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Research Article

Numerical Simulation of Fractional Order Dengue Disease with Incubation Period of Virus

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Abstract: Nowadays, numerical models have great importance in epidemiology. These helps us to understand the transmission dynamics of infectious diseases in a very comprehensive manner. In disease epidemiology, vector-host models are important because many diseases are spreading through vectors. Mosquitoes are vectors of dengue disease as these spread the disease in a population. The infectious vectors infect the hosts while infectious hosts infect to vectors .Two main groups of dengue patients are septic and contagious. The susceptible mosquitoes can get dengue infection from infectious humans but not from infected ones. Humans can be categorized into Susceptible, infected, infectious and recovered ones while mosquitoes are susceptible, infected and infectious. Susceptible individual can transfer dengue infection from diseased mosquitoes only. The transmission dynamics of "Fractional order dengue fever" with incubation period of virus has been analyzed in this paper. Using standard methods for analyzing a system, the stability of equilibrium points of the model has been determined. Finally, numerical simulation has been performed for the same problem for different values of discretization parameter 'h'.

Keywords: Dengue virus, incubation period**,** stability, fraction order numerical modeling

1. INTRODUCTION

Recently, diseases caused by dengue virus have become a major health problem in the world [1]. Most probably these diseases are found in tropical areas and also found in some sub-tropical areas [2]. These diseases are found in the following countries America, Africa, Western Pacific, South Asia and Eastern Mediterranean. Before 1970 there were only nine countries which were affected by the dengue disease but after 1995 it increased four times. Until 2001 there were 609,000 patients affected by this disease. This number of patients is double to the figure as in 1995. Now major population of the world is at risk due to this disease. World Health Organization estimated that 49 M (Million) patients can be affected each year by this disease. The attack rate of this disease is 40-50% that can reach up to 80%-90% very soon. The dengue disease can be classified into three different types which are dengue hemorrhagic fever, dengue shock syndrome and dengue fever. These types have different symptoms. Dengue Fever symptoms are less in appearance in case of children's while these appear in case of young and grown up children. Dengue hemorrhagic fever is one of the complex diseases which can turn into fatal condition. This type of disease occurs to the patient when a patient is prone to the dengue virus more than once. Dengue shock syndrome (DSS) is a severe type of which can lead the patient to the hospitalization.

1.1 Causes

It can be caused by DEN1 – 4 virus.

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1.2 Transmission

The main cause of dengue disease is *Aedes* mosquito. After biting the infected human, it becomes infected with this disease and then that disease will be transmitted to other human beings. There are two types of *Aedes* mosquitoes which cause this disease. These types are: *Aedes aegypti* and *Aedes albopictus*. When the mosquito *Aedes aegypti* recruits with a dengue infectious person, the dengue virus enters the blood and circulate in the blood (viremia) that carry on for roughly four to seven days [3, 4]. Dengue cannot be transmitted directly from one person to another person. It needs a mosquito as transmitter from one person to another. Mosquitoes are only infected by this disease by biting an infected person. Virus replicates within the mosquito during the incubation period of 8-12 days. After this, glands of mosquito become infected and then virus of this disease will be transmitted to other person after biting that person. Then virus replicates in this newly infected person during the incubation period [5].

1.3 Symptoms

Its symptoms are 4 to 7 days fever after a person has been attacked by mosquito infected with virus. Symptoms are retro-orbital pain nausea, severe headache, joint and muscle pain, rashes, vomiting & high fever. DHF includes all symptoms of dengue fever with some additional symptoms which are bleeding from the nose, gums, or under the skin. It is severe form of dengue which can lead to death.

2. MODEL FORMATION

2.1 Assumptions

- The number of human and mosquito remains same.
- DEN virus effete gets permanent protection from the particular virus but becomes sensitive for others.
- Death and birth rate of human and mosquito goes on side by side.

Variables for model are that represents some special notions are as follow:

$\bar{S}_h(t)$ represents the susceptible individuals with in time t, $\bar{X}_h(t)$ denotes the Infected individuals with in time t, $\bar{I}_h(t)$ gives the number of Infectious individuals with in time t and $\bar{R}_h(t)$ are Recovered individuals with in time t. Beside these $\bar{S}_v(t)$ are the Susceptible mosquitoes (vectors) with in time t, $\bar{X}_v(t)$ tells the Infected mosquitoes (vectors) with in time t, and $\bar{I}_v(t)$ gives the number of Infectious mosquitoes with in time t.

2.2 Mathematical Model

The transmission of dengue virus is shown by following flow chart [6].

The following equations describe transmission dynamics of Dengue infection in human and vector populations:

$$\left.\begin{aligned}
\frac{d\bar{S}_h}{dt} &= \lambda N_T - \beta_h \bar{S}_h \bar{I}_v - \mu_h \bar{S}_h \\
\frac{d\bar{X}_h}{dt} &= \beta_h \bar{S}_h \bar{I}_v - \alpha_h \bar{X}_h - \mu_h \bar{X}_h \\
\frac{d\bar{I}_h}{dt} &= \alpha_h \bar{X}_h - r\bar{I}_h - \mu_h \bar{I}_h \\
\frac{d\bar{R}_h}{dt} &= r\bar{I}_h - \mu_h \bar{R}_h \\
\frac{d\bar{S}_v}{dt} &= C - \beta_v \bar{I}_h \bar{S}_v - \mu_v \bar{S}_v \\
\frac{d\bar{X}_v}{dt} &= \beta_v \bar{I}_h \bar{S}_v - \alpha_v \bar{X}_v - \mu_v \bar{X}_v \\
\frac{d\bar{I}_v}{dt} &= \alpha_v \bar{X}_v - \mu_v \bar{I}_v
\end{aligned}\right\} \quad (1)$$

with $N_T = \bar{S}_h + \bar{X}_h + \bar{I}_h + \bar{R}_h$

and $N_v = \bar{S}_v + \bar{X}_v + \bar{I}_v$

where

N_T and N_v are constant. So,

$$\frac{dN_T}{dt} = 0 \text{ and } \frac{dN_v}{dt} = 0$$

which implies that $\lambda = \mu_h$

Now for vector population

$$\frac{dN_v}{dt} = \frac{d}{dt}(\bar{S}_v + \bar{X}_v + \bar{I}_v)$$

$$0 = \frac{d}{dt}\bar{S}_v + \frac{d}{dt}\bar{X}_v + \frac{d}{dt}\bar{I}_v$$

$$0 = C - \mu_v \bar{S}_v - \mu_v \bar{X}_v - \mu_v \bar{I}_v$$

$$0 = C - \mu_v(\bar{S}_v - \bar{X}_v - \bar{I}_v)$$

$$0 = C - \mu_v N_v \Rightarrow N_v = \frac{C}{\mu_v}$$

For normalization of system (1), we let

$$S = \frac{\bar{S}_h}{\mathrm{N_T}}, X = \frac{\bar{X}_h}{\mathrm{N_T}}, I = \frac{\bar{I}_h}{\mathrm{N_T}}, R = \frac{\bar{R}_h}{\mathrm{N_T}}$$

$$S_v = \frac{\bar{S}_v}{\mathrm{N_T}}, X_v = \frac{\bar{X}_v}{\mathrm{N_T}}, I_v = \frac{\bar{I}_v}{\mathrm{N_T}}$$

The system of differential equation for transmission dynamics of dengue fever in normalized form is

$$\left.\begin{aligned}
\frac{dS}{dt} &= \mu_h - \beta_h S I_v\,(\mathrm{C}/\mu_v) - \mu_h S \\
\frac{dX}{dt} &= \beta_h S I_v\,(\mathrm{C}/\mu_v) - \alpha_h X - \mu_h X \\
\frac{dI}{dt} &= \alpha_h X - rI - \mu_h I \\
\frac{dX_v}{dt} &= \beta_v I N_T(1 - X_v - I_v) - \alpha_v X_v - \mu_v X_v \\
\frac{dI_v}{dt} &= \alpha_v X_v - \mu_v I_v
\end{aligned}\right\} \quad (2)$$

subject to the conditions:

$$S + X + I + R = 1 \quad \text{and} \quad S_v + X_v + I_v = 1$$

The terminology used for the different parameters are given Table 1.

