Reliability and Quantile Analysis of Pareto Distribution

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Abstract

This paper presents the reliability and Quantile analysis of the Pareto distribution. The main interests are in the relationship between β and various percentiles lives that describe the spread of the values in a set of data. Here these quantiles models are presented graphically and mathematically.

Key Words: Pareto distribution, Quantile analysis, percentile life.

1. Introduction

Pareto distribution is one of the most important distribution in statistical theory that describe the allocation of wealth among individuals (Lorenz, 1905). This distribution has also many applications for wind data. This distribution has many applications in Socio-economic problems, like population studies, in natural resources problems; size of firms, distribution of income over populations, pareto distribution curves gives a good fit to the data towards the extremes cases (primal .M, 2000).

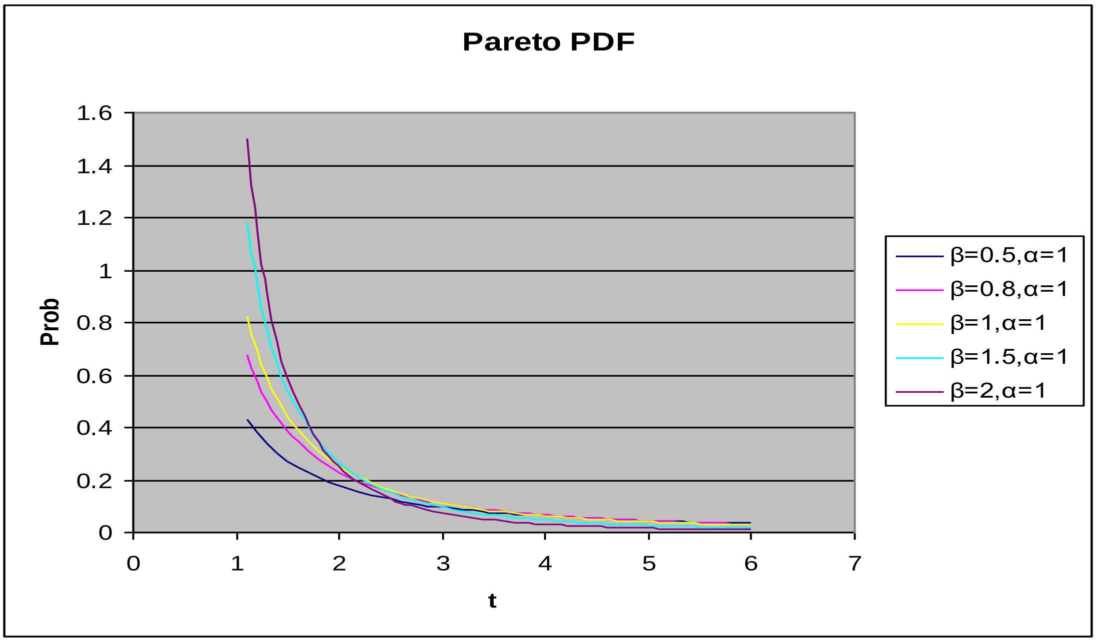
2. Pareto Model Analysis

2.1 Pareto Probability Distribution

The Pareto probability distribution has two parameters α and β. It can be used to represent the fatigue probability density function (PDF) is given by:

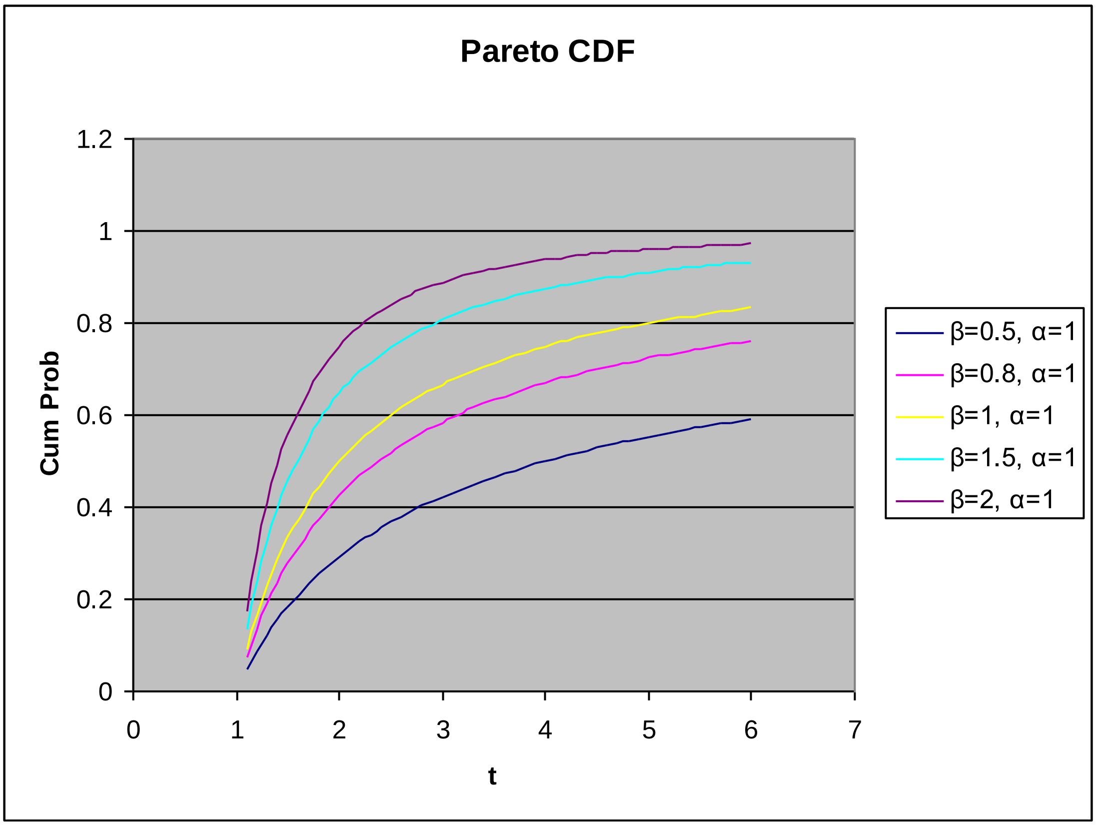
$$f_P(t) = \frac{\beta}{\alpha}\left(\frac{\alpha}{t}\right)^{\beta}, \qquad \alpha > 0, \beta > 0, t \geq \alpha \quad (2.1)$$

Fig2.1 The Pareto PDF



Where β is the shape parameter representing the different pattern of the Pareto PDF and is positive, η is a scale parameter and is also positive. It can be seen from the pareto probability density graph that different patterns shows a small amount of variations, and then decreases steadily as $t \to \infty$. This distribution is not limited to describing wealth or income distribution, but to many situations in which an equilibrium is found in the distribution of the "small" to the "large". Fig. 2.1 shows the steadily decreasing shapes of the Pareto PDF with the value of $\alpha = 1$ and β (=0.5, 0.8, 1, 1.5, 2).

Fig2.2 The Pareto CDF



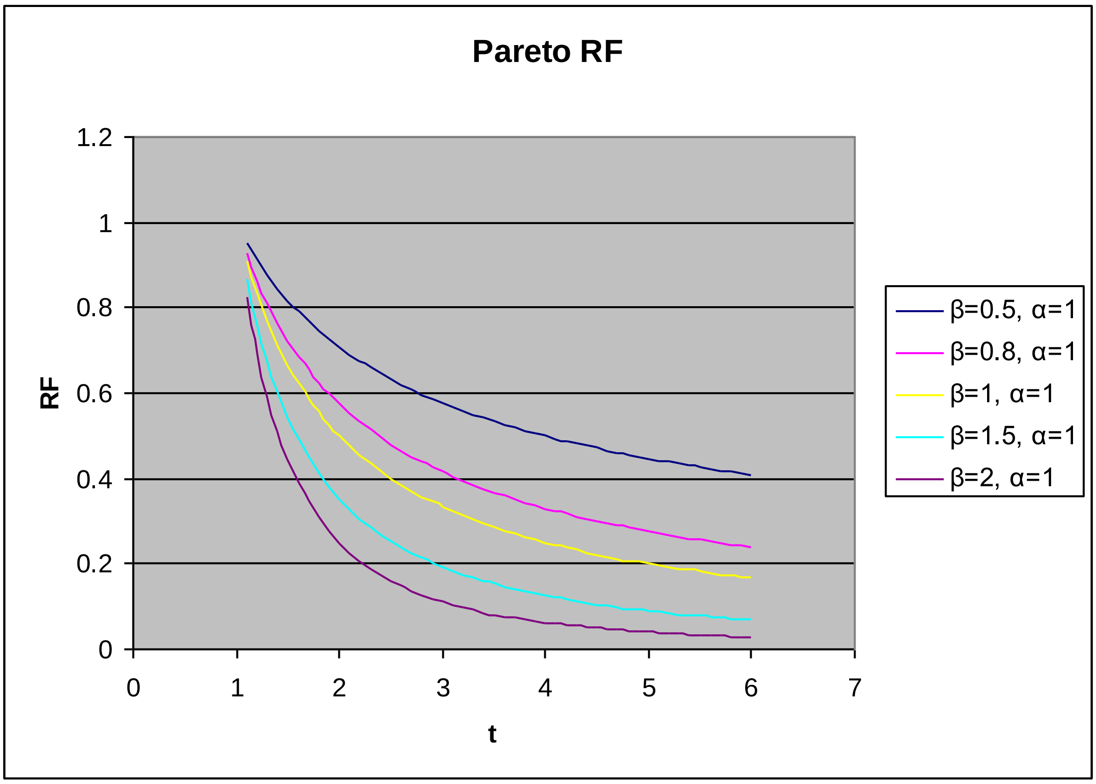
2.2 Pareto Cumulative Distribution Function

