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Research Article

Mass Spectrum and Decay Constants of Heavy Quarkonia

Tasawer Shahzad Ahmad[1*], Talab Hussain[1], and M. Atif Sultan[1]

[1]Centre for High Energy Physics, University of the Punjab, Lahore, Pakistan

Abstract: In this paper, a non-relativistic potential model is used to find the solution of radial Schrodinger wave equation by using Crank Nicolson discretization for heavy quarkonia ($c\bar{c}$,$b\bar{b}$). After solving the Schrodinger radial wave equation, the mass spectrum and hyperfine splitting of heavy quarkonia are calculated with and without relativistic corrections. The root means square radii and decay constants for S and P states of $c\bar{c}$ and $b\bar{b}$ mesons by using the realistic and simple harmonic oscillator wave functions. The calculated results of mass, hyperfine splitting, root means square radii and decay constants agreed with experimental and theoretically calculated results in the literature.

Keywords: Potential Model, Spectroscopy, Decay Constant, Schrodinger Wave Equation, Charmonium, Bottomonium

1. INTRODUCTION

Heavy Quarkonia, the bound states of heavy quark and anti-quark, are the excellent tools for understanding the properties of the strong interaction [1]. Important aspects of the nature of strong interaction/Quantum Chromodynamics (QCD) can be studied through threshold physics for quarkonium systems. The thresholds for the charmonium and bottomunium systems are approximately 3.71 and 10.50 *GeV*, respectively [2]. Many charmonium and bottomonium states below thresholds have been experimentally observed and theoretically studied through various relativistic and non-relativistic potential models [3-19] and approaches like chiral perturbation theory [20], heavy quark effective field theory, lattice QCD [21-31], QCD sum rules [32-38], NRQCD [39-41], and dynamical equations based approaches like Bethe Salpeter and Dyson-Schwinger equations [41-44]. The heavy quarkonia were observed so far, still have many puzzles [17].

The potential between quark and anti-quark cannot be explained through the first principle of QCD, and therefore there is no dominant potential which interprets the quarkonium spectrum exactly [2]. Potential models can be employed to explore the gross features of quarkonia. Much more theoretical efforts have been made to explain the characteristics of quarkonia. More characteristics can be studied by employing the effects of QCD corrections, coupled channels effects, etc. Though many potential models existed for studying relativistic [45, 46] and non-relativistic effects [47-49]. The non-relativistic potential approach was used in this work because it is fully supported by the lattice QCD and experimental results. The non-relativistic potentials have quite successfully described the spectrum of heavy flavor sector. However, agreements of the mass spectrum with experimental results do not ensure the validity of the models. A model can be better validated through a comparison of its observables with those yielded by other approaches (theoretical/experimental). In this regard, observables including decay constants, leptonic decay width, radiative decay width, etc. are very important. Decay constants are significant because they can be used to provide the information about the non-perturbative QCD dynamics, the short distance structure of the hadrons, estimation of the hadronic matrix elements. The root mean square (rms) radii can be used to find out the form factors along with magnetic polarizabilities and energy shifts for quarkonia [7, 12]. The precise knowledge of these properties is vital in various experimental measurements of quarkonia.

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*Corresponding Author: Tasawer Shahzad Ahmad <tasawershahzadahmad@gmail.com>

In the last decade, many renowned experiments such as BABAR, BELLE, CLEOIII, ATLAS, CMS, LHCb and BESIII have produced interesting results on Heavy Quarkonia. There are 37 and 21 experimentally discovered states of charmonium and bottomoniam system respectively. The so-called X, Y and Z states above the $D\bar{D}$ (in case of charmonium) and $B\bar{B}$ (in case of bottomonium) thresholds could be considered as exotic states, multi-quarks or hybrid states and mesonic molecules due to their unusual properties [2].

In this paper, the non-relativistic potential model provides realistic solution of the radial Schrodinger wave equation through Crank Nicolson discretization approach. The solution of Schrodinger wave equation provides radial wave function at origin which is used to calculate rms radii and decay constants of radially excited S and P states of conventional quarkonia. The mass spectra were computed without and with relativistic corrections by using the leading order perturbations of heavy quarkonia. The computed results are found to be quite consistent with the experimental and particle data group (PDG) averages [50].

