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Research Article

A Numerical Scheme for Solving Nonlinear Boundary Value Problems of Fractional Order $0 \leq \beta \leq \alpha < 1$

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Abstract: The primary objective of this research work is to find accurate numerical approximations for nonlinear fractional order boundary value problems (BVPs). To carry out this goal, central finite difference scheme of order four is used to approximate first- and second-order derivatives. Integrals are approximated using composite Trapezoidal rule in "the Caputo definition". The effectiveness of the proposed scheme is illustrated by solving nonlinear fractional order BVPs of order $0 \leq \beta \leq \alpha < 1$.

Keywords: Fractional differential equations, Boundary value problems, Trapezoidal rule, Central finite difference scheme.

1. INTRODUCTION

Fractional calculus provides a very supportive tool to describe natural phenomena more realistically by making beautiful and accurate modeling of physical phenomena [1]. There is much literature survey available which deals with the theory and applications of fractional differential equations [2, 3, 4, 5]. The applications of fractional derivative and fractional integral cover a broad field of complex systems including: chemistry, physics, visco-elasticity, signal processing, bioengineering, mathematical biology, and fluid mechanics, see, for example [6, 7, 8, 9]. Not only in applied mathematics, fractional calculus also has great applications in pure mathematics, see [10]. One of the hottest problems of fractional calculus is fractional differential equations with boundary conditions. These types of equations help to model many complex systems including: blood flow, thermo-elasticity, underground water flow, population dynamic, see, for example [11, 12, 13, 14, 15]. BVPs of fractional order are also applied in various physical processes of stochastic transport and many applications in the liquid filtration in a strongly porous medium, as described in [16].

Generally, numerical solution techniques are preferred when dealing with fractional models since the analytical solutions are available for a few simple cases. Following this numerous numerical techniques was developed to tackle fractional order boundary value problems

2. MATERIALS AND METHODS

Many algorithms have been developed and implemented for the numerical approximation of fractional order differential equations; see, for example [17, 18, 19, and 20]. Several new techniques are created for the solution of linear fractional order BVPs [21]. H. Demir have used shooting method to obtain solution of fractional order boundary value problem in [24]. Mohamed have investigated fractional Euler method and modified Trapezoidal rule in [25]. M.A. Anwar et al proposed the finite difference scheme for fractional order boundary value problem in [26]. Rahman dealt these boundary value problem by finite difference method in which discretization is done by 2nd order finite difference scheme and Caputo operator [27]. In this paper, we develop a numerical scheme for the approximation of nonlinear fractional order BVPs with high accuracy.

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2.1. Derivative Approximation

We approximate the derivatives in the developed scheme (Section 2.3) using central difference formulae. The stencil of fourth order implicit compact finite difference scheme used to approximate the first and second-order derivatives for the interior nodes is:

$$\left\{x_{i-1}, x_i, x_{i+1}\right\}, \quad for\ i = 2,3,...,n-2.$$

Nodes for central difference scheme are shown in Fig 1. It means that if we are at location i, then we need one grid node to the left of it and one grid point to the right of it. It is noticeable that, mutual distance between nodes is equal to $h = \frac{x_n - x_0}{n}$. We consider the following implicit compact finite difference scheme

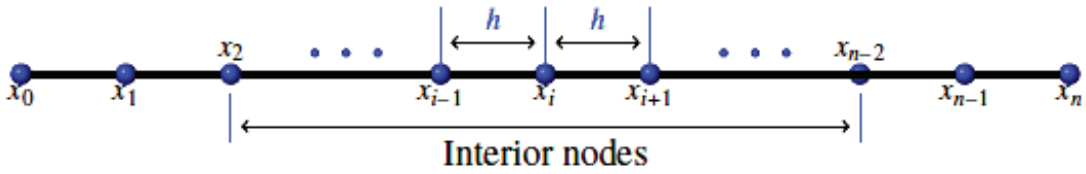

Fig. 1. Nodes for central difference scheme

$$\alpha_1 f''_{i-1} + f''_i + \alpha_2 f''_{i+1} = \frac{1}{h^2}\left(\beta_1 f_{i-1} + \beta_2 f_i + \beta_3 f_{i+1}\right). \tag{1}$$

