Proceedings of the Pakistan Academy of Sciences:
A: Physical and Computational Sciences 58(2): 11–18 (2021)

ISSN (Print): 2518-4245; ISSN (Online): 2518-4253
https://doi.org/10.53560/PPASA(58-2)585

Pakistan Academy of Sciences

Research Article

The Reve's Puzzle Revisited

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Abstract: The Reve's puzzle, introduced by the English puzzlist, H.E. Dudeney, is a mathematical puzzle with 10 discs of different sizes and four pegs, designated as S, P_1, P_2 and D. Initially, the n (≥ 1) discs rest on the source peg, S, in a tower (with the largest disc at the bottom and the smallest disc at the top). The objective is to move the tower from the peg S to the destination peg D, in a minimum number of moves, under the condition that each move can transfer only one disc from one peg to another such that no disc can ever be placed on top of a smaller one. This paper considers the solution of the dynamic programming equation corresponding to the Reve's puzzle.

Keywords: Classical Tower of Hanoi, Reve's Puzzle, Dynamic Programming, Recurrence Relation

1. INTRODUCTION

In 1883, the French number theorist, Lucas [1], introduced a mathematical puzzle, called the Tower of Hanoi. The puzzle, in its general form, is as follows: Given are three pegs, S, P and D, and n (≥ 1) discs of different sizes. Initially, the discs rest on the source peg, S, in a tower in small-on-large ordering (with the largest disc at the bottom, the second largest above it, and so on, and the smallest disc at the top). The problem is to transfer this tower from the peg S to the destination peg D, in a tower, in a minimum number of moves, under the following conditions:

1) only one (the topmost) disc can be transferred in each move,
2) ("divine" rule) no disc can ever be placed on top of a smaller one.

Denoting by $THP(n)$ the minimum number of moves required to solve the problem, it is well-known that $THP(n) = 2^n - 1$, $n \geq 1$.

An immediate generalization of the classical Tower of Hanoi problem is the Reve's (or Reeve's) puzzle, introduced by the English puzzlist, Dudeney [2], where in addition to the three pegs, there is a fourth one. For a detailed historical account of the classical Tower of Hanoi problem as well as the Reve's puzzle, we refer the reader to Hinz, Klavzar, and Petr [3].

Let $M(n)$ denote the minimum number of moves required to solve the Reve's puzzle with n (≥ 1) number of discs. Then, the Dynamic Programming Equation satisfied by $M(n)$ is

$$M(n) = \min_{1 \leq k \leq n-1} \left\{ 2M(k) + 2^{n-k} - 1 \right\}, \; n \geq 4, \tag{1.1a}$$

with

$$M(0) = 0, \tag{1.1b}$$

$$M(n) = 2n - 1 \text{ for all } 1 \leq n \leq 3. \tag{1.1c}$$

Recall that, to get the equation (1.1), the steps followed are as below:

Step 1: First, move the topmost k discs from the peg S to some auxiliary peg, say, P_1, using all the four pegs available, in (minimum) $M(k)$ moves.

Step 2: Next, shift the remaining $n-k$ discs (from the peg S) to the peg D, using the three pegs available, in $2^{n-k} - 1$ moves.

Step 3: Finally, move the k discs (from the peg P_1 to the peg D), in $M(k)$ moves.

The total number of moves involved in the scheme is $2M(k) + 2^{n-k} - 1$, where k is to be chosen so as to minimize the total number

Received: July 2020; Accepted: June 2021
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of moves. Thus, we get the recurrence relation (1.1).

So far as the author knows, the dynamic programming formulation of the Reve's puzzle, leading to the equation (1.1), first appeared in Roth [4], and later the same scheme was used by several researchers. However, the main objection to the dynamic programming formulation is that, it lacks the proof of the optimality of the scheme. That's why several researchers preferred to call $M(n)$ the presumed minimum solution (pms, for short). A proof of the optimality of the pms has been claimed by Bousche [5] (see also §5.5 in Hinz, Klavzar and Petr [3]), though back in 1994, the first proof of the optimality of the pms was put forward by Majumdar [6].