2. METHODOLOGY

The non-relativistic potential model of a conventional quarkonium is written as[7].

$$\boldsymbol{V_{q\bar{q}}(r)} = -\frac{4\alpha_s}{3r} + \boldsymbol{br} + \frac{32\pi\alpha_s}{9m_q^2}\left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2}\boldsymbol{S_q.S_{\bar{q}}} + \boldsymbol{V_{spin-dep}}, \text{ --------} \quad (1)$$

where $\boldsymbol{S_q.S_{\bar{q}}} = \frac{s(s+1)}{2} - \frac{3}{4}$ and $\boldsymbol{m_q}$ is the mass of the quark/anti-quark. In equation (1) the first term is due to one gluon exchange with the quark-gluon coupling constant α_s, the second term is the linear confining potential with string tension b, the third term is the Gaussian-smeared hyperfine interaction. The spin-dependent term in Eq. (1) is

$$V_{spin-dep} = \frac{1}{m_q^2}\left[\left(\frac{2\alpha_s}{r^3} - \frac{b}{r}\right)L.S + \frac{4\alpha_s}{r^3}T\right], \quad (2)$$

with

$$L.S = \frac{J(J+)-L(L+1)-S(S+1)}{2} \text{ --------------} \quad (3)$$

and

$$< L_J|T|^3L_J > = \begin{cases} \frac{-L}{6(2L+3)} & J = L+1 \\ \frac{1}{6} & J = L \quad \text{------------} \\ \frac{-L(L+1)}{6(2L-1)} & J = L-1 \end{cases} \quad (4)$$

Here S and L are spin and relative orbital angular momentum of the quarkonium respectively. For the angular momentum L = 0, the spin-orbit term and the tensor term both are equal to be zero. The parameters used in this potential model for the charmonia are α_s = 0.5461, b = 0.1425 GeV^2, m_c = 1.4794 GeV, σ = 1.096 GeV [17] and for botomonia α_s = 0.36, b = 0.1340 GeV^2, m_b = 4.825 GeV and σ = 1.34 GeV [10]. The radial form of Schrodinger wave equation is

$$-\frac{1}{2\mu}\frac{d^2}{dr^2} + \left(V_{q\bar{q}}(r) + \frac{L(L+1)}{2\mu r^2}\right)U(r) = EU(r) \quad (5)$$

Where $U(r) = rR(r)$ and μ is the reduced mass of the quark and anti-quark. Actual calculations requires regularizing the limit $U(r \to 0)$ because the resultant wave function becomes unstable in the limit $r \to 0$. When we include spin dependent terms in the potential which are $\propto 1/r^3$; ignoring these terms the $r \to 0$ limit of U(r) is easily manageable $\propto 1/r^{l+1}$. The required regularization can be achieved by replacing $1/r^3$ with $1/(r + r_c)^3$, with r_c being a finite constant and [14] did a similar regularization. The cutoff distance r_c, thus introduced, is determined by fitting it to the resulting mass of the $\chi_{(c/b)0}(1^3P_0)$. The fit parameter gave us $r_c = 0.034\,fm$ for both ($\boldsymbol{c\bar{c}, b\bar{b}}$). The resulting wave functions are used to calculate the decay constant of the S and P states. In these calculations, the decay constants found by using the realistic wave function and the fitted SHO. The rms radii and mass spectra are also calculated. In this work, the SHO wave function is fitted to the quark model wave function to obtain the SHO parameter (β)[15].

2.1 Solving Schrodinger's Equation with Crank Nicolson Discretization

The discretized time independent Schrodinger wave equation in the form of a matrix equation is

$$\sum_i H_{ij}U_j = E\sum_i U_i \quad (6)$$

where $Uj = U(r_j)$ and the Hamiltonian matrix has the elements

$$H_{ij} = -\frac{1}{2}\frac{\{\delta_{i+1,\ j}-2\delta_{i,j}+\delta_{i-1,j}\}}{\Delta^2} + V_{q\bar{q}}(r_i)\delta_{i,j} \quad (7)$$

After getting the H matrix, the energy eigenvalues were found for this matrix. Then the mass spectra were obtained after adding the rest mass energy of the quark and anti-quark in the energy eigenvalue.