We are interested in finding the values of unknowns in such a way that we can achieve fourth order accurate approximation of second-order derivative. On expanding equation (1) around x_1, we obtain the following system of algebraic equations:

$$\begin{aligned}
&\beta_1 + \beta_2 + \beta_3 = 0\\
&\beta_1 - \beta_3 = 0\\
&-\frac{\beta_1}{2} - \frac{\beta_3}{2} + \alpha_1 + \alpha_2 + 1 = 0\\
&-\frac{\beta_1}{6} - \frac{\beta_3}{6} - \alpha_1 + \alpha_2 = 0\\
&-\frac{\beta_1}{24} - \frac{\beta_3}{24} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 0
\end{aligned} \tag{2}$$

By solving the system of equations (2), we obtain

$$\alpha_1 = \frac{1}{10},\ \alpha_2 = \frac{1}{10},\ \beta_1 = \frac{6}{5},\ \beta_2 = \frac{12}{5},\ \beta_3 = \frac{6}{5} \tag{3}$$

To find the approximations of first-order derivatives at the interior nodes, we consider the following model

$$\alpha_1 f'_{i-1} + f'_i + \alpha_2 f'_{i+1} = \frac{1}{h}\left(\beta_1 f_{i-1} + \beta_2 f_i + \beta_3 f_{i+1}\right). \tag{4}$$

After expanding equation (4), we get the following system of algebraic equations:

$$\begin{aligned}
&\beta_1+\beta_2+\beta_3=0,\\
&\beta_1-\beta_3+\alpha_1+\alpha_2=0,\\
&-\frac{\beta_1}{2}-\frac{\beta_3}{2}-\alpha_1+\alpha_2=0,\\
&\frac{\beta_1}{6}-\frac{\beta_3}{6}+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}=0,\\
&-\frac{\beta_1}{24}-\frac{\beta_3}{24}-\frac{\alpha_1}{6}+\frac{\alpha_2}{6}=0.
\end{aligned} \tag{5}$$

Solving this system, we get

$$\alpha_1=\frac{1}{4},\ \alpha_2=\frac{1}{4},\ \beta_1=-\frac{3}{4},\ \beta_2=0,\ \beta_3=\frac{3}{4} \tag{6}$$

Similarly, for one sided approximation for boundary nodes and solving system of equations, we obtain the following coefficients:
for second-order derivative, we use the following scheme

$$f_1''+\alpha f_2''=\frac{1}{h^2}\left(\beta_1 f_1+\beta_2 f_2+\beta_3 f_3+\beta_4 f_4+\beta_5 f_5\right), \tag{7}$$

where,

$$\alpha=10,\ \beta_1=\frac{145}{12},\ \beta_2=-\frac{76}{3},\ \beta_3=\frac{29}{2},\ \beta_4=-\frac{4}{3},\ \beta_5=\frac{1}{12} \tag{8}$$

for first-order derivative, the proposed scheme is

$$f_1'+\alpha f_2'=\frac{1}{h}\left(\beta_1 f_1+\beta_2 f_2+\beta_3 f_3+\beta_4 f_4+\beta_5 f_5\right), \tag{9}$$

where,

$$\alpha=4,\ \beta_1=-\frac{37}{12},\ \beta_2=\frac{2}{3},\ \beta_3=3,\ \beta_4=-\frac{2}{3},\ \beta_5=\frac{1}{12} \tag{10}$$

2.2. Integral Approximation

We approximate the integrals in the developed scheme (Section 2.3) by the Trapezoidal method

$$\int_1^{x_{i-1}}\xi(s)ds\approx h\left(\frac{\xi_1}{2}+\xi_2+\cdots+\xi_{i-2}+\frac{\xi_{i-1}}{2}\right), \tag{11}$$

where, $\xi_j=\xi(x_j)$ and $h=x_i-x_{i-1}$ is a uniform step size.