Even though the dynamic programming formulation of the Reve's puzzle was available, till 1994, no serious extensive study of the dynamic programming equation (1.1) appeared in literature, though several researchers found some results, mostly based on observation on numerical values of $M(n)$. Some properties of the optimal value function $M(n)$ appeared for the first time in Majumdar [7], as a particular case of the more general p-peg Tower of Hanoi problem with $p \geq 4$; and based on these properties, the complete solution of (1.1) was presented.

It is rather surprising that, though the classical Tower of Hanoi problem is very simple, adding just one more peg to it makes it so complicated.

This paper reinvestigates the Reve's puzzle by confining attention to the corresponding dynamic programming equation (1.1). We try to find the solution by investigating the equation (1.1) only. This is done in Section 3. In Section 2, we give some background material. We conclude the paper with some remarks in the final Section 4.

2. BACKGROUND MATERIAL

This section gives some known properties of $M(n)$, most of which appeared in Majumdar [7] as particular cases of the p-peg Tower of Hanoi problem. For the Reve's puzzle, some of the proofs are simpler, and we give here the simple proofs.

For $n \geq 4$ fixed, we define

$$F(n, k) = 2M(k) + 2^{n-k} - 1,\ 0 \leq k \leq n-1, \quad (2.1)$$

so that the equation (1.1) may be expressed as

$$M(n) = \min_{0 \leq k \leq n-1} \{F(n,k)\}. \quad (2.2)$$

It is well-known that, for some n, $M(n)$ is attained at more than one value of k. We thus define the optimal partition numbers, $k_{min}(n)$ and $k_{max}(n)$, as follows:

$$k_{min}(n) = min\ \{k : 0 \leq k \leq n-1,\ M(n) = F(n, k)\}, \quad (2.3a)$$

$$k_{max}(n) = max\ \{k : 0 \leq k \leq n-1,\ M(n) = F(n, k)\}, \quad (2.3b)$$

with

$$k_{min}(1) = 0 = k_{max}(1). \quad (2.3c)$$

Thus, for $n \geq 1$ fixed, $k_{min}(n)$ and $k_{max}(n)$ denote respectively the minimum and the maximum number of discs to be stored on some auxiliary peg under the pms. Note that, for $n \geq 1$ fixed, $M(n)$ is attained at a unique value of k if and only if $k_{min}(n) = k_{max}(n)$.

The following lemmas give some properties satisfied by $M(n)$, $k_{min}(n)$ and $k_{max}(n)$.

Lemma 2.1: For $n \geq 1$, $M(n+1) - M(n) \geq 2$.

Proof: Using (1.1c), it can easily be verified that the result is true for $n = 1$. So, we assume that the result is true for some n (which implies that the result is true for all $m \leq n$). To prove by induction, we have to show that the result is true for $n+1$ as well.

We first observe that, if $M(n+1)$ is attained at $k = n$, then it is also attained at $k = n-1$, for otherwise,

$$M(n+1) = 2M(n) + 1 < 2M(n-1) + 3,$$

so that

$$M(n) - M(n-1) < 1,$$

which contradicts the induction hypothesis.

Now, since, for all $0 \le k \le n-1$,

$$2M(k)+2^{n+1-k}-1 \ge [2M(k)+2^{n-k}-1]+2,$$

it follows that

$$\min_{1\le k\le n-1}\{2M(k)+2^{n+1-k}-1\} \ge \min_{1\le k\le n-1}\{2M(k)+2^{n-k}-1\}+2,$$

which shows that the result is true for $n+1$.

Corollary 2.1**:** For any $n\ge 3$, $F(n, k)$ cannot attain its minimum at $k=n-1$.