Table 1. Different symbols and terminology.

Symbol	Terminology
N_T	Total population of human
β_h	Infectious rate of dengue virus from mosquitoes (vector) to human population
λ	Human population's birth rate
β_v	Infectious rate of dengue virus from human to mosquito (vector) population
α_h	Rate of changing the infected human population to infectious human population
R	Human population's recovery rate
μ_h	Human population death rate
α_v	Vector (mosquito) population death rate
C	Vector (mosquito) population constant recruitment rate

3. BEGINNINGS AND CYPHERS

In this segment, some simple explanations and chattels of the fractional calculus theory and Non-standard discretization are discussed.

3.1. Fundamentals of Fractional-order

Fractional calculus represents a generalization of the ordinary differentiation and integration to non-integer and complex order [22]. The generalization of differential calculus to non-integer orders of derivatives can be traced back to Leibnitz [24]. The main reason for using integer order models was the absence of solution methods for fractional differential equations. It is an emerging field in the area of applied mathematics and mathematical physics such as chemistry, biology, economics, image and signal processing and it has many applications in many areas of science and engineering [23] for example, viscoelasticity, control theory, heat conduction, electricity, chaos and fractals etc. [22]. Various applications, like in the reaction kinetics of proteins, the anomalous electron transport in amorphous materials, the dielectrical or mechanical relation of polymers, the modeling of glass forming liquids and others are successfully performed in numerous papers [24].

The physical and geometrical meaning of the non-integer integral containing the real and complex conjugate power-law exponent has been proposed. Finding examples of real systems describes by the fractional derivative is an open issue in the area of fractional calculus [22].

Since integer order differential equations cannot precisely describe the experimental and field measurement data, as an alternative approach non-integer order differential equation models are now being widely applied [19, 20].The advantage of fractional-order differential equation systems over ordinary differential equation systems is that they allow greater degrees of freedom and incorporate memory effect in the model. In other words, it provides an excellent tool for the description of memory and hereditary properties which were not taken into account in the classical integer order model [21].

The calculus of variations is widely applied for some disciplines like engineering, pure and applied mathematics. Moreover, the researchers have recently proved that the physical systems with dissipation can be clearly modeled more accurately by using fractional representations [23]. Recently, most of the dynamical systems based on the integer-order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer order modeling. Few of the nonlinear models are given in [28-30].

There are many definitions of fractional derivatives. Few of them are:

(1) Caputo's definition [25]

$$\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{1}{\Gamma(n-\phi)}\int_0^u (u-s)^{n-\phi-1}\frac{d^n g(s)}{ds^n}\,ds$$

(2) Riemann-Liouville definition [25]

$$\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{1}{\Gamma(n-\phi)}\frac{d^n}{du^n}\int_0^u (u-s)^{n-\phi-1}\,g(s)\,ds$$

(3) Jumarie's definition [25]

$$\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{1}{\Gamma(n-\phi)}\frac{d^n}{du^n}\int_0^u (u-s)^{n-\phi-1}\,[g(s)-g(0)]\,ds$$

(4) Xiao-Jun Yang's definition [25]

$$\mathfrak{D}_u^{\phi}\big(g(u_0)\big) = g^{\phi}(u_0) = \frac{d^{\phi}}{du^{\phi}}g(u)\Big|_{u=u_0} = \lim_{u\to u_0}\frac{\Delta^{\phi}(g(u)-g(u_0))}{(u-u_0)^{\phi}}$$

where $\Delta^{\phi}\big(g(u)-g(u_0)\big) \cong \Gamma\,(1+\phi)\Delta(g(u)-g(u_0))$.

(5) Chen's fractal derivative [25]

$$\frac{dg}{du^{\phi}} = \lim_{s\to u}\frac{g(u)-g(s)}{u^{\phi}-s^{\phi}}$$

(6) Ji-Huan He's fractal derivative [25]

$$\frac{\mathfrak{D}g}{\mathfrak{D}u^{\phi}} = \Gamma\,(1+\phi)\lim_{\Delta u=u_1-u_2\to L}\frac{g(u_1)-g(u_2)}{(u_1-u_2)^{\phi}}$$

where Δu does not tend to zero, it can be the thickness (L) of a porous medium. Applications of the fractal derivative to fractal media have attracted much attention, for example it can model heat transfer and water permeation in multi-scale of fabrics and wool fibers.

(7) Davidson-Essex derivative [26]

$$\mathfrak{D}_0^{\phi}\big(g(u)\big) = \frac{1}{\Gamma(1-\phi)}\frac{d^{n+1-k}}{du^{n+1-k}}\int_0^u (u-s)^{-\phi}\,\frac{d^k g(s)}{ds^k}\,ds$$

(8) Coimbra derivative [26]

$$\mathfrak{D}_0^{\phi(u)}\big(g(u)\big) = \frac{1}{\Gamma(1-\phi(u))}\left\{\int_0^u (u-s)^{-\phi(x)}\,\frac{dg(s)}{ds}\,ds + g(0)u^{-\phi(u)}\right\}$$

(9) Canavati derivative [26]

$${}_a\mathfrak{D}_u^{\xi}\big(g(u)\big) = \frac{1}{\Gamma(1-\phi)}\frac{d}{du}\int_0^u (u-s)^{\phi}\,\frac{d^n g(s)}{ds^n}\,ds$$

$n = |\xi|,\ \phi = n-\xi$

(10) Osler fractional derivative [26]

$${}_a\mathfrak{D}_u^{\xi}\big(g(u)\big) = \frac{\Gamma(1+\phi)}{2\pi\,\mathrm{i}}\int_{\Im(a,u^+)}\frac{g(s)}{(s-u)^{1+\phi}}\,ds$$

(11) k-fractional Hilfer derivative [26]

$${}^{k}\mathfrak{D}^{\phi,\xi}\big(g(u)\big) = I_k^{\xi(1-\phi)}\frac{d}{du}I_k^{(1-\xi)(1-\phi)}\,g(u)$$

where $I_k^{\phi}g(u)$ is the k-fractional Hilfer integral

$$I_k^{\phi}g(u) = \frac{1}{k\,\Gamma_k(\phi)}\int_0^u (u-s)^{\frac{\phi}{k}-1}g(u)du$$

(12) Caputo Fabrizio derivative [27]

Let $g \in H^1(a,b)$, $a < b$, $\phi \in [0,1]$ then, the new Caputo fractional derivative is

$$\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{M(\phi)}{1-\phi}\int_a^u g'(u)\exp\left(-\phi\frac{u-s}{1-\phi}\right)\,ds \qquad (*)$$

where $M(\phi)$ enotes a normalization function obeying $M(0) = M(1) = 1$. However, if the function does not belong to $H^1(a,b)$then, the derivative has the form

$$\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{\phi M(\phi)}{1-\phi}\int_a^u [g(u)-g(s)]\exp\left(-\phi\frac{u-s}{1-\phi}\right)\,ds \qquad (**)$$

If $\sigma = \frac{1-\phi}{\phi} \in [0,\infty]$, $\phi = \frac{1}{1+\sigma} \in [0,1]$, then eq. (**) assumes the form

$$\mathfrak{D}_u^{\sigma}\big(g(u)\big) = \frac{N(\sigma)}{\sigma}\int_a^u g'(u)\exp\left(-\frac{u-s}{\sigma}\right)ds\ , N(0) = N(\infty) = 1$$

(13) Atangana Baleanu Fractional Derivative in Riemann-Liouville sense[28]

Let $g \in H^1(a,b)$, $a < b$, $\phi \in [0,1]$ and not necessary differentiable then, the definition of the new fractional derivative is given as

$${}^{ABR}_{a}\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{B(\phi)}{1-\phi}\frac{d}{du}\int_a^u g(s)\,\mathrm{E}_{\phi}\left(-\phi\frac{(u-s)^{\phi}}{1-\phi}\right)\,ds$$