The cumulative distribution function of the Pareto distribution is denoted by $F_P(t)$ and is defined as:

$$F_P(t) = 1 - \left(\frac{\alpha}{t}\right)^{\beta} \quad (2.2)$$

It can be seen from the pareto cumulative distribution graph that different patterns of unreliability shows an increasing pattern as $t \to \infty$. It is important to note that as shape parameter increases the patterns of unreliability is increases. It is also important to note that there is no unreliability pattern between zero and one. Fig. 2.1 shows the increasing shapes of the Pareto CDF with the value of $\alpha = 1$ and β (=0.5, 0.8, 1, 1.5, 2).

Fig2.3 The Pareto RF



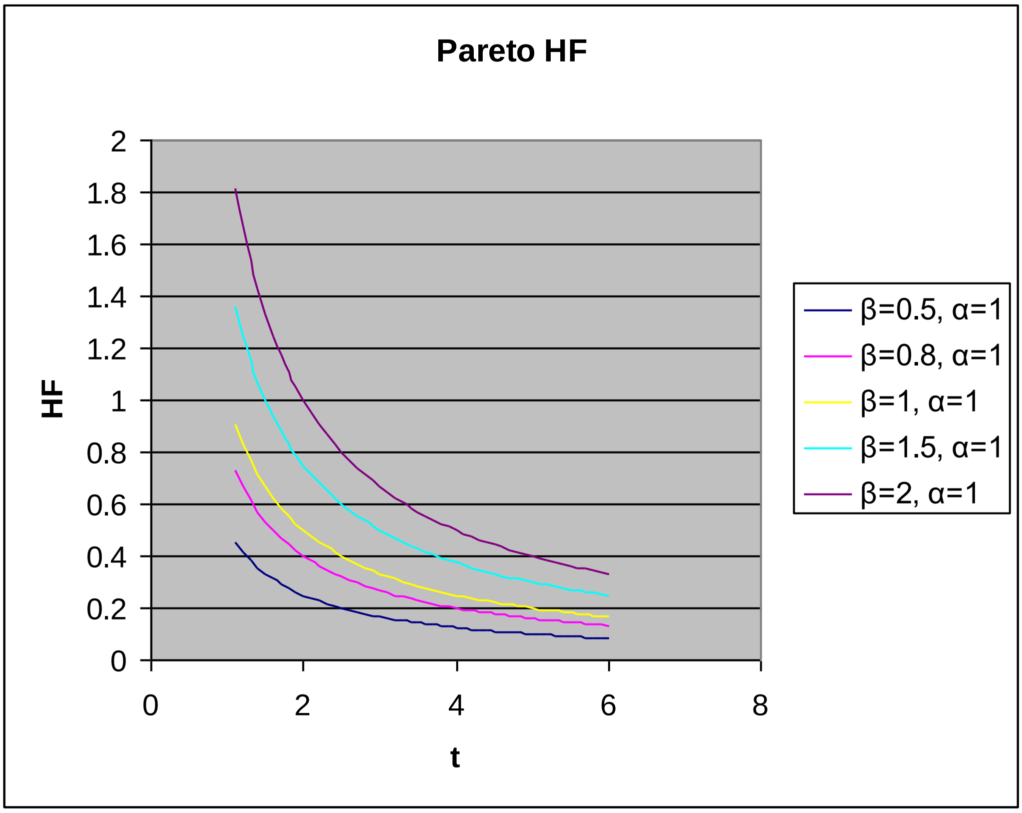
2.3 Pareto Reliability Function

The reliability function of the Pareto distribution is denoted by $R_P(t)$ and is defined as:

$$R_P(t) = \left(\frac{\alpha}{t}\right)^{\beta} \qquad (2.3)$$

It can be seen from the pareto reliability graph that different patterns of reliability shows decreasing pattern as $t \to \infty$. It is important to note that as shape parameter increases then the patterns of reliability is decreases. It is also important to note that there is no reliability pattern between zero and one. Fig. 2.1 shows the decreasing shapes of the Pareto RF with the value of $\alpha = 1$ and β (=0.5, 0.8, 1, 1.5, 2).

Fig2.4 The Pareto HF



2.4 Pareto Hazard Function

The Hazard function of the Pareto distribution is denoted by $H_P(t)$ and is defined as:

$$H_P(t) = \frac{\frac{\beta}{\alpha}\left(\frac{\alpha}{t}\right)^{\beta}}{\left(\frac{\alpha}{t}\right)^{\beta}} \quad (2.4)$$

It can be seen from the pareto hazard graph that different patterns of hazard shows decreasing pattern as $t \rightarrow \infty$. It is important to note that as shape parameter increases then the patterns of hazard is decreases. It is also important to note that there is no hazard pattern between zero and one. Fig. 2.1 shows the decreasing shapes of the Pareto HF with the value of $\alpha = 1$ and β (=0.5, 0.8, 1, 1.5, 2).

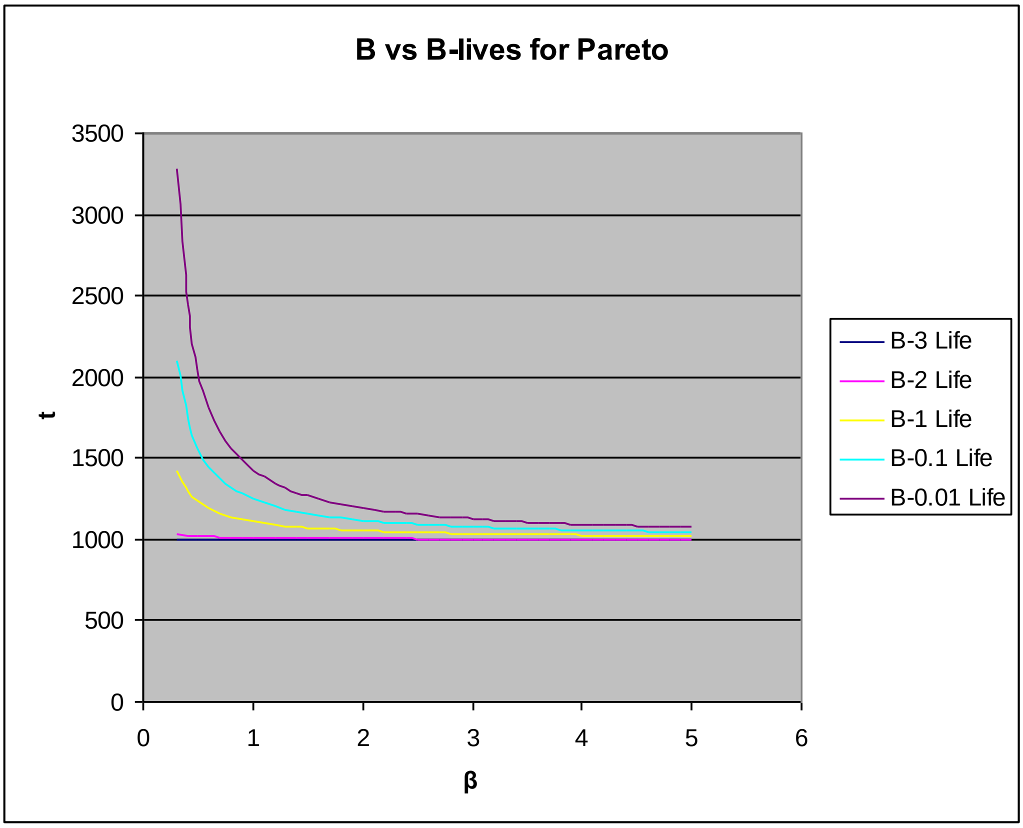
3. Quantile Analysis of the Pareto distribution