Above, the computed mass spectrum and state functions of quarkonium states by using the effective quark anti-quark potential for which the solution of Schrodinger equation did not encompass the significant relativistic effects. To study these relativistic corrections in mass spectrum of quarkonia, the lowest order relativistic corrections were applied to the Hamiltonian [7].

$$\Delta H_{rel} = \frac{-p^4}{4m_q^3} \quad (12)$$

2.2 Decay Constants

The decay constant is an essential physical property which depends on the wave function at origin. The Van Royen-Weisskopf relation between decay constant and wave function at origin, used to compute the decay constant [1, 9, 51] is given as

$$f_{V/P}^2 = \frac{3|R_{ns\,V/P}(0)|^2\overline{\bar{C}^2}\,(\alpha_s)}{\pi M_{V/P}} \quad (8)$$

where $M_{V/P}$ and $R_{nsV/P}(0)$ are the values of the mass and the wave function at origin for the spin (1) vector and spin (0) pseudoscalar mesons respectively. The QCD correction used for quarkonia is

$$\bar{\bar{C}}\,(\alpha_s) = 1 - \frac{\alpha_s}{\pi}\delta_{V/P} \quad (9)$$

where $\delta_V = \frac{8}{3}$ and $\delta_P = 2$ are used for vector and pseudoscalar quarkonia respectively.

The decay constants of P wave quarkonia can be calculated by using the following relations [52] (which depend on the values of the first derivative of the wave function at origin)

$$f_{\chi_0} = \sqrt{\frac{27|R'(0)^2|}{2\pi m_q^3}} \quad (10)$$

and

$$f_{\chi_1} = \sqrt{\frac{18|R'(0)^2|}{\pi m_q^3}} \quad (11)$$

where m_q is the mass of the corresponding quark. To calculate the rms radii, the normalized wave function is used in the following formula

$$\sqrt{< r^2 >} = \sqrt{\int U^*(r) r^2 U(r) dr} \quad (12)$$

3. RESULTS AND DISCUSSION

In this work, the computed mass spectrum results of charmonia and bottomonia with and without relativistic corrections and their comparison with experimental and theoretical results are reported in the Tables 1 and 2 respectively.

In case of $c\bar{c}$ mass spectra, the maximum of 0.79% and 0.88% mass differences between calculated masses without relativistic corrections and with relativistic corrections are found from the experimentally observed masses respectively. In case of $b\bar{b}$ mass spectra, the maximum differences found between computed masses and experimental masses are 1.16% and 1.30% with and without the relativistic corrections respectively.

The calculated S wave hyperfine splitting without and with relativistic corrections and their comparison with experimental and theoretically calculated results are tabulated in Tables 3 and 4. The hyperfine mass splitting values for (1S, 2S) and (1P, 2P) states of charmonia have very good agreement with the experimental results [50]. The mass splitting of bottomonia are found to be less than the experimental values [50] but these are consistent with the results reported by [12].

The calculated decay constants for pseudoscalar meson (f_P), vector meson (f_V) with QCD corrections (f'_P), (f'_V) for charmonia and bottomonia through Van Royen-Weisskopf formulae by using the realistic and SHO wave functions for nS ($n \leq 6$) and nP ($n \leq 5$) states, are listed in Tables 5, 6, 7, 8,9 and 10. In case of $c\bar{c}$, the difference in f_V for 1S states calculated by realistic wave function and SHO wave function are (17, 40) *MeV*, and for 2S states are (0.68, 14) *MeV* from the experimentally calculated f_V [50]. In case of higher excited states, the computed f'_V values with QCD corrections are found to be in good agreement with experimental results. For the n^3P_0 state the decay constants for charmonia calculated by realistic wave function are very large as compared to those using the SHO wave function and [53]. The calculated 1S states f_P by realistic and SHO wave function can be compared with only existing experimental pseudoscalar decay constants, and found to be in good agreement. In case of $b\bar{b}$, the calculated f_V for 2S, 3S, 4S and f'_V for 5S, 6S states by realistic wave functions are found to be in good agreement with the available experimental values. The decay constants for the P states for $c\bar{c}$ and $b\bar{b}$ calculated by realistic wave function increases towards higher radial excitation which agreed with [53]. The P state decay constants of bottomonia could not compare due to the unavailability of theoretical and experimental results. The computed results of rms radii for charmonia and bottomonia are listed in Table 1 and 2 respectively.