(14) Atangana Baleanu Fractional Derivative in Riemann-Caputo sense [28]

Let $g \in H^1(a,b)$, $a < b$, $\phi \in [0,1]$ and not necessary differentiable then, the definition of the new fractional derivative is given as

$${}^{ABC}_{a}\mathfrak{D}_u^{\phi}\big(g(u)\big) = \frac{B(\phi)}{1-\phi}\int_a^u g'(s)\,\mathrm{E}_{\phi}\left(-\phi\frac{(u-s)^{\phi}}{1-\phi}\right)\,ds$$

3.2. Grunwald-Letnikov (GL) Method

The GL method of approximation for the 1-D fractional derivative is as follows [11].

$$D^{\beta}x(\tau) = g\big(\tau, x(\tau)\big),\ x(0) = x_0,\ \tau \in [0,\tau_f], \qquad (3)$$

$$D^{\beta}x(\tau) = \lim_{h\to 0} h^{-\beta}\ \sum_{\boldsymbol{j=0}}^{\left[\frac{\tau_f}{h}\right]}(-1)^i\binom{\beta}{i}\,x(\tau - ih),$$

where $0 < \beta < 1$, D^{β} signifies the fractional derivative, h is the step size and $\left[\frac{\tau_f}{h}\right]$ represents the integer part of $\frac{\tau_f}{\boldsymbol{h}}$. Therefore, Eq. (3) is discretized in the next form,

$$\sum_{i=0}^{n} C_j^{\beta}\ x_{n-j} = f(\tau_n, x_n), \qquad n = 1,2,3,\ldots$$

where $\tau_n = nh$ and C_j^{β} are the GL coefficients defined as

$$C_i^{\beta} = \left(1 - \frac{1+\beta}{i}\right)C_{i-1}^{\beta}, \qquad C_0^{\beta} = h^{-\beta}, \qquad i = 1,2,3\ldots$$

The paper of Mickens [13] provides an all-purpose route for determining $\emptyset(h)$ for the ODEs. A specimen of the NSFD discretization procedure is its application to the decay equation

$$x' = -\lambda\, x$$

where λ is constant. The discretization scheme [13] is

$$\frac{x_{n+1}-x_n}{\emptyset} = -\lambda\, x_n, \qquad \emptyset(h,\lambda) = \frac{1-e^{-\lambda h}}{\lambda}$$

An alternate example is given by

$$x' = \lambda_1\, x - \lambda_2\, x^2$$

where the NSFD scheme is

$$\frac{x_{n+1}-x_n}{\emptyset} = \lambda_1 x_n - \lambda_2 x_n x_{n+1}$$

$$\emptyset(h,\lambda_1) = \frac{e^{\lambda_1 h}-1}{\lambda_1}$$

It should be renowned that the NSFD schemes for both ODEs are exact in the sense that $x_n = x(t_n)$ for all applicable values of $h > 0$.

4. FRACTIONAL ORDER DENGUE MODEL

Mathematical models have been used extensively in research into the epidemiology of dengue to help improve our understanding of the major contributing factors in a given epidemic. Main claim is that fractional model can give more realistic interpretation of natural phenomena. The use of fractional derivative allows us to model memory effects, and results in a more powerful approach to epidemiology models. A few papers have been written on fractional order dengue epidemiology [16, 17, 18]. So the system (2) in fractional order form is:

$$\left.\begin{aligned} \frac{d^{\gamma_1}S}{dt^{\gamma_1}} &= \mu_h - \beta_h S I_v\,(C/\mu_v) - \mu_h S \\ \frac{d^{\gamma_2}X}{dt^{\gamma_2}} &= \beta_h S I_v\,(C/\mu_v) - \alpha_h X - \mu_h X \\ \frac{d^{\gamma_3}I}{dt^{\gamma_3}} &= \alpha_h X - rI - \mu_h I \\ \frac{d^{\gamma_4}X_v}{dt^{\gamma_4}} &= \beta_v I N_T(1 - X_v - I_v) - \alpha_v X_v - \mu_v X_v \\ \frac{d^{\gamma_5}I_v}{dt^{\gamma_5}} &= \alpha_v X_v - \mu_v I_v \end{aligned}\right\} \quad (4)$$

with the initial conditions

$$S(0) = 0.1, \quad X(0) = 0, \qquad I(0) = 0, \quad X_v(0) = 0.1, \quad I_v(0) = 0.1$$

In order to analyze the model's stability, the theorem of stability on fractional order systems and fractional Ruth-Hurwitz stability conditions for fractional order differential equations are presented. The first theorem of stability has been given for fractional order systems.

4.1. Stability Analysis

Theorem 1. [14] Consider the fractional order system given below

$$D^{\alpha}\,U(t) = G(U), U(0) = U_0 \quad (5)$$

where $0 < \alpha \leq 1$ and $u \in R^n$. Equilibrium points of system (5) should be determined by cracking the $G(U) = 0$. These points will be non-globally asymptotically steady if all eigenvalues η matrix of the Jacobian $J = \frac{\partial G}{\partial U}$ evaluated at the equilibrium point mollify:

$$|\arg(\zeta)| > \frac{\alpha\,\pi}{2}$$

The Jacobian matrix J of system (4) with the equilibrium point $E = (s^*, x^*, i^*, {x_v}^*, {i_v}^*)$.

$$J(F^*) = \begin{bmatrix} L_1 & 0 & 0 & 0 & -L_4 \\ L_2 & L_3 & 0 & 0 & L_4 \\ 0 & \alpha_h & L_5 & 0 & 0 \\ 0 & 0 & L_6 & L_7 & L_8 \\ 0 & 0 & 0 & \alpha_v & -\mu_v \end{bmatrix} \quad (6)$$

$$L_1 = -\frac{\beta_h i_v{}^* C}{\mu_v} - \mu_h,\ L_2 = \frac{\beta_h i_v{}^* C}{\mu_v}, L_3 = -\mu_h - \alpha_h, L_4 = \frac{\beta_h s^* C}{\mu_v}, L_5 = -\mu_h - \mathrm{r},$$
$$L_6 = \beta_v N_T(1 - x_v{}^* - i_v{}^*), L_7 = -\alpha_v - \mu_v - \beta_v i^* N_T, L_8 = -\beta_v i^* N_T$$

4.2. Equilibrium Points

To find the equilibrium state points we set the right hand side of all equations in system (4) equated to zero. We found that the system has two possible equilibrium points i.e. the disease free equilibrium (DFE) and endemic equilibrium (EE).

Disease free equilibrium (DFE):

$$V_0 = (1, 0, 0,0,0)$$

Endemic equilibrium (EE):

$$V_1 = (S^*, X^*, I^*, X_v^*, I_v^*)$$

where

$$S^* = \frac{(\alpha_v + \mu_v)(WH\mu_h^2\mu_v + \alpha_h\,\gamma_v\,\mu_h)}{\alpha_h\,\gamma_v\,[\mu_h(\alpha_v + \mu_v) + \alpha_v\gamma_h\,]}$$

$$X^* = \frac{W\mu_h^2\mu_v(\alpha_v + \mu_v)(E_0 - 1)}{\alpha_h\alpha_h\,[\mu_h(\alpha_v + \mu_v) + \alpha_v\gamma_h\,]}$$

$$I^* = \frac{\mu_h\mu_v(\alpha_v + \mu_v)(E_0 - 1)}{\alpha_h\,[\mu_h(\alpha_v + \mu_v) + \alpha_v\gamma_h\,]}$$

$$X_v^* = \frac{\mu_v(WH\mu_h^3\mu_v)(E_0 - 1)}{\gamma_h\,\alpha_v(\alpha_h\,\gamma_v\,\mu_h + WH\mu_h^2\mu_v)}$$

$$I_v^* = \frac{WH\mu_h^3\mu_v}{\gamma_h\,(\alpha_h\,\gamma_v\,\mu_h + WH\mu_h^2\mu_v)}(E_0 - 1)$$

where

$$E_0 = \frac{\alpha_h\,\alpha_v\gamma_h\,\gamma_v}{\mu_v(r + \mu_h)(\alpha_h\, + \mu_h)(\alpha_v + \mu_v)}$$

$$\gamma_h\, = \frac{C\,\beta_h}{\mu_v},\ \gamma_v\, = N_T\beta_v,\ \ W = \frac{r+\mu_h}{\mu_h},\ H = \frac{\alpha_h\, + \mu_h}{\mu_h}$$

To see the non-global stability for each equilibrium phase can be determined by the insignia of all eigenvalues. If all eigenvalues have negative real part, then that equilibrium phase is non-global stability. We locate the eigenvalues for each equilibrium phase by setting

$$\det(J - \zeta I) = 0 \tag{7}$$

where J is the Jacobian matrix of the right hand side of (4) determined at the equilibrium phase.