One of the important properties of the Pareto distribution is the percentile life or B-life in engineering terminology is defined as

$$t_p = \frac{\alpha}{(1-p)^{\frac{1}{\beta}}} \quad (3.1)$$

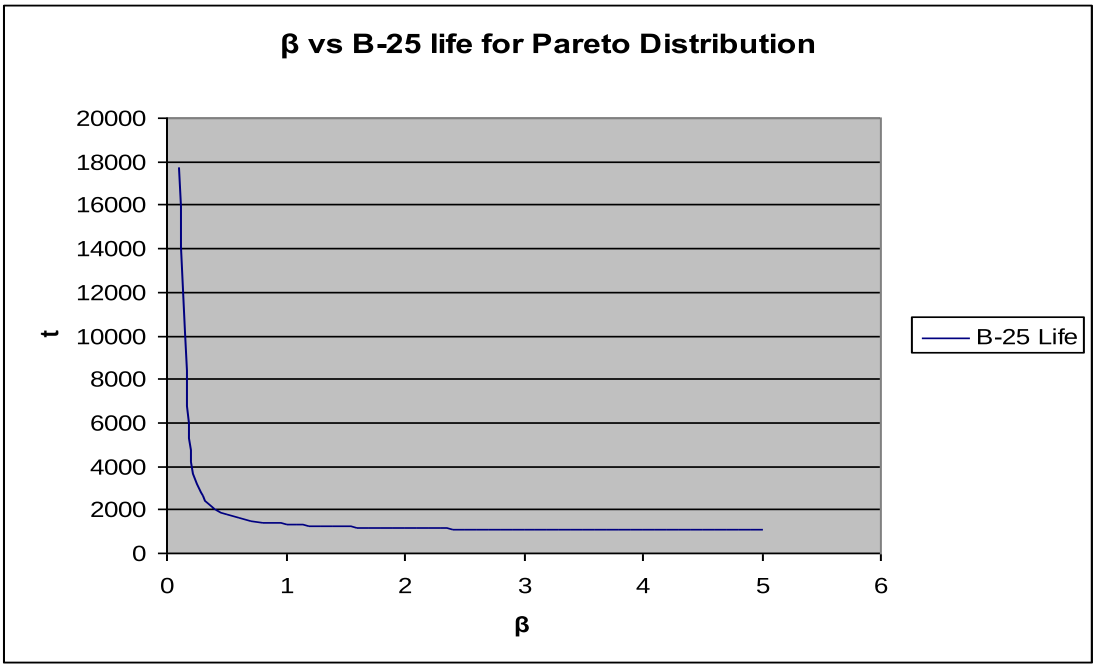
Fig. 3.1 shows the relationship between β and various values of B-lives (B-0.0l, B-0.1, B-1, B-2 and B-3 lives) for α = 1000. It is clear that the larger the value of β, the smaller the B-lives for the same value of α. The simulated results of the percentile life are shown in table A.

Fig. 3.1 β vs B-lives for α = 1000



We obtain the maximum value of B-3 life is 35401.33 for β =0.1 and the minimum value of B-3 life is 1073.941 for β =5. We obtain the maximum value of B-2 life is 9313.226 for β =0.1 and the minimum value of B-2 life is 1045.64 for β =5. We obtain the maximum value of B-1 life is 2867.972 for β =0.1 and the minimum value of B-1 life is 1021.296 for β =5. The relationship between β and percentile life for α = 1000 is shown in Fig. 3.1. Here we note that as $\beta \to \infty$ then percentile life has a minimum value asymptotically.

