrix can easily be obtained from the sets of shifts or from concurrence set(s) of shifts. For more detail see Iqbal et al. (2009) and Tahir et al. (2010).

The paper is organized as follows. In Section 2, some algorithms are given to search CPD($v, 3, \lambda; 3$)'s with $\lambda =$ 1,2,3,4,6,12. In Section 3, the sets of shifts are given for the construction of CPD($v, 3, \lambda; 3$)'s with $\lambda =$ 1,2,3,4,6,12, and a complete solution (in terms of cyclic BSAs) is given for $v \in \{21, 22, \ldots, 100\}$ treatments. The concluding remarks are given in Section 4.

2. Algorithms for the construction of CPD($v, 3, \lambda; 3$)'s

The method of cyclic shifts is described below:

Let $S_j = \{q_{1j}, q_{2j}\}; j = 1, 2, \ldots$ be the set of shifts, where $\alpha < q_{1j}, q_{2j} < v - \alpha$ and q_{1j} and q_{2j} may be repeated any number of times.

Then $S_j^* = \{q_{1j}, q_{2j}, q_{1j} + q_{2j} \bmod v, v - q_{1j}, v - q_{2j}, v - (q_{1j} + q_{2j}) \bmod v\}$.

Here $v - q_j$ is complement of q_j.

A design will be CPD with $\alpha = 3$, if

(i) S_j^* consists of $4, \ldots, v-4$ an equal number of times, say λ; and

(ii) $q_{1j} + q_{2j}$ mod $v \neq 0$.

Some algorithms for the construction of CPDs with $k = 3$ and λ =1, 2, 3, 4, 6,12 for joint distance $\alpha = 3$ are presented with the conditions that

(i) $4 \leq q_{1j}, q_{2j} \leq v - 4$; and

(ii) $(q_{1j} + q_{2j})$ mod $v \neq 0, 1, 2, 3, v-1, v-2, v-3$.

Algorithm 2.1A. A fractional CPD with $k = 3$ and $\lambda = 1$ for $\alpha = 3$ can be constructed if $v = 6i + 3$; $i > 2$ and integer, from the following $(i-1)$ sets of shifts along with the fractional part $S_i = [v/3, v/3]\,(1/3)$

$$S_j = \{q_{1j}, q_{2j}\}; j = 1, 2, \ldots, (i-1)$$

such that $4, 5, \ldots, v-4$ appear once among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod v and their complements.

Algorithm 2.1B. A CPD with $k = 3$ and $\lambda = 1$ for $\alpha = 3$ can be constructed if $v = 6i + 1$; $i > 3$ and integer, from the following $(i-1)$ sets of shifts

$$S_j = \{q_{1j}, q_{2j}\}; j = 1, 2, \ldots, (i-1)$$

such that $4, 5, \ldots, v-4$ appear once among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod v and their complements.

Algorithm 2.2A. A fractional CPD with $k = 3$ and $\lambda = 2$ for $\alpha = 3$ can be constructed if $v = 12i$; $i > 1$ and integer, through the following $(4i-3)$ sets of shifts along with the fractional part $S_{4i-2} = [v/3, v/3]\,(2/3)$

$$S_j = \{q_{1j}, q_{2j}\}; j = 1, 2, \ldots, (4i-3)$$

such that $4, 5, \ldots, (v-2)/2, (v+2/2, \ldots, v-5, v-4$ all appear twice but $(v/2)$ appears once among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod v and their complements.

Algorithm 2.2B. A CPD with $k = 3$ and $\lambda = 2$ for $\alpha = 3$ can be constructed if $v = 12i + 4$; $i > 1$ and integer, from the following $(4i-1)$ sets of shifts

$$S_j = \{q_{1j}, q_{2j}\};\ j = 1, 2, \ldots, (4i-1)$$

such that $4, 5, \ldots, (v-2)/2, (v+2/2, \ldots, v-5,\ v-4$ appears twice but $(v/2)$ appears only once among $q_{1j}, q_{2j}, (q_{1j}+q_{2j})$ mod v and their complements.

Algorithm 2.3. A CPD with $k=3$ and $\lambda=3$ for $\alpha=3$ can be constructed if $v=6i-1; i>3$ and integer, from the following $(3i-4)$ sets of shifts

$$S_j = \{q_{1j}, q_{2j}\};\ j = 1, 2, \ldots, (3i-4)$$

such that $4, 5, \ldots, v-4$ appear thrice among $q_{1j}, q_{2j}, (q_{1j}+q_{2j})$ mod v and their complements.

Algorithm 2.4A. A CPD with $k=3$ and $\lambda=4$ for $\alpha=3$ can be constructed if $v=12i-2; i>1$ and integer, from the following $(8i-6)$ sets of shifts

$$S_j = \{q_{1j}, q_{2j}\};\ j = 1, 2, \ldots, (8i-6)$$

such that $4, 5, \ldots, (v-2)/2, (v+2/2, \ldots, v-5,\ v-4$ appear four times but $(v/2)$ appears twice among $q_{1j}, q_{2j}, (q_{1j}+q_{2j})$ mod v and their complements.

Algorithm 2.4B. A fractional CPD with $k=3$ and $\lambda=4$ for $\alpha=3$ can be constructed if $v=12i+6; i>1$ and integer, from the following $(8i-1)$ sets of shifts along with the fractional part $S_{8i}=[v/3, v/3](1/3)$

$$S_j = \{q_{1j}, q_{2j}\};\ j = 1, 2, \ldots, (8i-1)$$

such that $4, 5, \ldots, (v-2)/2, (v+2/2, \ldots, v-5,\ v-4$ appear four times but $(v/2)$ appears twice among $q_{1j}, q_{2j}, (q_{1j}+q_{2j})$ mod v and their complements.