For the equilibrium phase

$$V_0 = (1, 0, 0,0,0)$$

Equation (6) reduces to

$$J(F^*) = \begin{bmatrix} -\mu_h & 0 & 0 & 0 & -\frac{\beta_h C}{\mu_v} \\ 0 & -\mu_h - \alpha_h & 0 & 0 & \frac{\beta_h C}{\mu_v} \\ 0 & \alpha_h & -\mu_h - \mathrm{r} & 0 & 0 \\ 0 & 0 & \beta_v N_T & -\alpha_v - \mu_v & 0 \\ 0 & 0 & 0 & \alpha_v & -\mu_v \end{bmatrix} \tag{8}$$

The characteristic equation is obtained by solving

$$\det(J - \zeta I) = 0$$

$$(\zeta + \mu_h)(\zeta^4 + b_3\zeta^3 + b_2\zeta^2 + b_1\zeta + b_0) = 0 \tag{9}$$

with

$$b_3 = \alpha_v + (W + H)\mu_h + 2\mu_v$$

$$b_2 = W\,H\mu_h^2\,\alpha_v + 2(W + H)\mu_h\mu_v + \mu_v^2 + \alpha_v(W + H)\mu_h + \mu_v$$

$$b_1 = \mu_h(\,\alpha_v(W\,H\mu_h + (W + H)\mu_v) + \mu_v(2W\,H\mu_h + (W + H)\mu_v))$$

$$b_0 = WH\mu_h^2\mu_v(1 - E_0)(\alpha_v + \mu_v)$$

So we have five eigenvalues corresponding to (9). We represent these eigenvalues by $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and ζ_5. Clearly $\zeta_1 = -\mu_h$ has negative real part. The other four eigenvalues can be obtained by solving

$$\zeta^4 + b_3\zeta^3 + b_2\zeta^2 + b_1\zeta + b_0 = 0.$$

These four eigenvalues have negative real part if they satisfy the Ruth-Hurwitz criteria [6,7],with $b_3 > 0, b_1 > 0, b_0 > 0$,and $b_1b_2b_3 > b_1^2 + b_3^2\,b_0$.

It can be seen that coefficients b_0, b_1, b_3 are greater than zero, when $E_0 < 1$. Evaluating

$$\begin{aligned} b_1b_2b_3 - (b_1^2 + b_3^2\,b_0) &= (\alpha_v + (W + H)\mu_h + 2\mu_v)\big(W\,H\mu_h^2\,\alpha_v + 2(W + H)\mu_h\mu_v + \mu_v^2 + \alpha_v(W + H)\mu_h \\ &\quad + \mu_v\big)\Big(\mu_h\big(\,\alpha_v(W\,H\mu_h + (W + H)\mu_v) + \mu_v(2W\,H\mu_h + (W + H)\mu_v)\big)\Big) \\ &\quad - \big(\big(\mu_h(\,\alpha_v(W\,H\mu_h + (W + H)\mu_v) + \mu_v(2W\,H\mu_h + (W + H)\mu_v)\big)^2 \\ &\quad + (\alpha_v + (W + H)\mu_h + 2\mu_v)^2(W\,H\mu_h^2\mu_v(1 - E_0)(\alpha_v + \mu_v))\big) \end{aligned}$$

$$\begin{aligned} &= \mu_h(W + H)(W\mu_h + \mu_v)(W\mu_h + \mu_v + \alpha_v)(H\mu_h + \mu_v)(H\mu_h + \mu_v + \alpha_v)(2\mu_v + \alpha_v) + \\ &\quad \alpha_v\alpha_h\gamma_h\gamma_v(\alpha_v + (W + H)\mu_h + 2\mu_v)^2 \end{aligned}$$

which tells us $b_1b_2b_3 > b_1^2 + b_3^2\,b_0$ or $b_1b_2b_3 - (b_1^2 + b_3^2\,b_0)$ is every time positive. So the disease-free equilibrium phase is non-global stability for $E_0 < 1$.

For the endemic equilibrium phase V_1 , the equation of characteristic is

$$\zeta^5 + a_4\zeta^4 + a_3\zeta^3 + a_2\zeta^2 + a_1\zeta + a_0 = 0 \tag{10}$$

$$a_4 = \alpha_v + (1 + W + H)\mu_h + 2\mu_v + \frac{\mu_h\mu_v(E_0 - 1)(\mu_v + \alpha_v)}{\alpha_v(\gamma_h + \mu_h) + \mu_h\mu_v} + \frac{(E_0 - 1)WH\mu_h^3\mu_v}{\alpha_h\gamma_v\mu_h + WH\mu_h^2\mu_v}$$

$$a_3 = \mu_v^2 + \mu_h\mu_v\left\{2(1+W+H) + \frac{\mu_v(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right\}$$
$$+ \mu_h^2\left\{W+H+WH + \frac{\mu_v(1+W+H)(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right\}$$
$$+ \frac{(E_0-1)WH\mu_h^3\mu_v\{(W+H)\mu_h+2\mu_v\}}{\alpha_h\gamma_v\mu_h+WH\mu_h^2\mu_v}$$
$$+ \frac{\mu_h\mu_v(E_0-1)(\mu_v+\alpha_v)}{(\alpha_h\gamma_v\mu_h+WH\mu_h^2\mu_v)(\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v)}$$
$$+ \left\{(1+W+H)\mu_h+\mu_v+\frac{\mu_h\mu_v(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}+\frac{(E_0-1)WH\mu_h^3\mu_v}{\alpha_h\gamma_v\mu_h+WH\mu_h^2\mu_v}\right\}\alpha_v$$

$$a_2 = \mu_h\Bigg\{WH\mu_h^2+2(WH+W+H)\mu_h\mu_v+(1+W+H)\mu_v^2$$
$$+ \frac{\mu_h\mu_v(E_0-1)(\mu_v+\alpha_v)\big((W+H+WH)\mu_h+(1+W+H)\mu_v\big)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}$$
$$+ \frac{WH\mu_h^2\mu_v(E_0-1)\{WH\mu_h^2+2(W+H)\mu_h\mu_v+\mu_v^2\}}{\alpha_h\gamma_v\mu_h+WH\mu_h^2\mu_v}$$
$$+ \frac{WH\mu_h^2\mu_v(E_0-1)\left\{\frac{\mu_h\mu_v(E_0-1)(\mu_v+\alpha_v)((W+H)\mu_h+\mu_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right\}}{\alpha_h\gamma_v\mu_h+WH\mu_h^2\mu_v}$$
$$+ \alpha_v\Bigg\{(WH+W+H)\mu_h+(1+W+H)\mu_v$$
$$+ \frac{\mu_v\mu_h(1+W+H)(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}$$
$$+ \frac{WH\mu_h^2\mu_v(E_0-1)\left\{(W+H)\mu_h+\mu_v+\frac{\mu_h\mu_v(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right\}}{\alpha_h\gamma_v\mu_h+WH\mu_h^2\mu_v}\Bigg\}\Bigg\}$$

$$a_1 = \mu_h^2 \left\{ \frac{\mu_h^2 \mu_v WH(E_0-1)(\mu_v+\alpha_v)\left(1+\frac{H(E_0-1)\mu_h^2\mu_v}{\alpha_h\gamma_v\mu_h+H\mu_h^2\mu_v}\right)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v} \right.$$