4. CONCLUSION

In this work, a non-relativistic potential model is used to find the realistic solution of the radial Schrodinger wave equation is used to find the mass spectrum and decay constants for nS ($n \leq 6$) and nP ($n \leq 5$) states of heavy quarkonia, hyperfine mass splitting, rms radii, and radial wave functions at origin. It is concluded that the computed mass values without relativistic corrections have less mass difference from the experimental values as compared to computed masses with relativistic corrections. The decay constants of P states of bottomonia are calculated first time in this work. It is noted that the decay constants for the S and P states for $c\bar{c}$ and $b\bar{b}$ calculated by realistic wave function increased towards higher radial excitation. It is also noted that the rms radii increased with radial and angular excited states. Decay constants and radii can be important to understand various important properties such as the short distance structure of hadrons, non perturbative QCD dynamics, energy shifts, magnetic polarizability and form factor.

5. CONFLICT OF INTEREST

The author declared no conflict of interest.

Table 1. Masses (in *GeV*) of ground and radially excited state charmonium mesons with and without relativistic corrections

Mesons	Our Work Without relativistic (*GeV*)	Our Work With relativistic (*GeV*)	[50] (*GeV*)	[7] Without relativistic (*GeV*)	[8] With relativistic (*GeV*)	rms radii (*fm*)
$\eta_c(1^1S_0)$	2.983	2.995	2.9839 ± 0.00005	2.981	2.992	0.366
$\psi(1^3S_1)$	3.091	3.096	3.0969 ± 0.000006	3.090	3.094	0.415
$\eta_c(2^1S_0)$	3.630	3.627	3.6375 ± 0.0012	3.630	3.625	0.833
$\psi(2^3S_1)$	3.672	3.669	3.6861 ± 0.000025	3.672	3.388	0.863
$\eta_c(3^1S_0)$	4.043	4.031		4.043	4.029	1.207
$\psi(3^3S_1)$	4.071	4.060	4.039 ± 0.001	4.071	4.059	1.229
$\eta_c(4^1S_0)$	4.383	4.357		4.384	4.059	1.531
$\psi(4^3S_1)$	4.406	4.382	4.421 ± 0.004	4.406	4.356	1.548
$\eta_c(5^1S_0)$	4.684	4.640		4.685	4.639	1.823
$\psi(5^3S_1)$	4.703	4.662		4.703	4.661	1.837
$\eta_c(6^1S_0)$	4.960	4.894		4.960	4.893	2.092
$\psi(6^3S_1)$	4.976	4.913		4.977	4.912	2.105
$h_c(1^1P_1)$	3.515	3.544	3.5254 ± 0.00011	3.516	3.544	0.674
$\chi_{c0}(1^3P_0)$	3.417	3.414	3.4147 ± 0.0003	3.423	3.472	0.568
$\chi_{c1}(1^3P_1)$	3.514	3.543	3.51067 ± 0.00005	3.501	3.543	0.668
$\chi_{c2}(1^3P_2)$	3.540	3.576	3.5562 ± 0.00007	3.549	3.586	0.707
$h_c(2^1P_1)$	3.933	3.951		3.933	3.951	1.069
$\chi_{c0}(2^3P_0)$	3.856	3.818		3.872	3.885	0.983
$\chi_{c1}(2^3P_1)$	3.933	3.949		3.920	3.951	1.067

Mesons	Our Work Without relativistic (*GeV*)	Our Work With relativistic (*GeV*)	[50] (*GeV*)	[7] Without relativistic (*GeV*)	[8] With relativistic (*GeV*)	rms radii (*fm*)
$h_c(3^1P_1)$	4.279	4.283		4.279	4.283	1.405
$\chi_{c0}(3^3P_0)$	4.210	3.149		4.230	4.219	1.328
$\chi_{c1}(3^3P_1)$	4.279	4.279		4.266	4.283	1.403
$\chi_{c2}(3^3P_2)$	4300	.319		4.309	4.328	1.433
$h_c(4^1P_1)$	4.584	4.571		4.585	4.571	1.705
$\chi_{c0}(4^3P_0)$	4.521	4.436		4.542	4.508	1.634
$\chi_{c1}(4^3P_1)$	4.585	4.566		4.572	4.570	1.703
$\chi_{c2}(4^3P_2)$	4.605	4.608		4.614	4.617	1.731
$h_c(5^1P_1)$	4.863	4.830		4.864	4.830	1.981
$\chi_{c0}(5^3P_0)$	4.804	4.694		4.826	4.766	1.914
$\chi_{c1}(5^3P_1)$	4.864	4.823		4.851	4.828	1.979
$\chi_{c2}(5^3P_2)$	4.883	4.867		4.893	4.876	2.006