2.3. Proposed Iterative Scheme

To describe the proposed iterative scheme, consider the following non-homogeneous nonlinear fractional order BVP

$$D^{-\alpha}y''+D^{-\beta}y+p(x)f(y)=g(x),\quad x\in(0,1),\ 0\le\beta\le\alpha<1, \tag{12}$$

with the boundary conditions:

$$y(0)=y(1)=0\,.$$

Where, $D^{-\alpha}$ and $D^{-\beta}$ are fractional orders derivatives in Caputo sense and is a nonlinear function. The fractional order differential equation (12) can also be written as

$$y'' = -D^{\alpha-\beta} y + D^{\alpha}(g(x) - p(x)f(y)). \tag{13}$$

For a given smooth function, we define

$$\begin{aligned} D^{-\alpha} w(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} w'(s) ds, \ \ \alpha > 0 \\ &= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \left(x^{1-\alpha} w'(0) + \int_0^x (x-s)^{1-\alpha} w''(s) ds \right) \end{aligned} \tag{14}$$

We can write equation (13) with the help of equation (14) as

$$\begin{aligned} y'' = &-\frac{1}{\Gamma(1-\gamma)} \left[\frac{x^{1-\gamma}}{1-\gamma} y'(0) + \int_0^x \frac{(x-s)^{1-\gamma}}{1-\gamma} y''(s) d \right] + \frac{1}{\Gamma(1-\alpha)} \\ &\left[\frac{x^{1-\alpha}}{1-\alpha} \left(g'(0) - p'(0)y(0) - p(0)y'(0) \right) + \int_0^x \frac{(x-s)^{1-\alpha}}{1-\alpha} \left(g''(s) - p''(s)y(s) - 2p'(s)y'(s) - p(s)y''(s) \right) ds \right] \end{aligned}$$

where, $\gamma = \alpha - \beta$.

We discretize $[0,1]$ for a given number of n nodes and compute a uniform step size

$$h = (1-0)/(n) = 1/(n)$$

Furthermore, we use central difference approximation of order four for the approximation of first and second-order derivative as given in Section 2.1. Whereas, integrals in our work are approximating by using composite trapezoidal method (11). The above equation can also be written as

$$\begin{aligned} y''(x_i) = &-\frac{1}{\Gamma(2-\gamma)} \left[x_i^{1-\gamma} y'(0) + I_1(x_i) \right] \\ &+ \frac{1}{(2-\alpha)} \left[x_i^{1-\alpha} \left(g'(0) - p'(0)y(0) - p(0)y'(0) \right) + I_2(x_i) + I_3(x_i) + I_4(x_i) + I_5(x_i) \right], \end{aligned} \tag{15}$$

where,

$$I_1(x_i) = \int_0^{x_i} (x_i - s)^{1-\gamma} y''(s) ds,$$

$$I_2(x_i) = \int_0^{x_i} (x_i - s)^{1-\alpha} g''(s) ds,$$

$$I_3(x_i) = \int_0^{x_i} (x_i - s)^{1-\alpha} p''(s) f \ ds,$$

$$I_4(x_i) = \int_0^{x_i} (x_i - s)^{1-\alpha} 2p'(s) f' \ ds,$$

$$I_5(x_i) = \int_0^{x_i} (x_i - s)^{1-\alpha} p(s) f'' \ ds.$$

Here, we implement a very robust iterative process. Equation (18) can also be written as

$$y_{n+1} = \Phi(y_n), \tag{16}$$

where, $y = [y_1, y_2, ..., y_n]^T$ is the *nth* approximation to the solution of discretized form of equation (18) and $\Phi(y)$ is the right-hand side.

3. RESULTS AND DISCUSSION

To quantify the quality, in terms of convergence and accuracy, of the above developed iterative scheme, we perform extensive numerical testing on a collection of test problems. In all our numerical testing, we approximate the numerical solution of five non-linear fractional order ($0 \le \beta \le \alpha < 1$) BVPs by solving iteratively equation (16) to obtain a sequence of presumably convergent vectors $y_0, y_1, y_2, ...,$ till $\|y_{n+1} - y_n\| \le$ some specified tolerance.

Problem 1

Consider the following nonlinear fractional differential equation:

$$D^{-1/4}y'' + D^{-1/8}y + x^2y^3 = -\frac{1024}{663}x^{13/4}\sqrt{2}\Gamma\left(\frac{3}{4}\right)(32x^2 - 17)\pi^{-1} - \frac{2097152}{45886995}x^{\frac{41}{8}}\sin\left(\frac{\pi}{8}\right)$$

$$\Gamma\left(\frac{7}{8}\right)\left(128x^2 - 133\right)\pi^{-1} + x^{17}\left(1 - x^2\right)^3$$

With boundary conditions:

$$y(0) = 0 = y(1)$$

Note that, the exact solution for this problem is $y(x) = x^5(1 - x^2)$.