Algorithm 2.5. A CPD with $k=3$ and $\lambda=6$ for $\alpha=3$ can be constructed if $v=12i+8; i>1$ and integer, from the following $(12i+1)$ sets of shifts

$$S_j = \{q_{1j}, q_{2j}\};\ j = 1, 2, \ldots, (12i+1)$$

such that $4, 5, \ldots, (v-2)/2, (v+2/2, \ldots, v-5,\ v-4$ appear six times but $(v/2)$ appears thrice among $q_{1j}, q_{2j}, (q_{1j}+q_{2j})$ mod v and their complements.

Algorithm 2.6. A CPD with $k=3$ and $\lambda=12$ for $\alpha=3$ can be constructed if $v=12i+2;\ i>1$ and integer, through the following $(24i-10)$ sets of shifts

$$S_j = \{q_{1j}, q_{2j}\};\ j = 1, 2, \ldots, (24i-10)$$

such that $4, 5, \ldots, (v-2)/2, (v+2/2, \ldots, v-5,\ v-4$ appear twelve times but $(v/2)$ appears six times among $q_{1j}, q_{2j}, (q_{1j}+q_{2j})$ mod v and their complements.

3. Construction of CPD $v, 3, \lambda; 3$)'s with $\lambda = 1, 2, 3, 4, 6, 12$

In this section, CPD($v, 3, \lambda; 3$)'s with $\lambda = 1, 2, 3, 4, 6, 12$ are constructed and a complete solution for $v \in \{21, 22, \ldots, 100\}$ treatments is given. Some fractional (or smaller) CPD($v, 3, \lambda; 3$)'s for $\lambda = 1, 2, 4$ have also been obtained. These CPDs exist under the necessary condition $\lambda = \beta \dfrac{k(k-1)}{[v-(2\alpha+1)]}$, where $\beta = \dfrac{\lambda(v-7)}{6}$ denotes the number of shifts required for a CPD with $\alpha = 3$.

The following Lemmas are given to complete constructions.

Lemma 3.1a. *There exists a fractional CPD(* $v,3,1;3$ *) if and only if* $v \equiv 3 \pmod 6$ *and* $v \geq 21$.

Proof. By using Algorithm 2.1A, the sets of shifts have been searched to construct non-fractional CPD($v,3,1;3$)'s for $v \in \{25, 31, 36, 43, 49, 55, 61, 67, 73, 79, 85, 91, 97\}$ treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=25: [4,9]+[5,6]+[7,8]
v=31: [4,7]+[5,10]+[6,12]+[8,9]
v=37: [4,7]+[5,14]+[6,10]+[8,12]+[9,13]
v=43: [4,16]+[5,13]+[6,11]+[7,8]+[9,12]+[10,14]
v=49: [4,9]+[5,21]+[6,14]+[7,17]+[8,11]+[10,12]+[15,16]
v=55: [4, 13]+[5, 15]+[6, 18]+[7, 23]+[8, 14]+[9, 12]+[10, 19]+[11, 16]
v=61: [4, 25]+[5, 17]+[6, 14]+[7, 21]+[8, 16]+[9, 18]+[10, 13]+[11, 15]+[12, 19]
v=67: [4, 23]+[5, 21]+[6, 9]+[7, 25]+[8, 16]+[10, 19]+[11, 17]+[12, 18]+[13, 20]
+[14, 22]
v=73: [4, 17]+[5, 27]+[6, 18]+[7, 26]+[8, 15]+[9, 30]+[10, 19]+[11, 20]+[12, 16]
+[13, 25]+[14, 22]
v=79: [4, 20]+[5, 21]+[6, 35]+[7, 29]+[8, 22]+[9, 25]+[10, 32]+[11, 17]+[12, 19]
+[13, 14]+[15, 18]+[16, 23]
v=85: [4, 32]+[5, 29]+[6, 20]+[7, 28]+[8, 17]+[9, 33]+[10, 14]+[11, 27]+[12, 18]
+[13, 31]+[15, 22]+[16, 23]+[19, 21]
v=91: [4, 27]+[5, 42]+[6, 33]+[7, 17]+[8, 28]+[9, 32]+[10, 16]+[11, 29]+[12, 22]
+[13, 15]+[14, 21]+[15, 30]+[18, 19]+[20, 23]
v=97: [4, 17]+[5, 26]+[6, 38]+[7, 12]+[8, 37]+[9, 25]+[10, 32]+[11, 29]+[13, 22]
+[14, 33]+[15, 28]+[16, 23]+[18, 30]+[20, 36]+[24, 27]

Lemma 3.1b. *There exists a fractional CPD(* $v,3,1;3$ *) if and only if* $v \equiv 1 \pmod 6$ *and* $v \geq 25$.

Proof. By using Algorithm 2.1B, the sets of shifts have been searched to construct fractional CPD($v,3,1;3$)'s for $v \in \{21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99\}$ treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=21: [4, 8]+[5, 6]+[7, 7]($(1/3)$
v=27: [4, 10]+[5, 6]+[7, 8]+[9, 9] $(1/3)$
v=33: [4, 12]+[5, 9]+[6, 7]+[8, 10]+[11, 11] $(1/3)$
v=39: [4, 14]+[5, 10]+[6, 11]+[7, 9]+[8, 12]+[13, 13] $(1/3)$
v=45: [4, 17]+[5, 11]+[6, 12]+[7, 13]+[8, 14]+[9, 10]+[15, 15] $(1/3)$
v=51: [4, 18]+[5, 9]+[6, 15]+[7, 12]+[8, 20]+[10, 16]+[11, 13]+[17, 17] $(1/3)$
v=57: [4, 16]+[5, 8]+[6, 17]+[7, 22]+[9, 15]+[10, 21]+[11, 14]+[12, 18]+[19, 19] $(1/3)$
v=63: [4, 22]+[5, 18]+[6, 28]+[7, 20]+[8, 9]+[10, 14]+[11, 19]+[12, 13]+[15, 16]
+[21, 21] $(1/3)$
v=69: [4, 29]+[5, 26]+[6, 12]+[7, 21]+[8, 17]+[9, 11]+[10, 22]+[13, 14]+[15, 24]
+[16, 19]+[23, 23] $(1/3)$
v=75: [4, 31]+[5, 19]+[6, 28]+[7, 13]+[8, 22]+[9, 27]+[10, 23]+[11, 18]+[12, 14]
+[15, 17]+[16, 21]+[25, 25] $(1/3)$
v=81: [4, 35]+[5, 25]+[6, 8]+[7, 17]+[9, 23]+[10, 21]+[11, 33]+[12, 26]+[13, 15]