Lemma 2.2**:** For any $n\ge 1$,

$$k_{min}(n)\le k_{min}(n+1),\ k_{max}(n)\le k_{max}(n+1).$$

Proof**:** Since

$$M(n+1)=2M(k_{min}(n+1))+2^{n+1-k_{min}(n+1)}-1,$$

$$M(n)\le 2M(k_{min}(n+1))+2^{n-k_{min}(n+1)}-1,$$

it follows that

$$M(n+1)-M(n)\ge 2^{n-k_{min}(n+1)}. \qquad (2.1)$$

Again, since

$$M(n+1)\le 2M(k_{min}(n))+2^{n+1-k_{min}(n)}-1,$$

$$M(n)=2M(k_{min}(n))+2^{n-k_{min}(n)}-1,$$

we get

$$M(n+1)-M(n)\le 2^{n-k_{min}(n)}. \qquad (2.2)$$

Combining (2.1) and (2.2), we get

$$2^{n-k_{min}(n+1)} \le M(n+1)-M(n)\le 2^{n-k_{min}(n)},$$

which proves that $k_{min}(n) \le k_{min}(n+1)$. The proof of the remaining part is similar, and is omitted here.

Lemma 2.3**:** For $n\ge 1$,

(1) **(a)** $k_{min}(n+1)\le k_{min}(n)+1$,

(b) $k_{max}(n+1)\le k_{max}(n)+1$,

(2) $M(n+1)-M(n)=2^s$ for some integer $s\ge 1$,

(3) $M(n+1)-M(n)\le M(n+2)-M(n+1) \le 2[M(n+1)-M(n)]$.

Proof: From (2.3), we see that

$$k_{min}(2)=0<1=k_{min}(1)+1,$$
$$k_{max}(2)=1=k_{max}(1)+1.$$

Again, from (1.1c), we have

$$M(3)-M(2)=2=M(2)-M(1).$$

Thus, the results are true for $n = 1$. So, to prove the results by induction on n, we assume that the results are true for some integer n (which, in turn, implies that the results are true for all $m\le n$).

(1) To show that $k_{min}(n+1)\le k_{min}(n)+1$, we assume, on the contrary, that $k_{min}(n+1) > k_{min}(n)+1$.

Now, since

$$M(n+1)<2M(k_{min}(n)+1)+2^{n-k_{min}(n)}-1,$$

$$M(n)=2M(k_{min}(n))+2^{n-k_{min}(n)}-1,$$

it follows that

$$M(n+1)-M(n)<2[M(k_{min}(n)+1)-M(k_{min}(n))]. \qquad \text{(i)}$$

Again, since

$$M(n+1)=2M(k_{min}(n+1))+2^{n+1-k_{min}(n+1)}-1,$$

$$M(n)\le 2M(k_{min}(n+1)-1)+2^{n-k_{min}(n+1)+1}-1,$$

we get

$$M(n+1)-M(n) \ge 2[M(k_{min}(n+1))-M(k_{min}(n+1)-1)]. \qquad \text{(ii)}$$

Combining (i) and (ii), we get

$$M(k_{min}(n)+1)-M(k_{min}(n)) > M(k_{min}(n+1))-M(k_{min}(n+1)-1),$$

which contradicts the induction hypothesis for part (3) of the lemma.

This contradiction establishes part (a). The proof of part (b) is similar and is omitted here.

(2) By part (1), together with Lemma 2.2, we need to consider the following two cases.

Case 1: $M(n+1)$ is attained at $k=k_{min}(n)$.
In this case,

$$M(n+1)-M(n)=2^{n-k_{min}(n)}. \qquad (2.3)$$

Case 2: $M(n+1)$ is attained at $k=k_{min}(n)+1$.

Here,

$$M(n+1)-M(n)$$
$$=2[M(k_{min}(n)+1)-M(k_{min}(n))].$$

Thus, in either case, $M(n+1) - M(n)$ is of the form 2^s, proving part (2) of the lemma for $n+1$.