Approximated solution and relative error of this problem are shown in Table 1. Analytic solution and approximated solution are shown graphically in Fig 2.

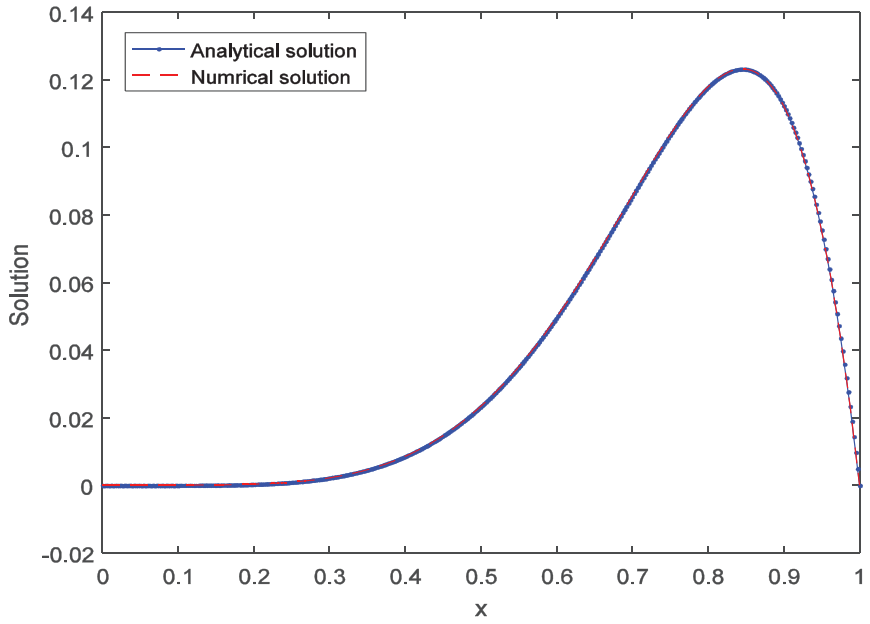

Fig. 2. Comparison between analytical and numerical solutions

Table 1. Approximate solution and relative error

X	Approximated Solution	Analytic Solution	Relative Error
0.2	2.98×10^{-4}	3.11×10^{-4}	4.05×10^{-2}
0.4	8.67×10^{-3}	8.70×10^{-3}	3.01×10^{-3}
0.6	4.94×10^{-2}	4.94×10^{-2}	7.75×10^{-4}
0.8	1.18×10^{-1}	1.18×10^{-1}	3.25×10^{-4}

Problem 2

Consider the following nonlinear fractional differential equation:

$$D^{-1/2}y'' + D^{-1/10}y + x^2e^y = -\frac{5120}{3003}\frac{x^{9/2}(224x^2-143)}{\sqrt{\pi}} - \frac{8000000000}{583657942329}x^{\frac{61}{10}}sin\left(\frac{1}{10\pi}\right)\Gamma\left(\frac{9}{10}\right)$$
$$\left(5600x^2-5751\right)\pi^{-1} + x^2e^{10x^6(1-x^2)}\left(1-x^2\right)^3$$

With boundary conditions:

$$y(0) = 0 = y(1)$$

Note that, the exact solution for this problem is $y(x) = 10x^6(1-x^2)$.

Approximated solution and relative error of this problem are shown in Table 2. Analytic solution and approximated solution are shown graphically in Fig 3.