+[16, 20]+[18, 29]+[19, 22]+[27, 27] (1/3)

v=87: [4, 37]+[5, 8]+[6, 25]+[7, 27]+[9, 33]+[10, 11]+[12, 16]+[14, 22]+[15, 24]
+[17, 23]+[18, 26]+[19, 30]+[20, 32]+[29, 29] (1/3)

v=93: [4, 37]+[5, 22]+[6, 26]+[7, 35]+[8, 39]+[9, 10]+[11, 25]+[12, 28]+[13, 21]
+[14, 29]+[15, 23]+[16, 17]+[18, 30]+[20, 24]+[31, 31] (1/3)

v=99: [4, 27]+[5, 42]+[6, 35]+[7, 30]+[8, 32]+[9, 20]+[10, 16]+[11, 38]+[12, 13]
+[14, 22]+[15, 39]+[17, 34]+[18, 28]+[19, 24]+[21, 23]+[33, 33] (1/3)

Lemma 3.2a. *There exists a fractional CPD($v, 3, 2; 3$) if and only if* $v \equiv 0 \pmod{12}$ *and* $v \geq 24$.

Proof. By using Algorithm 2.2A, the sets of shifts have been searched to construct fractional CPD($v, 3, 2; 3$)'s for $v \in \{24, 36, 48, 60, 72, 84, 96\}$ treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=24: [4, 6]+[4, 9]+[5, 7]+[5, 9]+[6, 7]+[8, 8] (2/3)

v=36: [4, 10](2)+[5, 11](2)+[6, 13](2)+[7, 8](2)+[9, 9]+[12, 12] (2/3)

v=48: [4, 15]+[4, 18]+[5, 7]+[5, 17]+[6, 11]+[6, 14]+[7, 12]+[8, 13]+[8, 15]+[9, 11]
+[9, 18]+[10, 13]+[10, 14]+[16, 16] (2/3)

v=60: [4, 24]+[4, 26]+[5, 17]+[5, 18]+[6, 17]+[6, 22]+[7, 11]+[7, 12]+[8, 13]+[8, 21]
+[9, 15]+[9, 16]+[10, 16]+[10, 19]+[11, 14]+[12, 15]+[13, 14]+[20, 20] (2/3)

v=72: [4, 18]+[4, 30]+[5, 14]+[5, 26]+[6, 21]+[6, 26]+[7, 16]+[7, 27]+[8, 12]+[8, 28]
+[9, 21]+[9, 23]+[10, 19]+[10, 25]+[11, 20]+[11, 22]+[12, 13]+[13, 15]
+[14, 15]+[16, 17]+[17, 18]+[24, 24] (2/3)

v=84: [4, 21]+[4, 32]+[5, 15]+[5, 35]+[6, 31]+[6, 33]+[7, 10]+[7, 33]+[8, 19]+[8, 23]
+[9, 20]+[9, 25]+[10, 24]+[11, 19]+[11, 30]+[12, 26]+[12, 27]+[13, 16]
+[13, 23]+[14, 18]+[14, 24]+[15, 22]+[16, 26]+[17, 18]+[21, 22]
+[28, 28] (2/3)

v=96: [4, 19]+[4, 24]+[5, 6]+[5, 31]+[6, 38]+[7, 22]+[7, 43]+[8, 29]+[8, 33]+[9, 38]
+[9, 39]+[10, 23]+[10, 31]+[11, 25]+[12, 18]+[12, 27]+[13, 21]+[13, 34]
+[14, 26](2)+[15, 20]+[15, 30]+[16, 28]+[16, 35]+[17, 20]+[17, 25]
+[18, 24]+[19, 27]+[21, 22]+[32, 32] (2/3)

Lemma 3.2b. *There exists a fractional CPD($v, 3, 2; 3$) if and only if* $v \equiv 4 \pmod{12}$ *and* $v \geq 28$.

Proof. By using Algorithm 2.2B, the sets of shifts have been searched to construct non-fractional CPD($v, 3, 2; 3$)'s for $v \in \{28, 40, 52, 64, 76, 88, 100\}$ treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=28: [4,7]+[4, 9]+[5, 6]+[5, 9]+[6, 10]+[7, 8]+[8, 10]

v=40: [4, 13]+[4, 14]+[5, 10]+[5, 11]+[6, 9]+[6, 14]+[7, 12](2)+[8, 9]+[8, 10]+[11, 13]

v=52: [4, 14]+[4, 20]+[5, 11]+[5, 16]+[6, 12]+[6, 17]+[7, 13]+[7, 15]+[8, 11]+[8, 17]
+[9, 19]+[9, 21]+[10, 13]+[10, 15]+[12, 14]

v=64: [4, 24]+[4, 25]+[5, 14](2)+[6, 20]+[6, 25]+[7, 15]+[7, 21]+[8, 16]+[8, 21]
+[9, 18](2)+[10, 12]+[10, 16]+[11, 12]+[11, 23]+[13, 17]+[13, 20]+[15, 17]

v=76: [4, 26](2)+[5, 24]+[5, 29]+[6, 8]+[6, 25]+[7, 8]+[7, 24]+[9, 18]+[9, 23]+[10, 22]
+[10, 27]+[11, 17]+[11, 22]+[12, 16]+[12, 21]+[13, 21]+[13, 23]+[14, 25]
+[15, 20]+[16, 19]+[17, 19]+18, 20]