Table 2. Masses (in *GeV*) of ground and radially excited state bottomonium mesons with and without relativistic corrections

Mesons	Our Work Without relativistic (*GeV*)	Our Work With relativistic (*GeV*)	[50] (*GeV*)	[12] Without relativistic (*GeV*)	[16] With relativistic (*GeV*)	rms radii (*fm*)
$\eta_b(1^1S_0)$	9.509	9.413	9.3987 ± 0.002	9.483	9.399	0.227
$\Upsilon(1^3S_1)$	9.531	9.433	9.4603 ± 0.00026	9.501	9.470	0.233
$\eta_b(2^1S_0)$	10.004	10.129	9.999 ± 0.0035	9.995	9.986	0.541
$\Upsilon(2^3S_1)$	10.010	10.094	10.0233 ± 0.00031	10.009	10.033	0.545
$\eta_b(3^1S_0)$	10.291	10.440		10.290	10.315	0.802
$\Upsilon(3^3S_1)$	10.295	10.444	10.3552 ± 0.0005	10.302	10.352	0.805
$\eta_b(4^1S_0)$	10.522	10.656		10.536	10.583	1.027
$\Upsilon(4^3S_1)$	10.525	10.658	10.5794 ± 0.0012	10.548	10.615	1.029
$\eta_b(5^1S_0)$	10.723	10.844		10.755	10.816	1.230
$\Upsilon(5^3S_1)$	10.725	10.847		10.765	10.845	1.232
$\eta_b(6^1S_0)$	10.905	11.016		10.953	11.024	1.417
$\Upsilon(6^3S_1)$	10.907	11.018		10.962	11.052	1.419
$h_b(1^1P_1)$	9.928	9.853	9.8993 ± 0.0001	9.852	9.864	0.711
$\chi_{b0}(1^3P_0)$	9.914	9.859	$9.8594 \pm 0.0008 \pm 0.00031$	9.837	9.837	0.701
$\chi_{b1}(1^3P_1)$	9.928	9.861	$9.8928 \pm 0.00026 \pm 0.00031$	9.807	9.85	0.711
$\chi_{b2}(1^3P_2)$	9.935	9.956	$9.9122 \pm 0.00026 \pm 0.00031$	9.823	9.877	0.717
$h_b(2^1P_1)$	10.221	10.376	$10.2598 \pm 0.0005 \pm 0.0011$	10.267	10.98	0.945
$\chi_{b0}(2^3P_0)$	10.212	10.368	$10.2325 \pm 0.0004 \pm 0.0005$	10.247	10.258	0.936
$\chi_{b1}(2^3P_1)$	10.221	10.376	$10.2555 \pm 0.00022 \pm 0.00050$	10.206	10.279	0.945
$\chi_{b2}(2^3P_2)$	10.227	10.382	$10.2686 \pm 0.00022 \pm 0.00050$	10.227	10.317	0.950
$h_b(3^1P_1)$	10.456	10.596		10.512	10.555	1.154
$\chi_{b0}(3^3P_0)$	10.448	10.589		10.486	10.503	1.145
$\chi_{b0}(3^3P_1)$	10.456	10.596	10.513 ± 0.00041	10.433	10.529	1.153
$\chi_{b0}(3^3P_2)$	10.461	10.601	10.5240 ± 0.00057	10.460	10.580	1.159
$h_b(4^1P_1)$	10.660	10.789		10.734	10.785	1.345

Mesons	Our Work Without relativistic (*GeV*)	Our Work With relativistic (*GeV*)	[50] (*GeV*)	[12] Without relativistic (*GeV*)	[16] With relativistic (*GeV*)	rms radii (*fm*)
$\chi_{b0}(4^3P_0)$	10.653	10.782		10.704	10.727	1.337
$\chi_{b0}(4^3P_1)$	10.661	10.788		10.644	10.756	1.345
$\chi_{b0}(4^3P_2)$	10.665	10.793		10.672	10.814	1.349
$h_b(5^1P_1)$	10.846	10.963		10.935	10.994	1.524
$\chi_{b0}(5^3P_0)$	10.839	10.957		10.903	10.930	1.516
$\chi_{b0}(5^3P_1)$	10.846	10.963		10.837	10.962	1.523
$\chi_{b0}(5^3P_2)$	10.850	10.967		10.871	11.026	1.528