(3) Let $M(n+1)$ be attained at $k=K$. Then,

$$M(n+2)+2M(n)$$
$$\le 2M(K)+2^{n-K+2}-1+2[2M(K)+2^{n-K}-1]$$
$$=3[2M(K)+2^{n-K+1}-1]$$
$$=3M(n+1),$$

which proves the r.h.s. inequality for $n+1$. To prove the other part of the inequality, we first show that $M(n+1)$ is attained at the point $k=k_{max}(n)$. Otherwise, $M(n+1)$ must be attained at the point $k=k_{max}(n)+1$. Then,

$$M(n+1)=2M(k_{max}(n)+1)+2^{n-k_{max}(n)}-1$$
$$<2M(k_{max}(n))+2^{n+1-k_{max}(n)}-1,$$
$$M(n)=2M(k_{max}(n))+2^{n-k_{max}(n)}-1$$
$$<2M(k_{max}(n)+1)+2^{n-k_{max}(n)-1}-1,$$

and we get the following chain of inequalities:

$$2^{n-k_{max}(n)-1}<M(n+1)-M(n)<2^{n-k_{max}(n)}.$$

But the above inequality contradicts the fact that $M(n+1) - M(n)$ is of the form 2^s for some integer s (≥ 1). Hence, $M(n+1)$ must be attained at the point $k=k_{max}(n)$, so that

$$M(n+1)-M(n)=2^{n-k_{max}(n)}$$
$$<2^{n-k_{max}(n)+1}$$
$$\le M(n+2)-M(n+1).$$

All these complete the proof of the lemma.

Corollary 2.2**:** For any $n \ge 1$, $M(n+1)$ is attained at the points $k=k_{max}(n)$, $k_{min}(n+2)$.

Proof **:** In course of proving Lemma 2.3, it has been shown that $M(n+1)$ is attained at the point $k = k_{max}(n)$. The proof of the remaining case is similar, and is left for the reader.

Corollary 2.3**:** For $n \ge 1$ fixed, $F(n, k)$ is minimized at (at most) two consecutive points.

Proof**:** Let $F(n, k)$ be minimized at the two points $k=K$, L ($>K$), so that

$$M(n)=2M(K)+2^{n-K}-1=2M(L)+2^{n-L}-1.$$

Then,

$$M(L)-M(K)=2^{n-L-1}(2^{L-K}-1).$$

Since $M(L) - M(K)$ is of the form 2^s, we must have $L-K=1$.

Corollary 2.3 shows that, if for some n (≥ 1) fixed, $F(n,k)$ is not attained at a unique point, then $k_{max}(n) = k_{min}(n) + 1$. From Corollary 2.2, we see that, if for some n (≥ 1), $M(n+1)$ is attained at the unique point $k=K$, then both $M(n)$ and $M(n+2)$ are attained at $k=K$, so that

$$M(n+2)-M(n+1)=2^{n+1-K}$$
$$=2[M(n+1)-M(n)]. \quad (2.4)$$

Lemma 2.4**:** For any $n \ge 4$ fixed, $F(n,k)$ is (strictly) convex in k in the sense that

$$F(n,k+2)-F(n,k+1)$$
$$>F(n,k+1)-F(n,k),\ 0\le k\le n-3.$$

Proof **:** Since

$$F(n,k+1)-F(n,k)$$
$$=2[M(k+1)-M(k)]-2^{n-k-1}, \quad (2.5)$$

we get

$$[F(n,k+2)-F(n,k+1)]$$
$$-[F(n,k+1)-F(n,k)]$$
$$=2[\{M(k+2)-M(k+1)\}-$$
$$\{M(k+1)-M(k)\}]+2^{n-k-2}.$$

Now, appealing to part (3) of Lemma 2.2, we get the desired result.

Lemma 2.5**:** Let, for some integers $n \ge 1$ and K ($0<K<n-2$),

$$F(n,K)<F(n,K+1),\ F(n,K)<F(n,K-1).$$

Then, $F(n, k)$ is minimized at the (unique) point $k=K$.