Fig. 3.2 β vs B-25 life for α = 1000

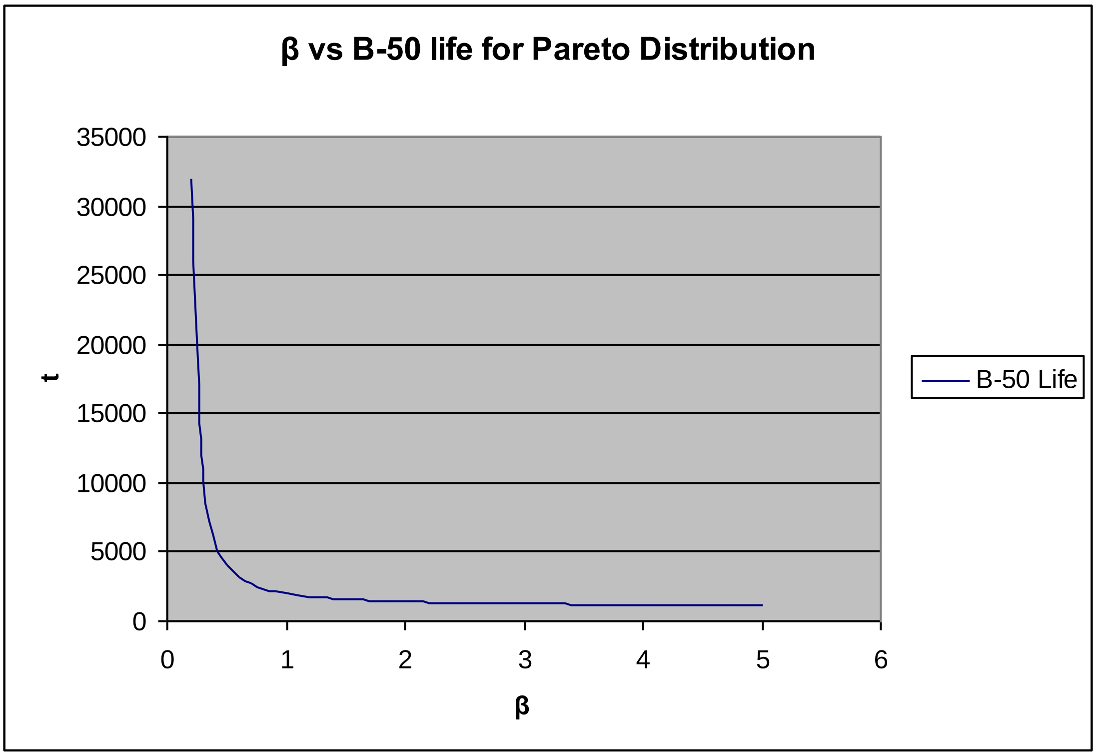


The first Quartle life (B-25 percentile) of the Pareto distribution is defined as

$$B_{25\,p} = \frac{\alpha}{\left(1-\frac{1}{4}\right)^{\frac{1}{\beta}}} \qquad (3.2)$$

For B-25 life mean this is the life for which the unit will have a failure probability of 25%. In fig 3.2 the shape parameter is taken along x-axis and its percentile life is taken along y-axis. In this percentile life the maximum life is 17757.73 for $\beta = 0.1$ and the minimum value of median life is 1059.224 for β =5. The relationship between β and B-25 life for α = 1000 is shown in Fig. 3.2. Here we note that as $\beta \to \infty$ then B-25 life has a minimum value asymptotically.