$$+\mu_v^2\left(W+H+WH+\frac{H(W+H)(E_0-1)\mu_h^2\mu_v}{\alpha_h\gamma_v\mu_h+H\mu_h^2\mu_v}\right)$$

$$+\mu_v\mu_h\left(2WH+\frac{\mu_v(WH+W+H)(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right.$$

$$\left.+\frac{H\mu_h^2\mu_v(E_0-1)\left\{2WH+\frac{\mu_v(E_0-1)(W+H)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right\}}{\alpha_h\gamma_v\mu_h+H\mu_h^2\mu_v}\right)$$

$$+\alpha_v\left(WH\mu_h+(W+H+WH)\mu_v+\frac{\mu_h\mu_v(E_0-1)(W+H+WH)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right.$$

$$\left.\left.+\frac{H\mu_h^2\mu_v(E_0-1)\left\{WH\mu_h+(W+H)\mu_v+\frac{\mu_h\mu_v(E_0-1)(W+H)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right\}}{\alpha_h\gamma_v\mu_h+H\mu_h^2\mu_v}\right)\right\}$$

$$a_0 = (\mu_v+\alpha_v)\mu_h\mu_v\left(WH\mu_h^2\left(1+\frac{H(E_0-1)\mu_h^2\mu_v}{\alpha_h\gamma_v\mu_h+H\mu_h^2\mu_v}\right)\left(1+\frac{\mu_h(E_0-1)(\mu_v+\alpha_v)}{\alpha_v(\gamma_h+\mu_h)+\mu_h\mu_v}\right)\right)$$

Equation (10) corresponds five eigenvalues. We represent these five eigenvalues by ζ_1, ζ_2, ζ_3, ζ_4 and ζ_5. These five eigenvalues have negative real parts if they satisfy the Routh-Hurwitz criteria [6, 7], that is,

$$a_i > 0, \quad \text{for } i = 0,1,2,3,4$$

$$a_2a_3a_4 > a_2^2 + a_4^2\, a_1$$

$$(a_1a_4 - a_0)(a_4a_3a_2 - a_2^2 - a_4^2\, a_1) > a_0(a_4a_3 - a_2)^2 + a_4a_0^2$$

If all these conditions satisfy and will satisfy (already checked) for $E_0 > 1$. Thus, the endemic equilibrium phase is non-global stability for $E_0 > 1$.

5. NSFD DISCRETIZATION FOR FRACTIONAL-ORDER DENGUE MODEL

In this section we shall construct Non Standard Finite Difference Scheme proposed by Mickens [2, 13], for the system (4) and swapping the step size h by a function $\psi(h)$ and using GL discretization technique, it can be seen that

$$\sum_{j=0}^{n+1} C_j^{\gamma_1}\, S^{n+1-j} = \mu_h - \beta_h S^{n+1} {I_v}^n \left(\frac{C}{\mu_v}\right) - \mu_h S^{n+1} \tag{11}$$

$$\sum_{j=0}^{n+1} C_j^{\gamma_2} X^{n+1-j} = \beta_h S^{n+1} {I_v}^n \left(\frac{C}{\mu_v}\right) - \alpha_h X^{n+1} - \mu_h X^{n+1} \tag{12}$$

$$\sum_{j=0}^{n+1} C_j^{\gamma_3} I^{n+1-j} = \alpha_h X^{n+1} - rI^{n+1} - \mu_h I^{n+1} \tag{13}$$

$$\sum_{j=0}^{n+1} C_j^{\gamma_4} {X_v}^{n+1-j} = \beta_v I^{n+1} N_T (1 - {X_v}^{n+1} - {I_v}^n) - \alpha_v {X_v}^{n+1} - \mu_v {X_v}^{n+1} \tag{14}$$

$$\sum_{j=0}^{n+1} C_j^{\gamma_5} {I_v}^{n+1-j} = \alpha_v {X_v}^{n+1} - \mu_v {I_v}^{n+1} \tag{15}$$

(11) $\Rightarrow$
$$S^{n+1} = \frac{\mu_h - \sum_{j=1}^{n+1} C_j^{\gamma_1} S^{n+1-j}}{(C_0^{\gamma_1} + \beta_h {I_v}^n \left(\frac{C}{\mu_v}\right) + \mu_h)} \tag{16}$$

(12) $\Rightarrow$
$$X^{n+1} = \frac{\beta_h S^{n+1} {I_v}^n \left(\frac{C}{\mu_v}\right) - \sum_{j=1}^{n+1} C_j^{\gamma_2} X^{n+1-j}}{C_0^{\gamma_2} + \alpha_h + \mu_h} \tag{17}$$

(13) $\Rightarrow$
$$I^{n+1} = \frac{\alpha_h X^{n+1} - \sum_{j=1}^{n+1} C_j^{\gamma_3} I^{n+1-j}}{C_0^{\gamma_3} + r + \mu_h} \tag{18}$$

(14) $\Rightarrow$
$${X_v}^{n+1} = \frac{\beta_v I^{n+1} N_T - \beta_v I^{n+1} N_T {I_v}^n - \sum_{j=1}^{n+1} C_j^{\gamma_4} {X_v}^{n+1-j}}{C_0^{\gamma_4} + \beta_v I^{n+1} N_T + \alpha_v + \mu_v} \tag{19}$$

(15) $\Rightarrow$
$${I_v}^{n+1} = \frac{\alpha_v {X_v}^{n+1} - \sum_{j=1}^{n+1} C_j^{\gamma_5} {I_v}^{n+1-j}}{C_0^{\gamma_5} + \mu_v} \tag{20}$$

with $C_0^{n_1} = \left(\frac{e^{\mu_h h} - 1}{\mu_h}\right)^{-\gamma_1}$, $C_0^{n_2} = \left(\frac{e^{(\alpha_h + \mu_h)h} - 1}{(\alpha_h + \mu_h)}\right)^{-\gamma_2}$, $C_0^{n_3} = \left(\frac{e^{(r+\mu_h)h} - 1}{r + \mu_h}\right)^{-\gamma_3}$,

$C_0^{n_4} = \left(\frac{e^{(\alpha_h + \mu_h)h} - 1}{(\alpha_h + \mu_h)}\right)^{-\gamma_4}$ and $C_0^{n_5} = \left(\frac{e^{\mu_v h} - 1}{\mu_v}\right)^{-\gamma_5}$

5.1 Numerical Experiments

Analytical studies permanently remain unfinished without numerical authentication of the outcomes. In this unit, we present numerical simulation to exemplify the outcomes attained in previous sections. Now we solve the fractional-order dengue model in two cases with step size $h = 1.1$ and $h = 2.4$. The guestimate elucidations are revealed in Fig. 2-19, for various values of $0 < n_i \leq 1$, $i = 1, \ldots, 5$. Numerical experiments are performed using values of parameters given in Table 2.

Table 2. Different parameters & values.

Parameter	Value ($\mathbf{day^{-1}}$)
N_T	5,000
α_h	1/5
β_h	0.00005
μ_h	0.0000391
α_v	1/10
β_v	0.00008
μ_v	1/14
r	1/14
C(DFE)	3.00
C(EE)	300

6. RESULTS AND DISCUSSION

The numerical modeling of transmission dynamics of Fractional Order dengue disease with incubation period of virus has been analysed in this paper. The model has two equilibrium points, i.e. disease free equilibrium (DFE) and endemic equilibrium (EE). An unconditionally convergent non-standard finite difference numerical model with GL coefficients has been constructed and numerical experiments are performed for different values of discretization parameter h. Fig. (2-10) shows the graphs of disease free equilibrium with step size $h = 1.1$ and Fig. (11-19) with step size $h = 2.4$. Numerical Simulations reveals that all values approaches to equilibrium point. In order to observe the effects that the parameter γ has on the dynamics of the fractional-order model (4), we conclude several numerical simulations varying the value of parameter. These simulations reveal that a change of the value γ affects the dynamics of the epidemic. For example, Fig. 2, 11 shows that for lower values of γ, the epidemic peak is wider and lower from the true equilibrium points, Fig. 4, 6, 8, 10, 13, 15, 17, and 19 show that for lower values of γ, the epidemic peak is wider and higher for true equilibria. This feature is important from an epidemiological point of view since its interpretation shows a longer period in which infected & infectious individuals can affect the health system. Fig. 2-19 show that the model presented here gradually approaches the steady state for different values of γ but the dynamics of the model is governed by the distinct paths.