Table 3. The hyperfine splitting (in *MeV*) for the S and P states charmonia

Hyperfine Splitting	Our Work Without Relativistic (*MeV*)	Our Work Without Relativistic (*MeV*)	[50] (*MeV*)	[7] Without Relativistic (*MeV*)	[7] With Relativistic (*MeV*)	[54] (*MeV*)
$M\,(1^3S_1) - M\,(1^1S_0)$	113	101	113 ± 0.7	102	108	108
$M\,(2^3S_1) - M\,(2^1S_0)$	44	42	46.7 ± 0.2	22	42	42
$M\,(3^3S_1) - M\,(3^1S_0)$	30	29	–	30	29	29
$M\,(4^3S_1) - M\,(4^1S_0)$	24	25	–	24	22	22
$M\,(5^3S_1) - M\,(5^1S_0)$	20	2	–	22	19	–
$M\,(6^3S_1) - M\,(6^1S_0)$	18	19	–	19	17	–
$M\,(1^3P_2) - M\,(1^3P_1)$	26.42	32	45.5 ± 0.2	41	44	51
$M\,(1^3P_1) - M\,(1^3P_0)$	96.86	129	95.5 ± 0.4	71	80	81
$M\,(2^3P_2) - M\,(2^3P_1)$	22.63	37	–	40	40	47
$M\,(2^3P_1) - M\,(2^3P_0)$	77.75	131	–	66	73	73
$M\,(3^3P_2) - M\,(3^3P_1)$	21.1	40	–	45	38	46
$M\,(3^3P_1) - M\,(3^3P_0)$	68.95	130	–	63	69	69

Table 4. The hyperfine splitting (in *MeV*) for the S and P states bottomonia

Hyperfine Splitting	Our Work Without Relativistic (*MeV*)	Our Work Without Relativistic (*MeV*)	[50] (*MeV*)	[8] Without Relativistic (*MeV*)	[8] With Relativistic (*MeV*)
$M\,(1^3S_1) - M\,(1^1S_0)$	21.95	20	62	18	71
$M\,(2^3S_1) - M\,(2^1S_0)$	6	35	24	14	47
$M\,(3^3S_1) - M\,(3^1S_0)$	3.8	4	-	12	37
$M\,(4^3S_1) - M\,(4^1S_0)$	3	2	-	12	32
$M\,(5^3S_1) - M\,(5^1S_0)$	2.4	3	-	10	29
$M\,(6^3S_1) - M\,(6^1S_0)$	2.1	2	-	9	28

Hyperfine Splitting	Our Work Without Relativistic (*MeV*)	Our Work Without Relativistic (*MeV*)	[50] (*MeV*)	[8] Without Relativistic (*MeV*)	[8] With Relativistic (*MeV*)
$M\ (1^3P_2)\ -\ M\ (1^3P_1)$	13.26	95	34	16	25
$M\ (1^3P_1)\ -\ M\ (1^3P_0)$	7.5	2	19	30	15
$M\ (2^3P_2)\ -\ M\ (2^3P_1)$	9.6	6	22	21	38
$M\ (2^3P_1)\ -\ M\ (2^3P_0)$	5.6	8	14	41	21
$M\ (3^3P_2)\ -\ M\ (3^3P_1)$	8.1	5	-	27	51
$M\ (3^3P_1)\ -\ M\ (3^3P_0)$	4.7	5	-	53	26