Proof **:** By Lemma 2.4, for all $i = 2, 3, \ldots, n-K-1$,

$$F(n,K+i)-F(n,k+i-1)$$

$$>F(n, K+1) - F(n, K) > 0,$$

where the last inequality follows from the given condition. Therefore,

$$F(n, K+i) > F(n, K+1) \quad \text{(iii)}$$

for all $i = 2, 3, \ldots, \quad n-K-1$.
Again, for all $i = 1, 2, \ldots, K-1$,

$$0 > F(n, K+1) - F(n, K)$$

$$>F(n, K-i) - F(n, K-i-1),$$

so that, for all $i = 1, 2, \ldots, K$,

$$F(n, K-i) > F(n, K). \quad \text{(iv)}$$

From (iii) and (iv), we see that $F(n, k)$ is minimized at the (unique) point $k = K$.

Using the findings of this section, we give the complete solution of the dynamic programming equation (1.1) in the next section.

3. MAIN RESULT

This section derives, exploiting the results of the last section, the explicit expressions of $M(n)$, $k_{min}(n)$ and $k_{max}(n)$. This is done in the following theorem.

Theorem 3.1: Let,

$$\frac{s(s+1)}{2} < n < \frac{(s+1)(s+2)}{2},$$

for some integer $s \in \{1, 2, \ldots\}$,
so that, for some integer $1 \le R \le s$,

$$n = \frac{s(s+1)}{2} + R.$$

Then,

(1) $M\left(\frac{s(s+1)}{2}\right)$ is attained at the unique point $k = \frac{s(s-1)}{2}$ with

$$M\left(\frac{s(s+1)}{2}\right) = 2^s(s-1) + 1,$$

(2) $M\left(\frac{s(s+1)}{2} + R\right)$ is attained at the two points $k = \frac{s(s-1)}{2} + R, \frac{s(s-1)}{2} + R - 1$, with

$$M\left(\frac{s(s+1)}{2} + R\right) = 2^s(R + s - 1) + 1.$$

Proof: We first prove part (1) of the theorem. The proof is by induction on s. Since $M(1)$ is attained at the unique point $k = 0$, we see that the result is true for $s = 1$. So, we assume that the result is true for some integer $s-1$. We have to show that the result is true for s.

By the induction hypothesis, $M\left(\frac{s(s-1)}{2}\right)$ is attained at the (unique) point $k = \frac{(s-1)(s-2)}{2}$; and by Corollary 2.2, $M\left(\frac{s(s-1)}{2} - 1\right)$ is also attained at the point $k = \frac{(s-1)(s-2)}{2}$. Therefore, by (2.4),

$$M\left(\frac{s(s-1)}{2}\right) - M\left(\frac{s(s-1)}{2} - 1\right) = 2^{s-2}.$$

Now, using (2.3), we get

$$F\left(\frac{s(s+1)}{2}, \frac{s(s-1)}{2}\right) - F\left(\frac{s(s+1)}{2}, \frac{s(s-1)}{2} - 1\right)$$

$$= 2\left[M\left(\frac{s(s-1)}{2}\right) - M\left(\frac{s(s-1)}{2} - 1\right)\right] - 2^s$$

$$= 2^{s-1} - 2^s < 0.$$

Again, since (by Corollary 2.2) $M\left(\frac{s(s-1)}{2} + 1\right)$ is attained at the point $k = \frac{(s-1)(s-2)}{2}$, we get

$$M\left(\frac{s(s-1)}{2} + 1\right) - M\left(\frac{s(s-1)}{2}\right) = 2^{s-1}.$$

Therefore,

$$F\left(\frac{s(s+1)}{2}, \frac{s(s-1)}{2} + 1\right) - F\left(\frac{s(s+1)}{2}, \frac{s(s-1)}{2}\right)$$

$$= 2\left[M\left(\frac{s(s-1)}{2} + 1\right) - M\left(\frac{s(s-1)}{2}\right)\right] - 2^{s-1}.$$

$$= 2^s - 2^{s-1} > 0.$$

It then follows from Lemma 2.5 that $M\left(\frac{s(s+1)}{2}\right)$ is attained at the unique point $k = \frac{s(s-1)}{2}$. Therefore, using the induction hypothesis, we get

$$M\left(\frac{s(s+1)}{2}\right) = 2M\left(\frac{s(s-1)}{2}\right) + 2^s - 1$$

$$= 2[2^{s-1}(s-2) + 1] + 2^s - 1$$

$$= 2^s(s-1) + 1.$$

This proves part (1) of the theorem for s.