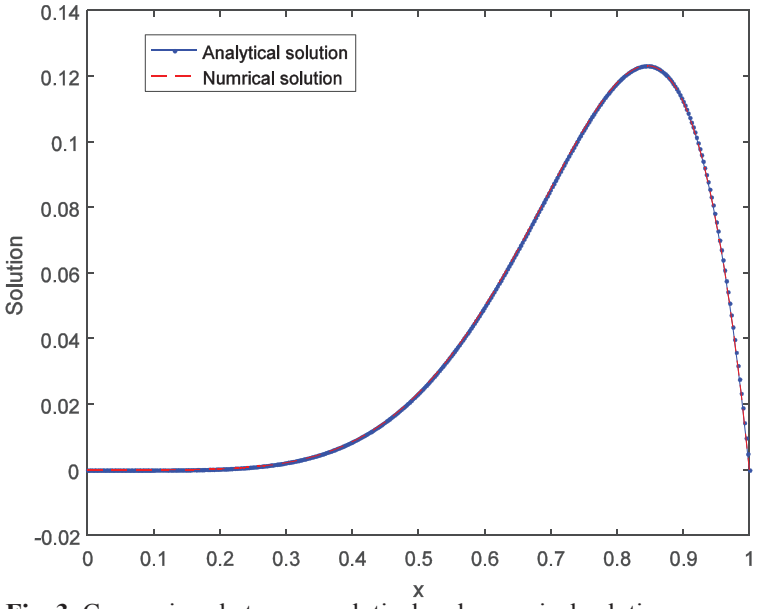

Fig. 3. Comparison between analytical and numerical solutions.

Table 2. Approximate solution and relative error

X	**Approximated Solution**	**Analytic Solution**	**Relative Error**
0.2	-1.14×10^{-4}	6.24×10^{-4}	1.18×10^{0}
0.4	3.35×10^{-2}	3.49×10^{-2}	4.16×10^{-2}
0.6	2.94×10^{-1}	2.96×10^{-1}	7.15×10^{-3}
0.8	9.40×10^{-1}	9.42×10^{-1}	2.39×10^{-3}

Problem 3

Consider the following nonlinear fractional differential equation:

$$D^{-1/2}y'' + D^{-1/10}y + x^2y^3 = \frac{128}{3003}\frac{x^{7/2}(896x^3-2912x^2+3432x-1287)}{\sqrt{\pi}} + \frac{40000000}{583657942329}x^{\frac{51}{10}}sin\left(\frac{1}{10\pi}\right)$$
$$\Gamma\left(\frac{9}{10}\right)\left(112000x^3-453600x^2+690120x-350811\right)\pi^{-1} + x^{14}(x-1)^3(1-(1-x)^3)^3$$

With boundary conditions:

$$y(0) = 0 = y(1)$$

Note that, the exact solution for this problem is $y(x) = x^4(x-1)(1-(1-x)^3)$.

The numerical results of this problem are shown in Fig 4 and Table 3.