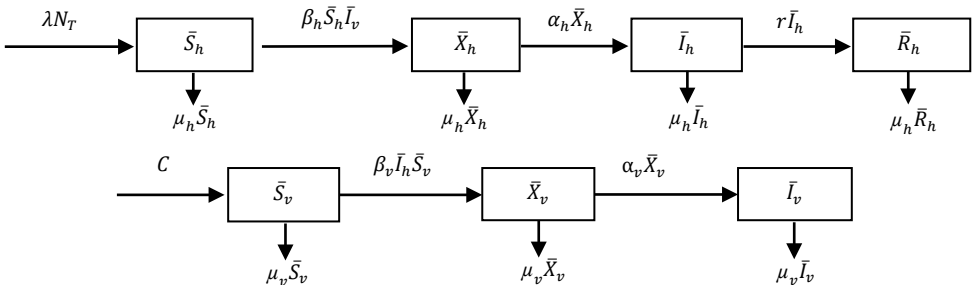

Fig.1. Flow diagram.

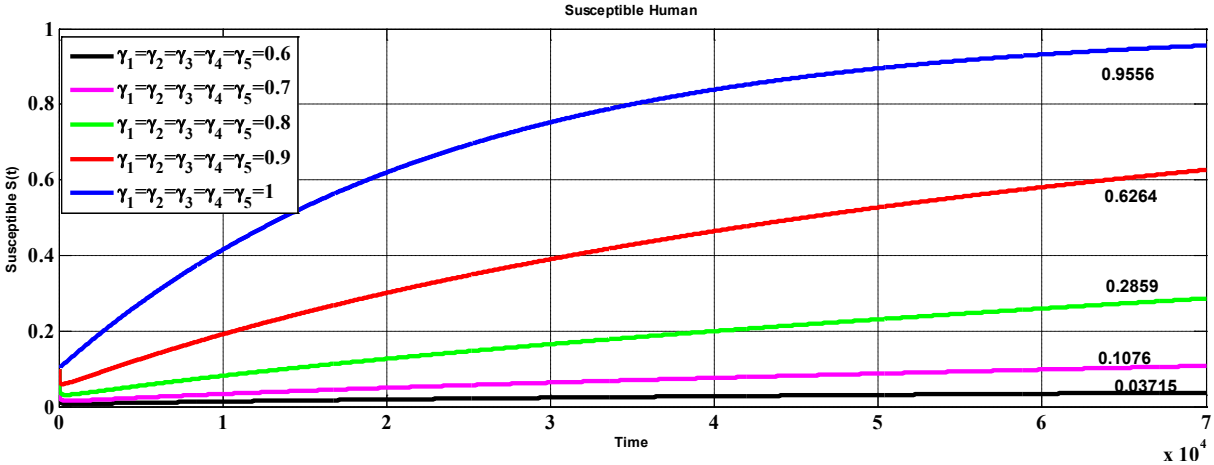

Fig. 2. The susceptible humans for different values of γ *'s* with $h = 1.1$.

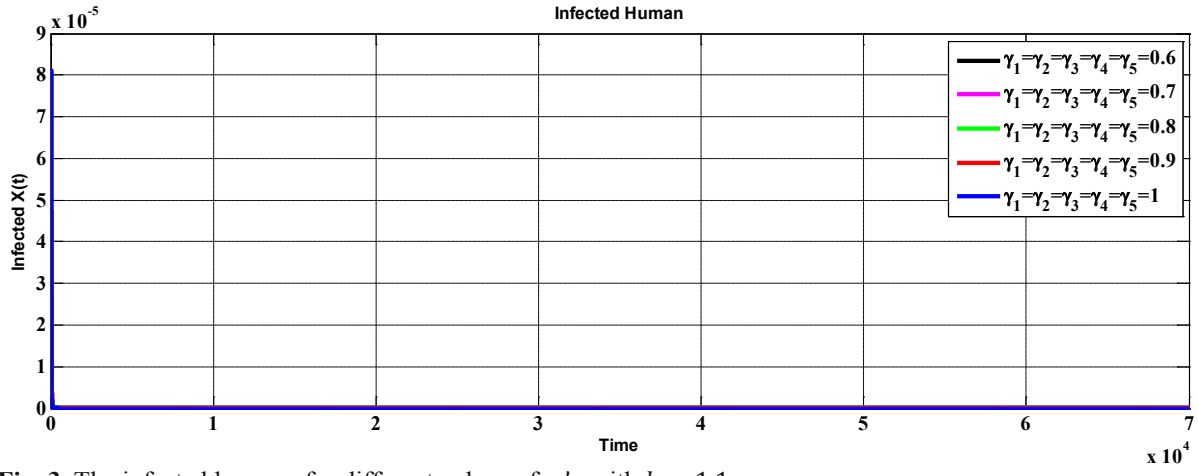

Fig. 3. The infected humans for different values of γ 's with $h = 1.1$.

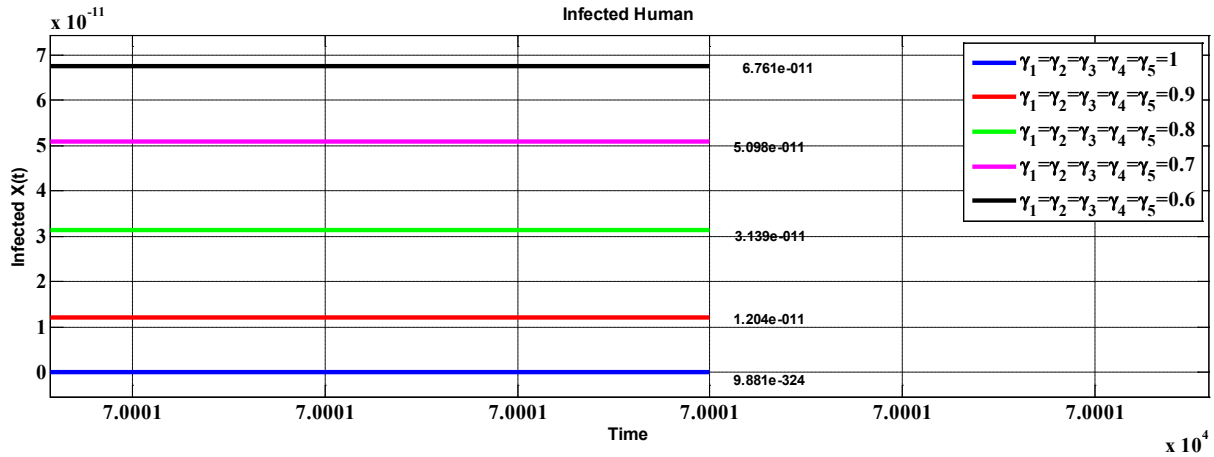

Fig. 4. The infected humans, in zoom.