Now, by Corollary 2.2, $M\left(\frac{s(s+1)}{2} + 1\right)$ is attained at the two points $k = \frac{s(s-1)}{2}$,

$\frac{s(s-1)}{2}+1$. Thus, part (2) of the theorem is true for $R=1$. To proceed by induction, we assume that the result is true for some integer R, that is, we assume that $M\left(\frac{s(s+1)}{2}+R\right)$ is attained at the (two) points $k=\frac{s(s-1)}{2}+R$, $\frac{s(s-1)}{2}+R-1$. Then, by Corollary 2,2, $M\left(\frac{s(s+1)}{2}+R+1\right)$ is attained at the point $k=\frac{s(s-1)}{2}+R$, and then the other point at which $M\left(\frac{s(s+1)}{2}+R+1\right)$ is attained is $k=\frac{s(s-1)}{2}+R+1$. Now,

$$M\left(\frac{s(s+1)}{2}+R+1\right)-M\left(\frac{s(s+1)}{2}+R\right)=2^s,$$

so that, using the induction hypothesis,

$$M\left(\frac{s(s+1)}{2}+R+1\right)=[2^s(R+s-1)+1]+2^s$$
$$=2^s(R+s)+1.$$

Thus, the result is true for $R+1$.

4. REMARKS

This paper finds the complete solution of the dynamic programming equation (1.1), related to the Reve's puzzle, after closely studying the equation itself. The expressions of the optimal value function, $M(n)$, and the optimal partition numbers, $k_{min}(n)$ and $k_{max}(n)$, are given in Theorem 3.1 in Section 3. The necessary background materials are given in Section 2.

Some of the properties satisfied by $M(n)$, $k_{min}(n)$ and $k_{max}(n)$ are given in five lemmas and three corollaries in Section 2, which follow completely from the dynamic programming equation (1.1). From the proof of Theorem 3.1, we see that, these are the minimum number of results necessary to reach the solution of the equation (1.1). These results are proved using the unique characteristic of the equation (1.1). Some of the results are only true for the Reve's puzzle; for example, from Corollary 2.3, we see that $M(n)$ is attained either at a unique point or else at two points. This result does not hold true for the p-peg Tower of Hanoi problem with $p\geq 5$. As such, Lemma 2.5 is not valid for the p-peg problem with $p\geq 5$.

In addition to the results given in Section 3, the Reve's puzzle satisfies other properties also which are not shared by the p-peg problem with $p\geq 5$. Some of these are given in Majumdar [7, 8]. In Majumdar [8], the following result has been established.

***Lemma 4.1*:** Let, for some integer n (≥ 1), $F(n, k)$ be minimized at the unique point $k = K$. Then, the next uniquely minimized function is $F(N, k)$, which is (uniquely) minimized at $k=n$, where $N=2n-K+1$.

By virtue of Lemma 4.1, it now follows that, part (1) of Theorem 3.1 is, in fact, a consequence of Lemma 4.1. To see this, we first note that, trivially, $F(1, k)$ is uniquely minimized at $k = 0$, and the next uniquely minimized function is $F(3, k)$, which is minimized at the point $k=1$. Repeating the argument, we see that, for any integer $s\geq 1$, $M\left(\frac{s(s+1)}{2}\right)$ is attained at the unique point $k=\frac{s(s-1)}{2}$. Lemma 4.1 is an interesting result, explaining why $M\left(\frac{s(s+1)}{2}\right)$ is attained at the unique triangular numbers $\frac{s(s-1)}{2}$. Lemma 4.1 has been exploited in Majumdar [8] to prove Theorem 3.1; however, this paper shows that the same results can be proved without making use of Lemma 4.1.