v=88: [4, 26]+[4, 38]+[5, 10]+[5, 33]+[6, 21]+[6, 22]+[7, 30]+[7, 35]+[8, 19]+[8, 31]
+[9, 23]+[9, 35]+[10, 24]+[11, 21]+[11, 25]+[12, 24]+[12, 28]+[13, 18] +[13, 20]
+[14, 25]+[14, 29]+[15, 26]+[16, 18]+[16, 29]+[17, 23]+[17, 20]+[19, 22]
v=100: [4, 22]+[4, 33]+[5, 22]+[5, 35]+[6, 23]+[6, 24]+[7, 13]+[7, 35]+[8, 24] +[8, 33]
+[9, 38]+[9, 43]+[10, 31]+[10, 39]+[11, 18]+[11, 34]+[12, 34] +[12, 38]+[13, 31]
+[14, 23]+[14, 28]+[15, 25]+[15, 36]+[16, 27]+[16, 28]+[17, 19]+[17, 30]
+[18, 21]+[19, 26]+[20, 32]+[21, 25]

Lemma 3.3 *There exists a fractional CPD(*$v,3,3;3$*) if and only if* $v \equiv 5 \pmod 6$ *and* $v \geq 23$.

Proof. By using Algorithm 2.3, the sets of shifts have been searched to construct non-fractional CPD($v,3,3;3$)'s for $v \in \{23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95\}$ treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=23: [4, 6]+[4, 8]+[4, 9]+[5, 6]+[5, 7]+[5, 9]+[6, 7]+[7, 8]
v=29: [4, 8](2)+[4, 10]+[5, 6]+[5, 9]+[5, 11]+[6, 7]+[6, 11]+[7, 8]+[7, 9]+[9, 10]
v=35: [4, 8]+[4, 9]+[4, 12]+[5, 10]+[5, 11](2)+[6, 8]+[6, 9]+[6, 12]+[7, 10]+[7, 11]
+[7, 13]+[8, 13]+[9, 10]
v=41: [4, 11]+[4, 14](2)+[5, 8]+[5, 14]+[5, 15]+[6, 9]+[6, 10]+[6, 16]+[7, 10](2)+[7, 13]
+[8, 9]+[8, 11]+[9, 12]+[11,12]+[12,13]
v=47: [4, 11]+[4, 15]+[4, 18]+[5, 12]+[5, 16]+[5, 17]+[6, 7]+[6, 14]+[6, 17]+[7, 11]
+[7, 12]+[8, 10]+[8, 12]+[8, 13]+[9, 11]+[9, 14]+[9, 16]+[10, 14]+[10, 16]
+[13, 15]
v=53: [4, 13]+[4, 17]+[4, 19]+[5, 15](2)+[5, 19]+[6, 10]+[6, 16]+[6, 18]+[7, 15]+[7, 18]
+[7, 19]+[8, 14]+[8, 16]+[8, 18]+[9, 11]+[9, 12](2)+[10, 13](2)+[11 , 14]
+[11, 17]+[12, 14]
v=59: [4, 19]+[4, 20](2)+[5, 13]+[5, 17]+[5, 21]+[6, 8]+[6, 21](2)+[7, 9]+[7, 17] +[7, 18]
+[8, 14]+[8, 15]+[9, 13]+[9, 19]+[10, 15]+[10, 18]+[10, 20]+[11 , 15]+[11, 16]
+ [11, 23]+[12, 14]+[12, 17]+[12, 19]+[13, 16]
v=65: [4, 7]+[4, 18](2)+[5, 10]+[5, 26](2)+[6, 19](2)+[6, 23]+[7, 11]+[7, 20]+[8, 14]
+[8, 25]+[8, 26]+[9, 21](3)+[10, 17](2)+[11, 13]+[12, 16](3)+[13, 20](2)
+[14, 15](2)+[17, 24]+[19, 23]
v=71: [4, 19]+[4, 29](2)+[5, 24]+[5, 26](2)+[6, 15](2)+[6, 22]+[7, 10]+[7, 27](2)+[8, 18]
+[8, 24]+[8, 27]+[9,16](2)+[9, 25]+[10, 13](2)+[11, 13]+[11, 17](2)+[12, 18](2)
+[12, 20]+[14, 19]+[14, 22](2)+[15, 16]+[19, 20]+[20, 21]
v=77: [4, 21]+[4, 35](2)+[5, 27]+[5, 31](2)+[6, 18](2)+[6, 21]+[7, 15]+[7, 30](2)+[8, 20]
+[8, 26](2)+[9, 14](2)+[9, 19]+[10, 16]+[10, 22](2)+[11, 16]+[11, 19]+[11, 28]
+[12, 17](2)+[12, 24]+[13, 20](2)+[13, 21]+[14, 17]+[15, 18]+[15, 25]+[16, 19]
+[23, 25]
v=83: [4, 17](2)+[4, 36]+[5, 20]+[5, 26](2)+[6, 29]+[6, 33](2)+[7, 17]+[7, 23](2)+[8, 29]
+[8, 32](2)+[9, 18]+[9, 19]+[9, 22]+[10, 23]+[10, 24](2)+[11, 15]+[11, 27](2)
+[12, 18]+[12, 25](2)+[13, 21]+[13, 22](2)+[14, 18]+[14, 28](2)+[15, 16]
+[15, 29]+[16, 20](2)+[19, 19]
v=89: [4, 29]+[4, 30](2)+[5, 17]+[5, 35]+[5, 19]+[6, 15]+[6, 27](2)+[7, 29]+[7, 32]
+[7, 35]+[8, 14]+[8, 31](2)+[9, 11](2)+[9, 22]+[10, 28](2)+[10, 23]+[11, 23]
+[12, 16]+[12, 25]+[12, 32]+[13, 24](2)+[13, 27]+[14, 29](2)+[15, 21](2)
+[16, 20]+[16, 26]+[17, 21]+[17, 25]+[18, 23](2)+[18, 26]+[19, 22]+[19, 25]
v=95: [4, 17]+[4, 35]+[4, 41]+[5, 28]+[5, 31]+[5, 34]+[6, 21](2)+[6, 31]+[7, 20]
+[7, 37](2)+[8, 23]+[8, 40](2)+[9, 17]+[9, 36](2)+[10, 23](2)+[10, 34]
+[11, 19](3)+[12, 22]+[12, 29](2)+[13, 22](2)+[13, 29]+[14, 24]+[14, 25]