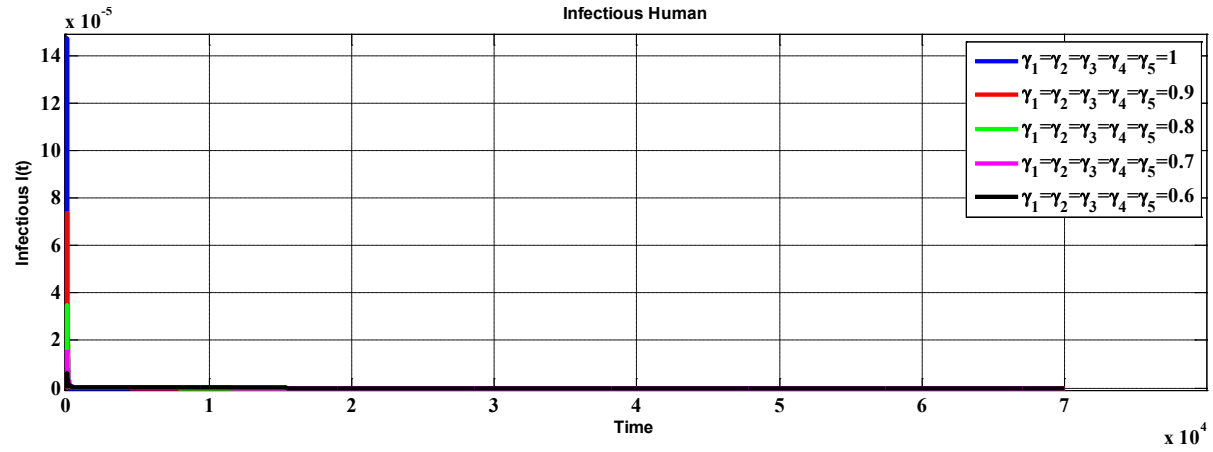

Fig. 5. The infectious humans for different values of γ 's with $h = 1.1$.

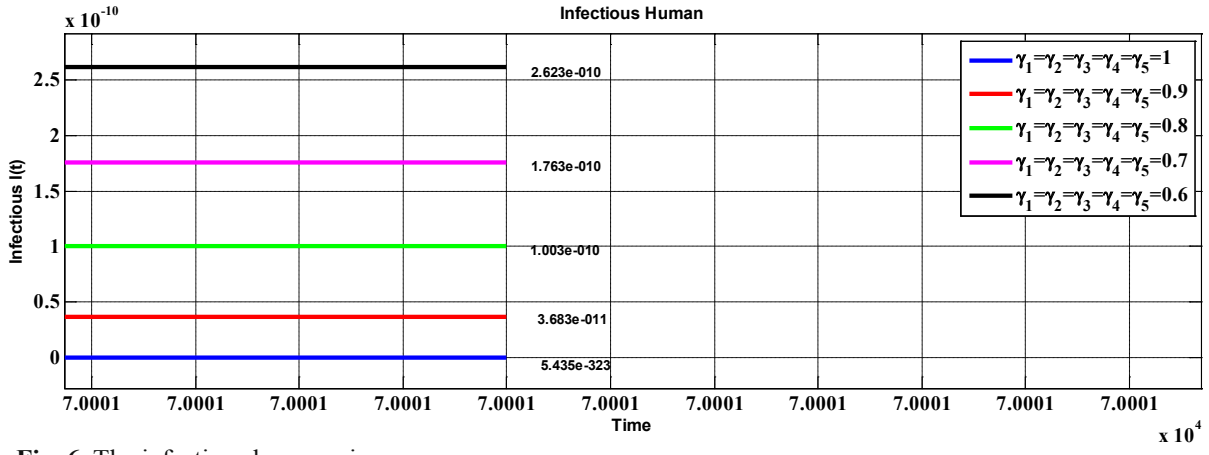

Fig. 6. The infectious humans, in zoom.

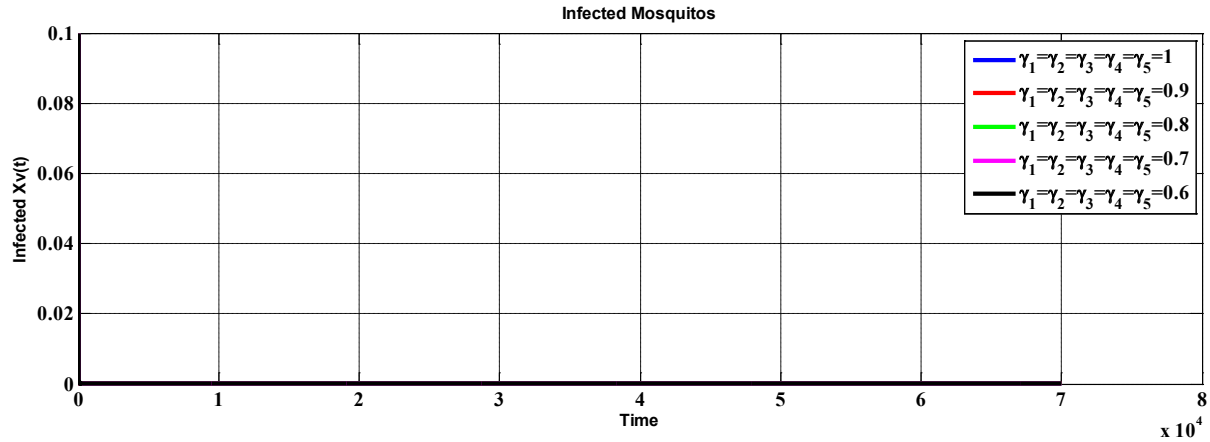

Fig. 7. The graph of infected mosquitoes for different values of γ *'s* with $h = 1.1$.

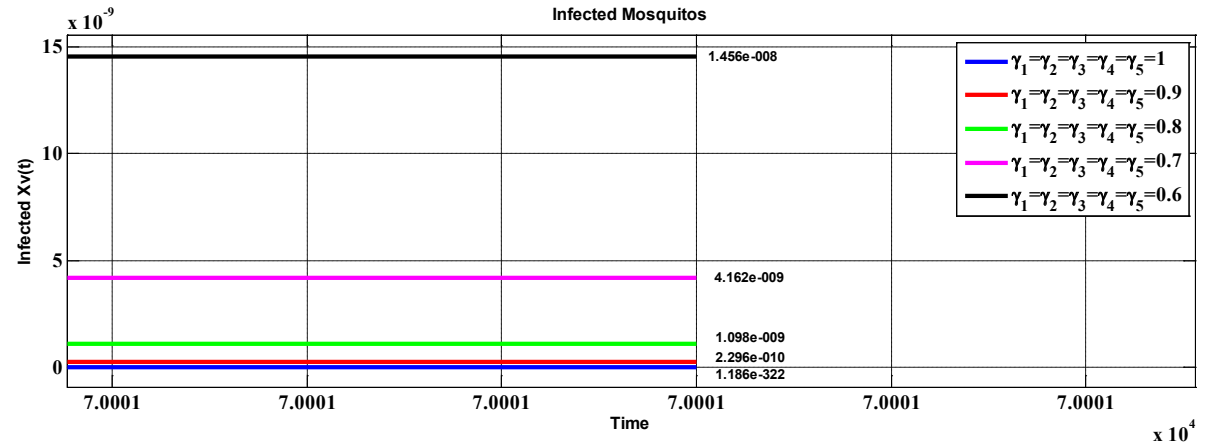

Fig. 8. The infected mosquitoes, in zoom.

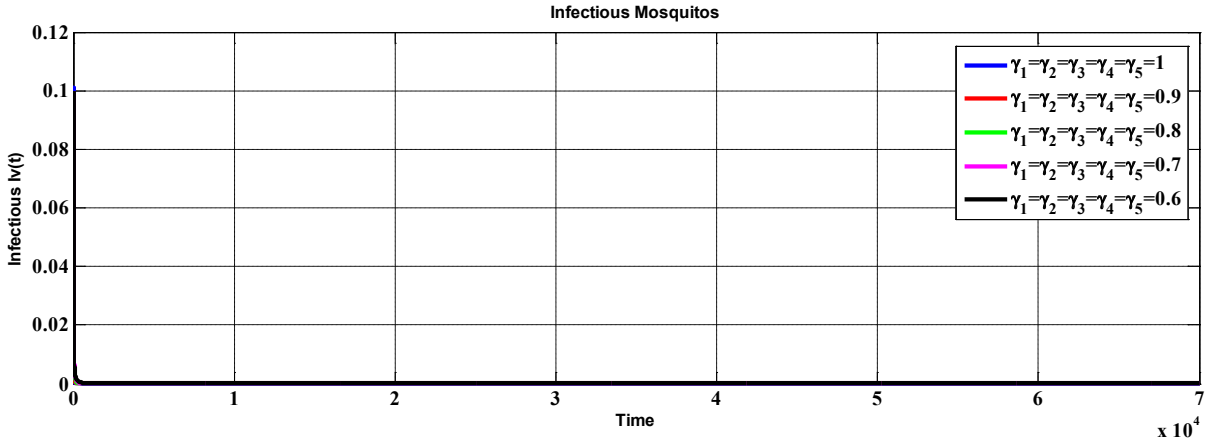

Fig. 9. The infectious mosquitoes for different values of γ 's with $h = 1.1$.