From Lemma 2.4, the following lemma follows readily.

***Lemma 4.2*:** The results below hold.

(a) $F(n,k+i)-F(n,k+i-1)>F(n,k+1)$
$-F(n,k)$ for all $2\leq i\leq n-k-1$,

(b) $F(n,k-i+1)-F(n,k-i)<F(n,k+1)$
$-F(n,k)$ for all $1\leq i\leq k-1$.

Equipped with Lemma 4.2, we can prove the result below.

***Lemma 4.3*:** Let, for some integer $n\geq 1$, $F(n, k)$ satisfy the following condition :

$F(n,K)=F(n,K+1)$ for some integer K.

Then, $F(n, k)$ is minimized at the two points $k = K, K+1$.

Proof **:** By assumption,

$$F(n, K) \equiv 2M(K) + 2^{n-K} - 1 = 2M(K+1) + 2^{n-K-1} - 1 \equiv F(n, K+1).$$

Therefore,

$$M(K+1) - M(K) = 2^{n-K-2}.$$

Now,

$$F(n,K+2) - F(n,K+1) = 2[M(K+2) - M(K+1)] - 2^{n-K-2} \geq 2[M(K+1) - M(K)] - 2^{n-K-2} > 0,$$

so that

$$F(n,K+2) > F(n,K+1) = F(n,K).$$

But then, for all $3 \leq i \leq n-k-1$,

$$F(n,K+i) - F(n,K+i-1) > F(n,K+2) - F(n,K+1) > 0,$$

showing that

$$F(n,k) > F(n,K+1) \text{ for all } k > K+1.$$

Again,

$$F(n,K) - F(n,K-1) = 2[M(K) - M(K-1)] - 2^{n-K} \leq 2[M(K+1) - M(K)] - 2^{n-K} < 0,$$

so that

$$F(n,K-1) > F(n,K) = F(n,K+1).$$

Now, since, for all $2 \leq i \leq K-1$,

$$F(n,K-i+1) - F(n,K-i) < F(n,K) - F(n,K-1) < 0,$$

it follows that

$$F(n,k) > F(n,K) \text{ for all } 1 \leq k \leq K-1.$$

Lemma 4.3 provides an alternative method to prove part (2) of Theorem 3.1. This is shown below.

***Lemma 4.4*:** For any integer R with $1 \leq R \leq s$, $M\left(\frac{s(s+1)}{2} + R\right)$ is attained at the two points $k = \frac{s(s-1)}{2} + R, \frac{s(s-1)}{2} + R - 1$.

***Proof*:** The proof is by induction on s. When $s = 1$, $R = 1$, and $M(2)$ is attained at the two points $k = 0, 1$. Thus, the result is true for $s = 1$. So, we assume that the result is true for some integer $s - 1$, and we have to establish the validity of the result for s.

Now,

$$F\left(\frac{s(s+1)}{2} + R, \frac{s(s-1)}{2} + R\right) - F\left(\frac{s(s+1)}{2} + R, \frac{s(s-1)}{2} + R - 1\right)$$

$$= 2\left[M\left(\frac{s(s-1)}{2} + R\right) - M\left(\frac{s(s-1)}{2} + R - 1\right)\right] - 2^{s}.$$

If $R \neq 0$, then each of $M\left(\frac{s(s-1)}{2} + R\right)$ and $M\left(\frac{s(s-1)}{2} + R - 1\right)$ is attained at the point $k = \frac{(s-1)(s-2)}{2} + R - 1$, so that

$$M\left(\frac{s(s-1)}{2} + R\right) - M\left(\frac{s(s-1)}{2} + R - 1\right) = 2^{s-1}.$$

Then,

$$F\left(\frac{s(s+1)}{2} + R, \frac{s(s-1)}{2} + R\right) - F\left(\frac{s(s+1)}{2} + R, \frac{s(s-1)}{2} + R - 1\right) = 0.$$

This completes the induction.

5. CONFLICT OF INTEREST

The author declares no conflict of interest.

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