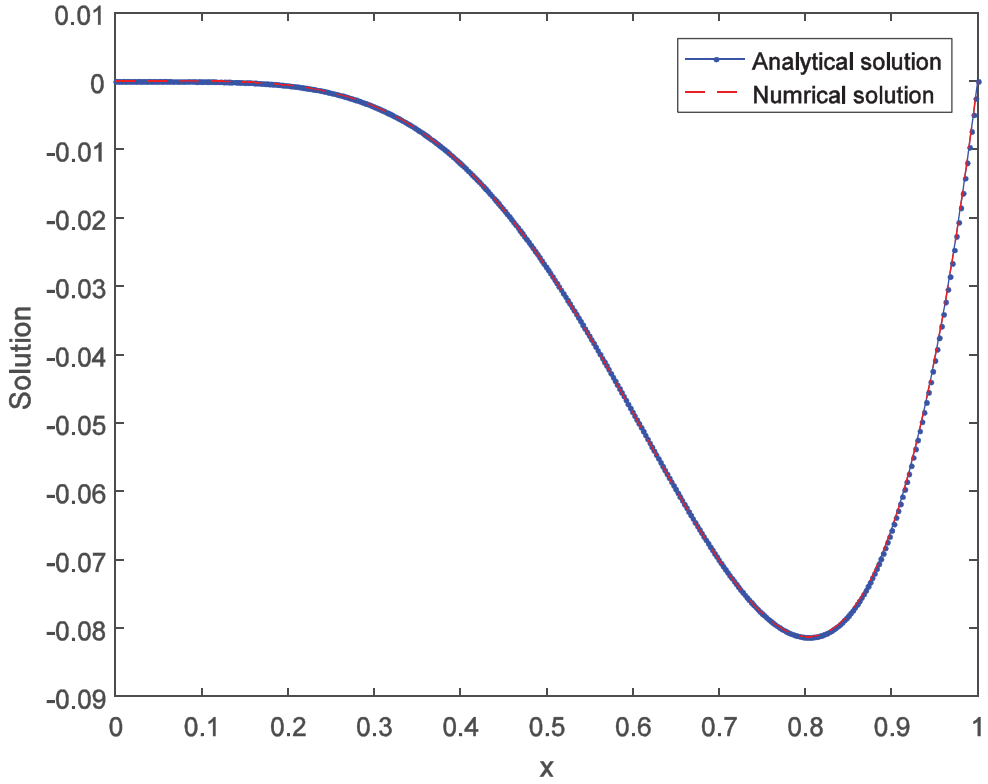

Fig. 4. Comparison between analytical and numerical solutions

Table 3. Numerical results of Problem 3

X	Approximated Solution	Analytic Solution	Relative Error
0.2	-6.07×10^{-4}	-6.32×10^{-4}	3.99×10^{-2}
0.4	-1.21×10^{-2}	-1.22×10^{-2}	4.44×10^{-3}
0.6	-4.82×10^{-2}	-4.83×10^{-2}	1.50×10^{-3}
0.8	-8.12×10^{-2}	-8.13×10^{-2}	7.48×10^{-4}

Problem 4

Consider the following nonlinear fractional differential equation:

$$D^{-1/2}y'' + D^{-1/10}y + x^2 ye^y = -\frac{128}{51051}\frac{x^{7/2}(21504x^5-91392x^4+152320x^3-123760x^2+48620x-7293)}{\sqrt{\pi}}$$
$$-\frac{40000000}{766342878277977}x^{\frac{51}{10}}\sin\left(\frac{1}{10\pi}\right)\Gamma\left(\frac{9}{10}\right)$$
$$\frac{\left(144000000x^5-727200000x^4+1470560000x^3-1488942000x^2+755106300x-153538281\right)}{\pi}$$
$$+x^7(1-x)^5 e^{x^5(1-x)^5}$$

With boundary conditions:

$$y(0) = 0 = y(1)$$

Note that, the exact solution for this problem is $y(x) = x^5(1-x)^5$.

The numerical results of this problem are shown in Fig 5 and Table 4.

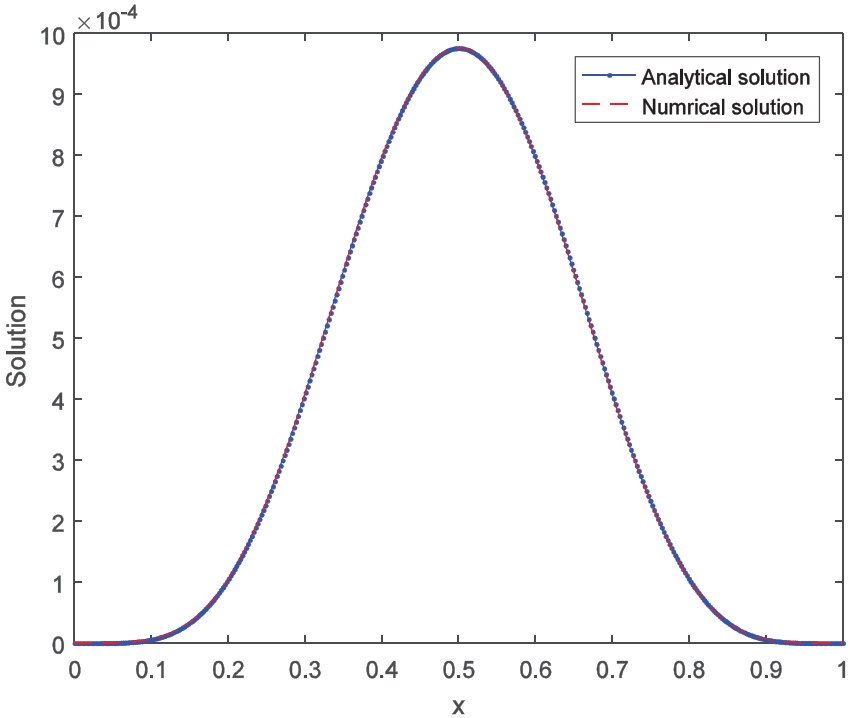

Fig. 5. Comparison between analytical and numerical solutions

Table 4. Numerical results of Problem 4

X	Approximated Solution	Analytic Solution	Relative Error
0.2	1.06×10^{-4}	1.06×10^{-4}	1.40×10^{-3}
0.4	7.99×10^{-4}	7.99×10^{-4}	1.19×10^{-4}
0.6	8.00×10^{-4}	7.99×10^{-4}	6.94×10^{-4}
0.8	1.06×10^{-4}	3.27×10^{-4}	3.27×10^{-4}