+[14, 32]+[15, 28](2)+[15, 32]+[16, 24]+[16, 26](2)+[17, 32]+[18, 20](2)
+[18, 25]+[24, 25]

Lemma 3.4a *There exists a fractional CPD(* $v,3,4;3$ *) if and only if* $v \equiv 10 \pmod{12}$ *and* $v \geq 22$.

Proof. By using Algorithm 2.4A, the sets of shifts have been searched to construct non-fractional CPD($v,3,4;3$)'s for $v \in$ {22, 34, 46, 58, 70, 82, 94} treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=22: [4, 6]+[4, 7]+[4, 8]+[4, 9]+[5, 6]+[5, 7](2)+[5, 8]+[6,7]+[6, 8]
v=34: [4, 8]+[4, 10]+[4, 12](2)+[5, 7]+[5, 9]+[5, 10]+[5, 13]+[6, 7]+[6, 8]+[6, 9] +[6, 11]
+[7, 10]+[7, 11]+[8, 11]+[8, 13]+[9, 10]+[9, 11]
v=46: [4, 8]+[4, 13](2)+[4, 17]+[5, 15](2)+[5, 18](2)+[6, 13](2)+[6, 14](2)+[7, 9](2)
+[7, 15](2)+[8, 9]+[8, 10](2)+[9, 12]+[10, 12](2)+[11, 14](2)+[11, 16](2)
v=58: [4, 20](3)+[4, 24]+[5, 21](4)+[6, 16](4)+[7, 11](4)+[8, 12]+[8, 19](3)+[9, 14](4)
+[10, 15](3)+[10, 19]+[12, 13]+[12, 15]+[12, 17]+[13, 17](3)
v=70: [4, 20](3)+[4, 25]+[5, 14]+[5, 25]+[5, 26]+[5, 30]+[6, 21](2)+[6, 26](2)+[7, 13]
+[7, 18]+[7, 22]+[7, 23]+[8, 11](3)+[8, 29]+[9, 22](3)+[9, 23]+[10, 17](2)
+[10, 24]+[10, 26]+[11, 21]+[12, 16](4)+[13, 17]+[13, 21]+[13, 23]
+[14, 15](2)+[14, 23]+[15, 18](2)+[17, 18]
v=82: [4, 12]+[4, 29]+[4, 32](2)+[5, 11](2)+[5, 30](2)+[6, 13](2)+[6, 27](2)+[7, 25](2)
+[7, 31](2)+[8, 21]+[8, 26](2)+[8, 29]+[9, 19](2)+[9, 22](2)+[10, 20](2)
+[10, 24](2)+[11, 17](2)+[12, 21]+[12, 26](2)+[13, 23](2)+[14, 23](2)
+[14, 27](2)+[15, 20](2)+[15, 25](2)+[16, 29]+[17, 22](2)+[18, 21](2)+[18, 24](2)
v=94: [4, 21]+[4, 24]+[4, 30](2)+[5, 30](2)+[5, 39](2)+[6, 35](2)+[6, 37](2)+[7, 26]
+[7, 32]+[7, 36](2)+[8, 10](2)+[8, 20](2)+[9, 29](2)+[9, 33]+[9, 39] +[10, 11](2)
+[11, 20](2)+[12, 19](2)+[12, 34](2)+[13, 19](2)+[13, 23](2)+[14, 15] +[14, 26]
+[14, 33](2)+[15, 26](2)+[15, 29]+[16, 22](2)+[16, 24](2)+[17, 25](2) +[17, 27]
+[17, 33]+[18, 27](2)+[21, 27]+[22, 23](2)+[24, 28]+[25, 32]

Lemma 3.4b *There exists a fractional CPD(* $v,3,4;3$ *) if and only if* $v \equiv 6 \pmod{12}$ *and* $v \geq 30$.

Proof. By using Algorithm 2.4B, the sets of shifts have been searched to construct fractional CPD($v,3,4;3$)'s for $v \in$ {30, 42, 54, 66, 78, 90} treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=30: [4, 5]+[4, 8]+[4, 11]+[4, 13]+[5, 7](2)+[5, 11]+[6, 8](2)+[6, 9]+[6, 10]+[7, 10]
+[7, 11]+[8, 9]+[9, 10]+[10, 10] (1/3)
v=42: [4, 14]+[4, 15](3)+[5, 8]+[5, 13](3)+[6, 11](3)+[6, 14]+[7, 8]+[7, 9](2)+[7, 10]
+[8, 12]+[8, 14]+[9, 12](2)+[10, 10]+[10, 16]+[11, 12]+[14, 14] (1/3)
v=54: [4, 6]+[4, 21](2)+[4, 23]+[5, 14](2)+[5, 15](2)+[6, 16](2)+[6, 17]+[7, 13](2)
+[7, 19](2)+[8, 14](2)+[8, 16](2)+[9, 12](2)+[9, 17](2)+[10, 15](2)+[10, 17]
+[11, 12](2)+[11, 13](2)+[18, 18]+[18, 18] (1/3)
v=66: [4, 17](2)+[4, 20](2)+[5, 20]+[5, 24](2)+[5, 27]+[6, 25](2)+[6, 26](2)+[7, 8]
+[7, 11]+[7, 21](2)+[8, 11]+[8, 15](2)+[9, 11]+[9, 19](2)+[9, 25] +[10, 17](2)
+[10, 23](2)+[11, 19]+[12, 14](2)+[12, 15]+[12, 18]+[13, 16](2) +[13, 18](2)
+[14, 16](2)+[22, 22]+[22, 22] (1/3)
v=78: [4, 6](2)+[4, 33](2)+[5, 28](2)+[5, 30](2)+[6, 18](2)+[7, 13](2)+[7, 16](2)+[8, 24]
+[8, 28](2)+[8, 30]+[9, 22](2)+[9, 25](2)+[10, 21](2)+[11, 19]+[11, 23](2)