Fig. 3.3 β vs B-50 life for α = 1000

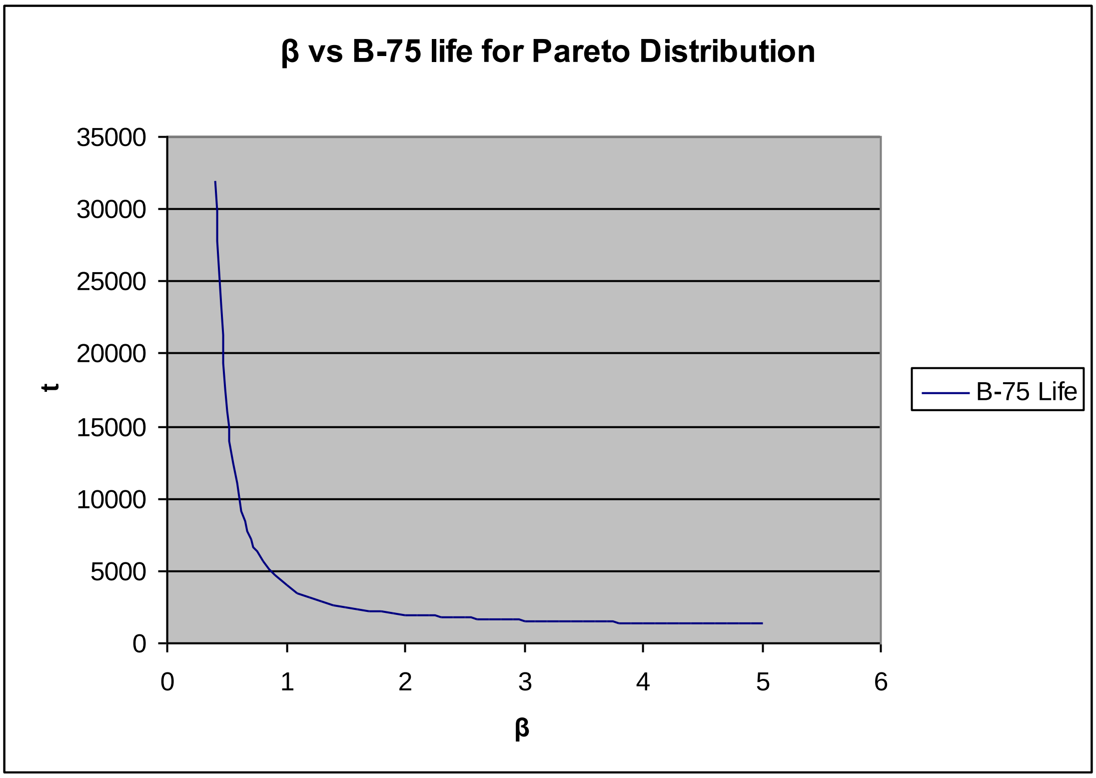


The second Quartle life (50^{th} percentile) of the Pareto distribution is defined as

$$B_{50\,P} = \frac{\alpha}{\left(1-\frac{1}{2}\right)^{\frac{1}{\beta}}} \qquad (3.3)$$

This is the life by which 50% of the units will be expected to have failed, and so it is the life at which 50% of the units would be expected to still survive. We obtain the maximum value of median life is 1024000 for β =0.1 and the minimum value of median life is 1148.698 for β =5. The relationship between β and B-50 life for α = 1000 is shown in Fig. 3.3. Here we note that as $\beta \to \infty$ then B-50 life has a minimum value asymptotically.

Fig. 3.4 β vs B-75 life for α = 1000



The upper Quartile life (75^{th} percentile) of the Pareto distribution is defined as

$$B_{75P} = \frac{\alpha}{\left(1-\frac{3}{4}\right)^{\frac{1}{\beta}}} \qquad (3.4)$$

This is the life by which 75% of the units will be expected to have failed, and so it is the life at which 25% of the units would be expected to still survive. We obtain the maximum value of upper quartile life is 1048576000 for β =0.1 and the minimum value of upper quartile life is 1319.50791for β =5. The relationship between β and B-75 life for α = 1000 is shown in Fig. 3.4. Here we note that as $\beta \to \infty$ then B-75 life has a minimum value asymptotically.

4. Summary and conclusions

The quantile comprehensive study of the Pareto quantile modeling is predicted for finding the life time of the wealth distribution study. These properties of the Pareto distribution for Quantile analysis are used for measuring fatigue data. These patterns of β and various B-lives are helpful for finding the life of Socio-economic problems.