Table 5. Decay constant of vector mesons of the charmonia

Decay Constant	Our Work Realistic (*MeV*)	Our Work SHO (*MeV*)	[50] (*MeV*)	[9] (*MeV*)	[1] (*MeV*)	$\|R_V(0)\|^2$ Realistic $(GeV)^3$	$\|R_V(0)\|^2$ SHO $(GeV)^3$	β_{SHO}
$\psi(1^3S_1)$								
f_V	433.237	376.178	416 ± 6	418.639	325.876	0.609	0.459	0.588
f'_V	232.412	201.803	–	291.717				
$\psi(2^3S_1)$								
f_V	304.678	318.347	304 ± 4	275.845	257.340	0.358	0.386	0.485
f'_V	163.446	170.779		192.215				
$\psi(3^3S_1)$								
f_V	265.45	278.069	187 ± 8	226.130	229.857	0.298	0.327	0.426
f'_V	142.402	149.172		160.562				
$\psi(4^3S_1)$								
f_V	241.212	252.64	161 ± 10	205.404	212.959	0.269	0.295	0.391
f'_V	129.399	135.53		143.179				
$\psi(5^3S_1)$								
f_V	225.779	237.852		188.852	200.848	0.251	0.279	0.369
f'_V	121.121	127.597		131.596				
$\psi(6^3S_1)$								
f_V	213.972	225.966		176.626	191.459	0.239	0.266	0.352
f'_V	114.786	121.221		123.077				

Table 6. Decay constant of pseudoscalar mesons of the charmonia

Decay Constant	Our Work Realistic (*MeV*)	Ourt Work SHO (*MeV*)	[50] (*MeV*)	[9] (*MeV*)	[53] (*MeV*)	$\|R_P(0)\|^2$ Realistic $(GeV)^3$	$\|R_P(0)\|^2$ SHO $(GeV)^3$	β_{SHO}
$\eta_c(1^1S_0)$								
f_P	560.531	468.54	335 ± 75	402.058	519	0.980	0.678	0.672
f'_P	365.66	305.648		310.637				
$\eta_c(2^1S_0)$								
f_P	337.154	351.38		268.926	318.1	0.433	0.471	0.518
f'_P	219.94	229.22		207.777				
$\eta_c(3^1S_0)$								
f_P	283.244	292.595		226.130	267.5	0.339	0.363	0.441
f'_P	184.772	190.872		174.712				

Decay Constant	Our Work Realistic (*MeV*)	Ourt Work SHO (*MeV*)	[50] (*MeV*)	[9] (*MeV*)	[53] (*MeV*)	$\lvert R_P(0)\rvert^2$ Realistic *(GeV)³*	$\lvert R_P(0)\rvert^2$ SHO *(GeV)³*	β_{SHO}
$\eta_c(4^1S_0)$								
f_P	254.869	263.17		202.432	240.8	0.298	0.318	0.401
f'_P	166.243	171.0		156.403				
$\eta_c(5^1S_0)$								
f_P	236.22	244.196		186.537	223.2	0.273	0.293	0.375
f'_P	154.103	159.299		154.09				
$\eta_c(6^1S_0)$								
f_P	222.58	231.216		174.783	210.2	0.257	0.278	0.357
f'_P	145.19	150.832		135.040				

Table 7. Decay constant of vector mesons of the bottomonia

Decay Constant	Our Work Realistic *(MeV)*	Our Work SHO *(MeV)*	[50] *(MeV)*	[55] *(MeV)*	[51] *(MeV)*	$\lvert R_V(0)\rvert^2$ Realistic *(GeV)³*	$\lvert R_V(0)\rvert^2$ SHO *(GeV)³*	β_{SHO}
$\Upsilon(1^3S_1)$								
f_V	815	513.531	715 ± 5	831	847	6.587	2.612	1.05
f'_V	566	356.607	–	645	685			
$\Upsilon(2^3S_1)$								
f_V	513	405.597	498 ± 5	566	700	2.766	1.726	0.799
f'_V	356	281.656		439	566			
$\Upsilon(3^3S_1)$								
f_V	439	343.35	430 ± 4	507	641	2.0977	1.278	0.671
f'_V	305	238.43		393	518			
$\Upsilon(4^3S_1)$								
f_V	403	312.576	336 ± 18	481	605	1.808	1.082	0.603
f'_V	280	217.06		373	489			
$\Upsilon(5^3S_1)$								
f_V	379	292.783	269 ± 42	458	578	1.639	0.975	0.56
f'_V	263	203.315		356	468			
$\Upsilon(6^3S_1)$								
f_V	363	282.98	240 ± 28	439	558	1.526	0.914	0.531
f'_V	252	196.508		341	451			