+[11, 27]+[12, 17](2)+[12, 25](2)+[13, 14]+[13, 27]+[14, 15](2)+[14, 24]
+[15, 17](2)+[16, 19](2)+[18, 21](2)+[19, 27]+[20, 22](2)+[26, 26]
+[26, 26] $(1/3)$

v=90: [4, 24](3)+[4, 42]+[5, 18]+[5, 32](2)+[5, 38]+[6, 10]+[6, 15]+[6, 27]+[6, 39]
+[7, 17]+[7, 35]+[7, 36]+[7, 41]+[8, 21](2)+[8, 33](2)+[9, 31](2)+[9, 35]+[9, 36]
+[10, 17]+[10, 19](2)+[11, 20]+[11, 26](2)+[11, 39]+[12, 22](4)+[13, 14]
+[13, 25](2)+[13, 27]+[14, 25](2)+[14, 28]+[15, 17](2)+[15, 21]+[16, 19](2)
+[16, 33]+[18, 20]+[18, 26](2)+[20, 23](2)+[23, 31]+[30, 30]+[30, 30] $(1/3)$

Lemma 3.5 *There exists a fractional CPD(*$v,3,6;3$*) if and only if* $v \equiv 8 \pmod{12}$ *and* $v \geq 33$.

Proof. By using Algorithm 2.5, the sets of shifts have been searched to construct non-fractional CPD($v,3,6;3$)'s for $v \in$ {32, 44, 56, 68, 80, 92} treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=32: [4, 9](2)+[4, 11](3)+[4, 12]+[5, 7](3)+[5, 8]+[5, 10]+[5, 11]+[6, 7]+[6, 8](2)+[6, 9]
+[6, 10]+[6, 12]+[7, 8]+[7, 11]+[8, 10](2)+[9, 10](2)+[9, 11]

v=44: [4, 13](2)+[4, 14]+[4, 15]+[4, 18](2)+[5, 12](4)+[5, 14](2)+[6, 10](3)+[6, 15](3)
+[7, 9](3)+[7, 14](3)+[8, 10](3)+[8, 11](3)+[9, 13]+[9, 15](2)+[11, 13](3)
+[12, 12]

v=56: [4, 18](3)+[4, 23](3)+[5, 16](3)+[5, 19](3)+[6, 9](3)+[6, 19](3)+[7, 13](3)+[7, 14]
+[7, 20]+[7, 23]+[8, 9]+[8, 14](2)+[8, 17](3)+[9, 17](2)+[10, 11](2)+[10, 12]
+[10, 14](3)+[11, 12](2)+[11, 16](2)+[12, 18](3)+[13, 15](3)+[16, 20]

v=68: [4, 21](3)+[4, 23](3)+[5, 10]+[5, 16](2)+[5, 23](3)+[6, 24](3)+[6, 26](3)
+[7, 15](3)+[7, 26](3)+[8, 8]+[8, 14](3)+[8, 27]+[9, 16]+[9, 18]+[9, 19](3)+[9, 21]
+[10, 15](2)+[10, 24](3)+[11, 16]+[11, 18](5)+[12, 19](3)+[12, 20](3) +[13, 16]
+[13, 17](3)+[13, 20](2)+[14, 17](3)

v=80: [4, 18]+[4, 23]+[4, 27](2)+[4, 28](2)+[5, 20]+[5, 30](3)+[5, 32](2)+[6, 15](3)
+[6, 27](3) +[7, 18](3)+[7, 23]+[7, 24]+[7, 25]+[8, 18](2)+[8, 28]+[8, 29](3)
+[9, 10](3)+[9, 29](3)+[10, 13]+[10, 20](2)+[11, 23](3)+[11, 24](2)+[11, 26]
+[12, 13]+[12, 24](2)+[12, 28](3)+[13, 21](3)+[13, 32]+[14, 17](3) +[14, 19](3)
+[15, 24] +[15, 26](2)+[16, 20](3)+[16, 22](2)+[16, 26]+[17, 22](3)

v=92: [4, 18]+[4, 20]+[4, 24](2)+[4, 39](2)+[5, 16](2)+[5, 27]+[5, 38](3)+[6, 32]
+[6, 35](2)+[6, 40](3)+[7, 20](2)+[7, 21]+[7, 33](3)+[8, 29](3)+[8, 34](3)
+[9, 21](3)+[9, 30](3)+[10, 22](2)+[10, 34](3)+[10, 39]+[11, 17]+[11, 18](2)
+[11, 25](3)+[12, 19](3)+[12, 23]+[12, 26](2)+[13, 16]+[13, 22](3)
+[13, 23](2)+[14, 23](3)+[14, 31](3)+[15, 17](2)+[15, 26](4)+[16, 17](3)
+[18, 24](3)+[19, 25](3)+[20, 27](3)+[28, 28]

Lemma 3.6 *There exists a fractional CPD(*$v,3,12;3$*) if and only if* $v \equiv 2 \pmod{12}$ *and* $v \geq 33$.