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Appendix

Table A. Relationship Between β vs B-lives for Pareto Distribution

β	B-0.01	B-0.1	B-1	B-2	B-3	B-25	B-50	B-75
0.1	1010.055	1105.727	2867.972	9313.226	35401.33	17757.73	1024000	1048576000
0.2	1005.015	1051.536	1693.509	3051.758	5949.902	4213.992	32000	1024000
0.3	1003.341	1034.069	1420.774	2103.94	3283.522	2608.93	10079.37	101593.667
0.4	1002.504	1025.444	1301.349	1746.928	2439.242	2052.801	5656.854	32000
0.5	1002.003	1020.304	1234.568	1562.5	2040.816	1777.778	4000	16000
0.6	1001.669	1016.892	1191.962	1450.497	1812.049	1615.218	3174.802	10079.3684
0.7	1001.43	1014.461	1162.433	1375.444	1664.518	1508.287	2691.8	7245.78931
0.8	1001.251	1012.642	1140.767	1321.714	1561.807	1432.76	2378.414	5656.85425
0.9	1001.112	1011.23	1124.195	1281.38	1486.323	1376.641	2160.119	4666.11616
1	1001.001	1010.101	1111.111	1250	1428.571	1333.333	2000	4000
1.1	1000.91	1009.179	1100.519	1224.898	1382.993	1298.915	1877.862	3526.36502
1.2	1000.834	1008.41	1091.77	1204.366	1346.124	1270.912	1781.797	3174.8021
1.3	1000.77	1007.761	1084.421	1187.261	1315.695	1247.69	1704.361	2904.84571
1.4	1000.715	1007.205	1078.162	1172.793	1290.162	1228.123	1640.671	2691.80039
1.5	1000.667	1006.723	1072.766	1160.397	1268.434	1211.414	1587.401	2519.8421
1.6	1000.626	1006.301	1068.067	1149.658	1249.723	1196.979	1542.211	2378.41423
1.7	1000.589	1005.929	1063.938	1140.265	1233.442	1184.386	1503.407	2260.23157
1.8	1000.556	1005.599	1060.281	1131.98	1219.149	1173.304	1469.734	2160.11948
1.9	1000.527	1005.304	1057.019	1124.619	1206.5	1163.475	1440.247	2074.31009
2	1000.5	1005.038	1054.093	1118.034	1195.229	1154.701	1414.214	2000
2.1	1000.477	1004.797	1051.452	1112.11	1185.121	1146.818	1391.066	1935.06356
2.2	1000.455	1004.579	1049.056	1106.751	1176.007	1139.699	1370.351	1877.86182
2.3	1000.435	1004.379	1046.874	1101.881	1167.747	1133.238	1351.707	1827.11218
2.4	1000.417	1004.196	1044.878	1097.436	1160.226	1127.348	1334.84	1781.79744

2.5	1000.4	1004.028	1043.045	1093.362	1153.349	1121.955	1319.508	1741.10113
2.6	1000.385	1003.873	1041.356	1089.615	1147.038	1117	1305.512	1704.36079
2.7	1000.371	1003.729	1039.794	1086.157	1141.225	1112.432	1292.685	1671.0336
2.8	1000.357	1003.596	1038.346	1082.956	1135.853	1108.207	1280.887	1640.67071
2.9	1000.345	1003.472	1036.999	1079.984	1130.875	1104.288	1269.999	1612.89796
3	1000.334	1003.356	1035.744	1077.217	1126.248	1100.642	1259.921	1587.40105
3.1	1000.323	1003.247	1034.571	1074.636	1121.937	1097.243	1250.566	1563.91412
3.2	1000.313	1003.146	1033.473	1072.221	1117.91	1094.066	1241.858	1542.21083
3.3	1000.303	1003.05	1032.443	1069.958	1114.141	1091.089	1233.733	1522.09732
3.4	1000.294	1002.96	1031.474	1067.832	1110.604	1088.295	1226.135	1503.40665
3.5	1000.286	1002.876	1030.561	1065.832	1107.281	1085.667	1219.014	1485.99429
3.6	1000.278	1002.796	1029.699	1063.946	1104.151	1083.191	1212.326	1469.73449
3.7	1000.27	1002.72	1028.885	1062.165	1101.198	1080.854	1206.034	1454.51739
3.8	1000.263	1002.648	1028.114	1060.48	1098.408	1078.645	1200.103	1440.24654
3.9	1000.257	1002.58	1027.384	1058.885	1095.768	1076.553	1194.503	1426.83698
4	1000.25	1002.516	1026.69	1057.371	1093.265	1074.57	1189.207	1414.21356
4.1	1000.244	1002.454	1026.031	1055.934	1090.89	1072.687	1184.192	1402.30958
4.2	1000.238	1002.396	1025.403	1054.566	1088.633	1070.896	1179.434	1391.06562
4.3	1000.233	1002.34	1024.805	1053.264	1086.485	1069.192	1174.916	1380.42861
4.4	1000.227	1002.287	1024.235	1052.022	1084.439	1067.567	1170.62	1370.35098
4.5	1000.222	1002.236	1023.69	1050.837	1082.487	1066.017	1166.529	1360.79
4.6	1000.218	1002.187	1023.169	1049.705	1080.623	1064.537	1162.629	1351.70714
4.7	1000.213	1002.141	1022.67	1048.622	1078.842	1063.121	1158.908	1343.06759
4.8	1000.208	1002.096	1022.193	1047.586	1077.138	1061.766	1155.353	1334.83985
4.9	1000.204	1002.053	1021.735	1046.592	1075.506	1060.468	1151.953	1326.9953
5	1000.2	1002.012	1021.296	1045.64	1073.941	1059.224	1148.698	1319.50791