Table 8. Decay constant of pseudoscalar mesons of the bottomonia

Decay Constant	Our Work Realistic (*MeV*)	Our Work SHO (*MeV*)	[50] (*MeV*)	[55] (*MeV*)	[51] (*MeV*)	$\lvert R_P(0)\rvert^2$ Realistic *(GeV)³*	$\lvert R_P(0)\rvert^2$ SHO *(GeV)³*	β_{SHO}
$\eta_c(1^1S_0)$								
f_P	850	541.945		834	824	7.115	2.89	1.086
f'_P	655	417.74		694	706			
$\eta_c(2^1S_0)$								
f_P	517	517		567	686	2.807	1.799	0.810
f'_P	399	319.503		472	587			
$\eta_c(3^1S_0)$								
f_P	434	348.266		508	629	2.116	1.307	0.676
f'_P	341	268.45		422	539			
$\eta_c(4^1S_0)$								
f_P	406	318.903		481	594	1.821	1.121	0.61
f'_P	313	245.816		401	509			

Decay Constant	Our Work Realistic (*MeV*)	Our Work SHO (*MeV*)	[50] (*MeV*)	[55] (*MeV*)	[51] (*MeV*)	$\|R_P(0)\|^2$ Realistic $(GeV)^3$	$\|R_P(0)\|^2$ SHO $(GeV)^3$	β_{SHO}
$\eta_c(5^1S_0)$								
f_P	383	296.299			569	1.650	0.985	0.562
f'_P	295	228.393			488			
$\eta_c(6^1S_0)$								
f_P	363	269.075			549	1.536	0.909	0.53
f'_P	282	207.408			471			

Table 9. Decay constant of P wave of the charmonia

Mesons	$\|R'(0)\|^2$ (SHO) $(GeV)^3$	$\|R'(0)\|^2$ (Realistic) $(GeV)^3$	Our Work (SHO) (*MeV*)	Our Work (Realistic) (*MeV*)	[53] (*MeV*)	β_{SHO}
$\chi_{c0}(1^3P_0)$	0.078	7.296	321.346	3111.7	754.7	0.553
$\chi_{c0}(1^3P_1)$	0.0337	0.197	244.48	591.4	788.3	0.468
$\chi_{c0}(2^3P_0)$	0.097	7.724	358.50	3139.0	841.5	0.481
$\chi_{c0}(2^3P_1)$	0.053	0.258	307.37	675.78	913.2	0.427
$\chi_{c0}(3^3P_0)$	0.096	7.448	356.28	3144.0	899.6	0.429
$\chi_{c0}(3^3P_1)$	0.063	0.300	332.55	728.67	994.9	0.394
$\chi_{c0}(4^3P_0)$	0.0974	2.745	359.47	3145.4	949.1	0.397
$\chi_{c0}(4^3P_1)$	0.070	0.334	352.79	769.18	1059.4	0.372
$\chi_{c0}(5^3P_0)$	0.099	2.745	363.09	3145.62	992.1	0.374
$\chi_{c0}(5^3P_1)$	0.077	0.364	368.03	802.743	1113.7	0.355

Table 10. Decay constant of P wave of the bottomonia

Mesons	$\|R'(0)\|^2$ (SHO) $(GeV)^3$	$\|R'(0)\|^2$ (Realistic) $(GeV)^3$	Our Work (SHO) (*MeV*)	Our Work (Realistic) (*MeV*)	β_{SHO}
$\chi_{b0}(1^3P_0)$	0.341	2.170	114.16	288.16	0.743
$\chi_{b0}(1^3P_1)$	0.295	1.216	122.703	249.01	0.722
$\chi_{b0}(2^3P_0)$	0.482	2.457	135.768	306.57	0.663
$\chi_{b0}(2^3P_1)$	0.433	1.441	148.626	271.10	0.649
$\chi_{b0}(3^3P_0)$	0.255	2.656	98.7889	318.78	0.522
$\chi_{b0}(3^3P_1)$	0.238	1.597	110.286	285.42	0.515
$\chi_{b0}(4^3P_0)$	0.315	2.825	109.733	328.73	0.502
$\chi_{b0}(4^3P_1)$	0.296	1.727	122.956	296.88	0.496
$\chi_{b0}(5^3P_0)$	0.383	2.97	121.121	337.35	0.49
$\chi_{b0}(5^3P_1)$	0.350	1.844	133.524	306.70	0.481

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