Proof. By using Algorithm 2.6, the sets of shifts have been searched to construct non-fractional CPD($v,3,12;3$)'s for $v \in$ {26, 38, 50, 62, 74, 86, 98} treatments, which satisfy the necessary condition $\lambda = [r(k-1)]/(v-7)$.

v=26: [4, 6](4)+[4, 7](2)+[4, 8](2)+[4, 10](2)+[4, 11](2)+[5, 6](2)+[5, 7]+[5, 8](4)
+[5, 9](3)+[5, 10](2)+[6, 7](2)+[6, 8](3)+[6, 9]+[7, 7]+[7, 8]+[7, 9](4)+[8, 9](2)

v=38: [4, 7]+[4, 9](2)+[4, 10](3)+[4, 12](2)+[4, 14](4)+[5, 8](2)+[5, 10]+[5, 11]

+[5, 12](4)+[5, 13](2)+[5, 14](2)+[6, 7]+[6, 8]+[6, 9](2)+[6, 10](3)+[6, 11](4)
+[6, 12]+[7, 9](2)+[7, 10](2)+[7, 11](4)+[7, 12](2)+[8, 8](2)+[8, 12]+[8, 13](2)
+[8, 15](2)+[9, 10](2)+[9, 13](2)+[9, 14](2)+[10, 13]+[11,12](2)

v=50: [4, 16](2)+[4, 18](5)+[4, 20](5)+[5, 11]+[5, 12](3)+[5, 13](3)+[5, 14]+[5, 16](3)
+[5, 19]+[6, 8](2)+[6, 9]+[6, 13](3)+[6, 14](3)+[6, 17](3)+[7, 8]+[7, 10]+[7, 11]
+[7, 12](3)+[7, 15](3)+[7, 18](3)+[8, 11](2)+[8, 14](3)+[8, 15]+[8, 16](3)
+[9, 11](2)+[9, 12](6)+[9, 15](3)+[10, 11](3)+[10, 13](5)+[10, 17](3)+[11, 14](3)
+[13, 15]+[15, 16](2)+[16, 17]

v=62: [4, 19](12)+[5, 7]+[5, 12](5)+[5, 24](5)+[5, 28]+[6, 7](5)+[6, 24](6)+[6, 28]
+[7, 13](6)+[8, 14](6)+[8, 21](6)+[9, 18](6)+[9, 22](6)+[10, 15](6)+[10, 16](4)
+[10, 18](2)+[11, 13]+[11, 16](6)+[11, 17](5)+[12, 20](4)+[12, 25](2)
+[14, 14](2)+[14, 16](2) +[15, 21](6)+[17, 20](2+[18, 18](2)

v=74: [4, 31](12)+[5, 19](10)+[5, 20](2)+[6, 15]+[6, 21](11)+[7, 25](2)+[7, 26](10)
+[8, 28](12)+[9, 9]+[9, 14]+[9, 15](2)+[9, 18]+[9, 25](6)+[10, 15](2)
+[10, 20](10)+[10, 12](11)+[11, 15]+[12, 14]+[13, 16](12)+[14, 18](10)
+[15, 22](6)+[17, 17](6)+[19, 22](2)+[22, 22](2)

v=86: [4, 21](6)+[4, 37](6)+[5, 31](6)+[5, 32](6)+[6, 33](12)+[7, 17](6)+[7, 28](6)
+[8, 18](5)+[8, 20]+[8, 26](6)+[9, 13](2)+[9, 14](3)+[9, 18]+[9, 21](6)
+[10, 10](2)+[10, 20]+[10, 24](6)+[10, 26]+[11, 17](6)+[11, 18](6)
+[12, 20](6)+[12, 31](6)+[13, 23](3)+[13, 30]+[13, 29](6)+[14, 27](5)
+[14, 30](2)+[14, 31]+[14, 36]+[15, 23](6)+[15, 25](6)+[16, 19](6)
+[16, 22](6)+[19, 27](6)+[20, 22](3)

v=98: [4, 26](6)+[4, 46](6)+[5, 36](5)+[5, 37](6)+[5, 40]+[6, 23](6)+[6, 34](6)
+[7, 25](5)+[7, 28](6)+[7, 32]+[8, 38](6)+[8, 39](6)+[9, 33](6)+[9, 35](6)
+[10, 20](5)+[10, 31](6)+[10, 38]+[11, 11](2)+[11, 15]+[11, 18](6)+[11, 25]
+[12, 21](6)+[12, 24]+[12, 38](5)+[13, 15](6)+[13, 17]+[13, 21](5)+[14, 20]
+[14, 22](5)+[14, 23](6)+[15, 20](5)+[16, 27](6)+[16, 31](6)+[17, 22](5)
+[17, 32](6) +[18, 27](6)+[19, 24](6)+[19, 25]+[19, 26](5)+[20, 21]+[20, 24](5)

4. Concluding remarks

Hedayat et al. (1988b, p.577) first constructed CPD (in terms of BSA) for joint distance $\alpha = 2$. Stufken (1993) introduced the existence of CPDs (in terms of BSAs) for joint distance $\alpha \geq 2$ and constructed CPDs for $\alpha = 2$. Wei (2002) suggested the use of Langford sequence for the existence and construction of CPDs (in terms of BSAs) with $k = 3$ and $\lambda = 1, 2$ for arbitrary α.

Zhang and Chang (2005) gave the construction of CPDs (in terms of BSAs) for $k = 3$, $\alpha = 3$ and $\lambda = 1, 2, 3, 4, 6, 12$ for some v. The base blocks and the triples for CPD($v, 3, \lambda; 3$)'s are given below in Table 1.

Table 1: CPD ($v, 3, \lambda; 3$)'s by Zhang and Chang (2005) for $\lambda = 1, 2, 3, 4, 6, 12$

λ	v (base blocks)	v (Triples using Langford sequence)
1	(25, 31, 37, 43)	(21, 27, 33, 39, 57, 63, 81, 87, 105, 111, 129)
2	(24), (28, 40, 52, 64, 76, 88)	(36, 48)
3	???	23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 89, 95)
4	22	(34, 46, 58), (30, 42, 54)
6	???	(32, 44)
12	26	(38, 50, 62)

Mandal et al. (2008) presented a catalog of CPDs (in terms of BSAs) with $k = 3$ and $\alpha = 3$ for some v. The detail of CPD($v, 3, \lambda; 3$)'s with $\lambda = 1, 2, 3, 4, 6, 12$ is given below in Table 2.