Problem 5

Consider the following nonlinear fractional differential equation:

$$D^{-1/2}y'' + D^{-1/10}y + x^2y^3 = \frac{512}{969969}\frac{x^{9/2}(118272x^5-204288x^4+217056x^2-176358x+46189)}{\sqrt{\pi}}$$
$$+\frac{800000000}{9451562165428383}x^{\frac{61}{10}}sin\left(\frac{\pi}{10}\right)\Gamma\left(\frac{9}{10}\right)$$
$$\frac{\left(88000000x^5-177600000x^4+272053600x^2-275454270x+93129777\right)}{\pi}$$
$$+x^{20}(1-x)^6(x^3-x+1)^3$$

With boundary conditions:

$$y(0) = 0 = y(1)$$

Note that, the exact solution for this problem is $y(x) = x^6(1-x)^2(x^3-x+1)$.

The numerical results of this problem are shown in Fig 6 and Table 5.

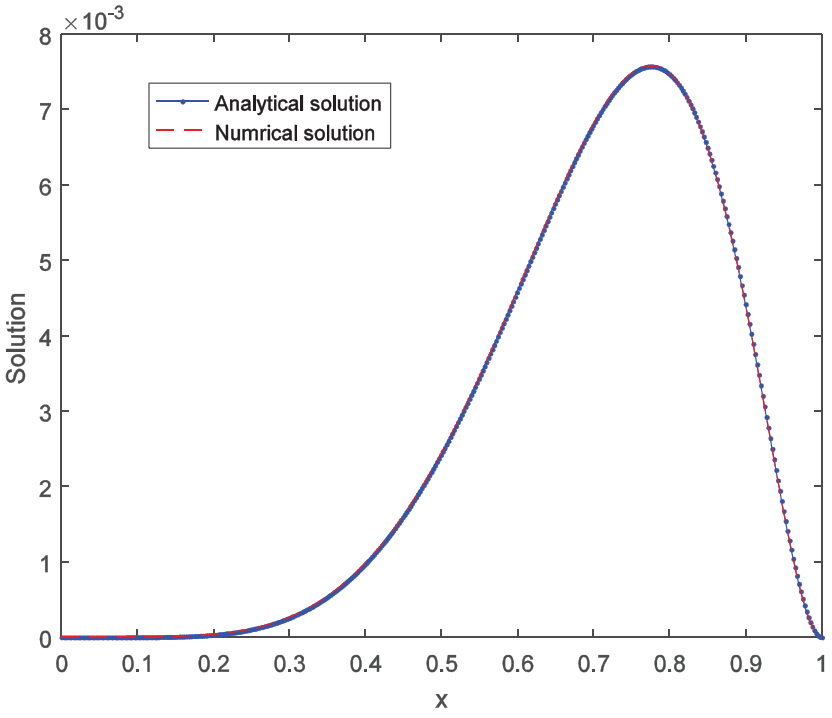

Fig. 6. Comparison between analytical and numerical solutions

Table 5. Numerical results of Problem 5

X	Approximated Solution	Analytic Solution	Relative Error
0.2	3.54×10^{-5}	3.35×10^{-5}	5.46×10^{-2}
0.4	9.93×10^{-4}	9.90×10^{-4}	2.88×10^{-3}
0.6	4.58×10^{-3}	4.57×10^{-3}	9.29×10^{-4}
0.8	7.48×10^{-3}	7.47×10^{-3}	1.28×10^{-3}

4. CONCLUSION

A new iterative scheme for the numerical approximation of nonlinear fractional order BVPs involving Caputo's derivative is proposed and hence, successfully applied in this paper. We used implicit compact finite difference scheme of order four for first and second-order derivatives and trapezoidal rule for numerical computation of integrals. Numerical experiments are performed on a collection of five nonlinear fractional orders BVPs. For the five test problems considered in this paper, convergence of the proposed iterative scheme till reaching optimal accuracy is achieved after no more than 20 iterations. We believe that the optimal accuracy can further be improved by using higher order finite difference schemes for the derivatives involved and using other numerical integration techniques for numerical computation of the integrals.

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