Table 2: CPD ($v, 3, \lambda; 3$)'s by Mandal et al. (2008) for $\lambda = 1, 2, 3, 4, 6, 12$

λ	v (Initial blocks using linear programming)
1	25, 31, 37
2	???
3	27, 29, 33
4	22, 34
6	23, 24, 36
12	30

It is noted from Table 1 and Table 2 that the CPD($v, 3, \lambda; 3$)'s with $\lambda = 1, 2, 3, 4, 6, 12$ are available only for limited v.

In this paper, CPD($v, 3, \lambda; 3$)'s with $\lambda = 1, 2, 3, 4, 6, 12$ are constructed and a solution for $v \in \{21, 22, \Lambda, 100\}$ treatments is given. The proposed CPDs are given in Table 3. Some fractional (or smaller) CPD($v, 3, \lambda; 3$)'s for $\lambda = 1, 2, 4$ have also been obtained.

Table 3: Proposed CPD ($v, 3, \lambda; 3$)'s with $\lambda = 1, 2, 3, 4, 6, 12$ for $v \in \{21, 22, \Lambda, 100\}$

λ	v (CPDs by using the method of cyclic shifts)	Existence
1 f	21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99	$v \equiv 3 \pmod 6$
1	25, 31, 37, 43, 49, 55, 61, 67, 73, 79, 85, 91, 97	$v \equiv 1 \pmod 6$
2 f	24, 36, 48, 60, 72, 84, 96	$v \equiv 0 \pmod{12}$
2	28, 40, 52, 64, 76, 88, 100	$v \equiv 4 \pmod{12}$
3	23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95	$v \equiv 5 \pmod 6$
4	22, 34, 46, 58, 70, 82, 94	$v \equiv 10 \pmod{12}$
4 f	30, 42, 54, 66, 78, 90	$v \equiv 6 \pmod{12}$
6	32, 44, 56, 68, 80, 92	$v \equiv 8 \pmod{12}$
12	26, 38, 50, 62, 74, 86, 98	$v \equiv 2 \pmod{12}$

where f stands for fractional (or smaller) CPDs

References

Christman, M.C. (1997). Efficiency of some sampling designs for spatially clustered populations. *Environmetrics*, 8, 145--166.

Cochran, W. G. (1977). *Sampling Techniques*. Wiley, New York.

Frank, D.C. and O'Shughnessy, C.D. (1974). Polygonal designs. *The Canadian Journal of Statistics*, 2, 45-59.

Hedayat, A.S., Rao, C.R. and Stufken, J. (1988a). Sampling plans excluding contiguous units. *Journal of Statistical Planning and Inference*, 19, 159-170.

Hedayat, A.S., Rao, C.R. and Stufken, J. (1988b). Designs in survey sampling avoiding contiguous units. In: P.R. Krishnaiah and C.R. Rao (Eds.), *Handbook of Statistics 6*: *Sampling*, Elsevier, Amsterdam, pp.575-583.

Iqbal, I., Tahir, M.H., Akhtar, M., Ghazali, S.S.A, Shabbir, J. and Bukhari, N.S. (2009). Generalized polygonal designs with block size 3 and $\lambda = 1$. *Journal of Statistical Planning and Inference*, 139, 3200-3219.

Mandal, B.N., Parsad, R. and Gupta, V.K. (2008). Computer-aided construction of balanced sampling plans excluding contiguous units. *Journal of Statistics and Applications*, 3, 67-93.

Mandal, B.N., Parsad, R., Gupta, V.K. and Sud, U.C. (2009). A family of distance balanced sampling plans. *Journal of Statistical Planning and Inference*, 139, 860-874.

See, K. and Song, S.Y. (2002). Spatially constrained sampling. In: A.H.El-Shaarwi and W.W. Piegorsch (Eds.), *Encyclopedia of Environmetrics*, Vol. 4, Wiley, pp.2080-2083.

See, K., Nobel, R.B. and Bailer, A.J. (2007a). Comparison of relative efficiencies of sampling plans excluding certain neighboring units: a simulation study. *Journal of Statistical Computation and Simulation*, 77, 605-612.

See, K., Nobel, R.B. and Bailer, A.J. (2007b). Computing inclusion probabilities to obtain Horvitz-Thompson estimators for sampling plans excluding neighboring units. *Journal of Statistical Computation and Simulation*, 77, 955-967.

See, K., Song, S.Y., Stufken, J. and Bailer, A.J. (2000). Relative efficiencies of sampling plans for selecting a small number of units from a rectangular region. *Journal of Statistical Computation and Simulation*, 66, 273-294.

Stufken, J. (1993). Combinatorial and statistical aspects of sampling plans to avoid the selection of adjacent units. *Journal of Combinatorial and Information System Sciences*, 18, 149-160.

Stufken, J., Song, S.Y., See, K. and Driessel, K.R. (1999). Polygonal designs: Some existence and non-existence results. *Journal of Statistical Planning and Inference*, 77, 155-166.

Stufken, J. and Wright, J.H. (2001). Polygonal designs with blocks of size $k \leq 10$. *Metrika*, 54, 179-184.

Wei, R. (2002). Cyclic BSEC of block size 3. *Discrete Mathematics*, 250, 291-298.

Wright, J. (2005). Variance estimation under balanced sampling plans excluding adjacent units. *Proceedings Survey Research Methods Section, ASA*, pp. 3676--3682.
Available from http://www.amstat.org/sections/SRMS/Proceedings/y2005/Files/JSM2005-000095.pdf

Zhang, J. and Chang, Y. (2005). The spectrum of cyclic BSA($v, 3, \lambda; \alpha$) with $\alpha = 2, 3$. *Journal of Combinatorial Designs*, 13, 313-335.