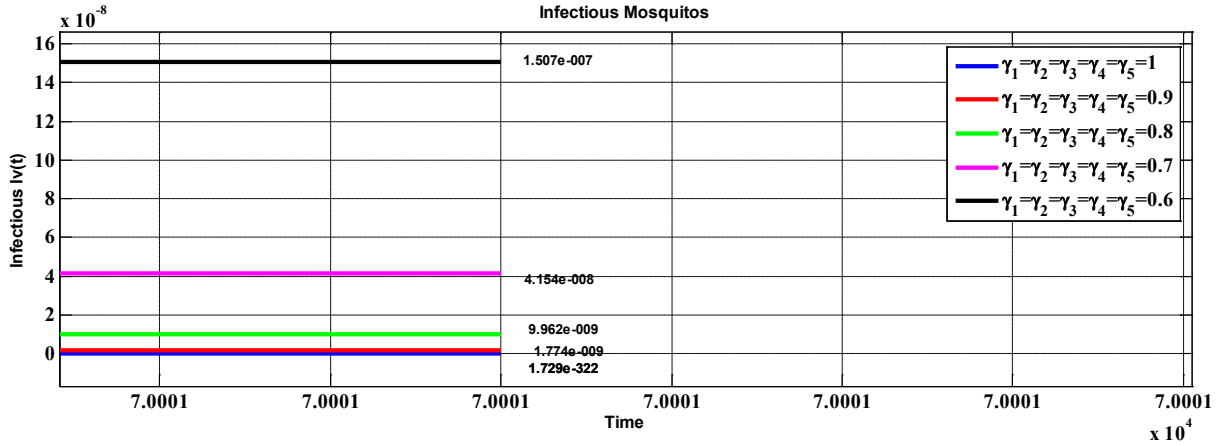

Fig. 10. The infectious mosquitoes in zoom.

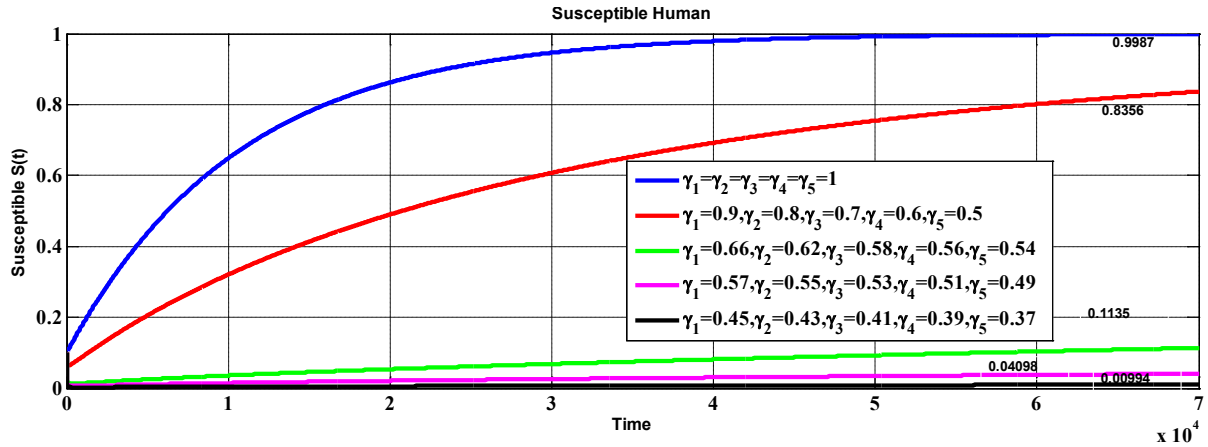

Fig. 11. The susceptible humans for different values of γ 's with $h = 2.4$.

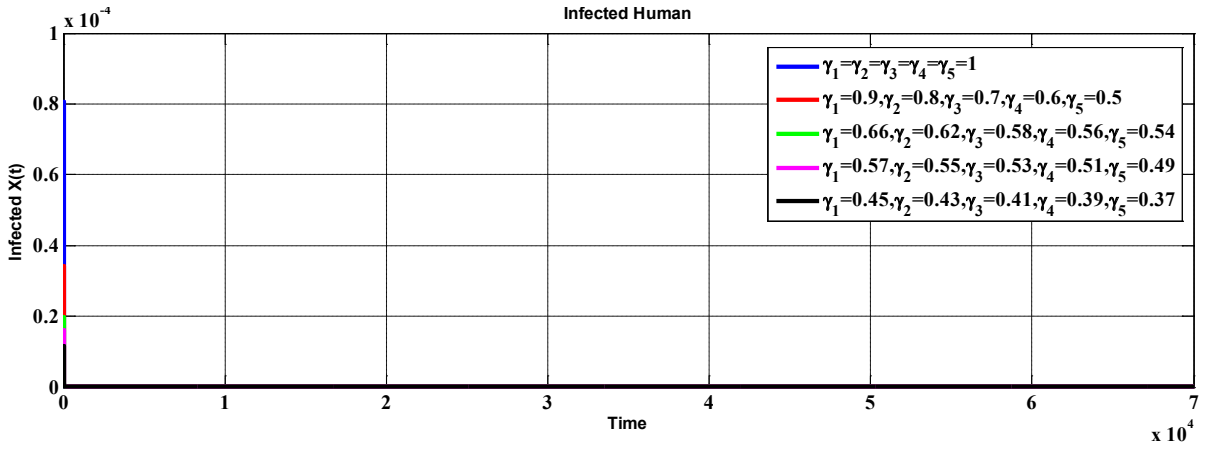

Fig. 12. The infected humans for different values of γ *'s* with $h = 2.4$.

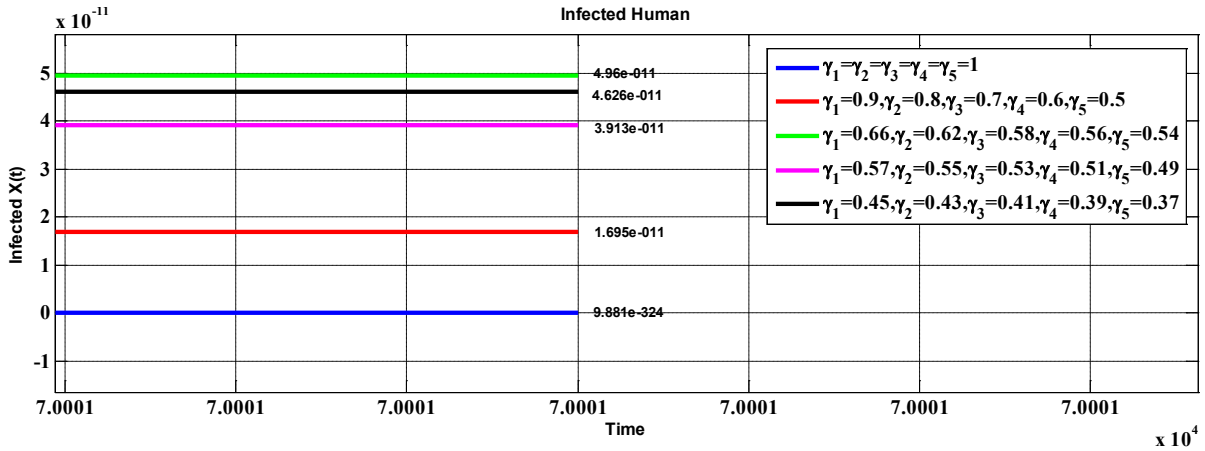

Fig. 13. The infected humans, in zoom.

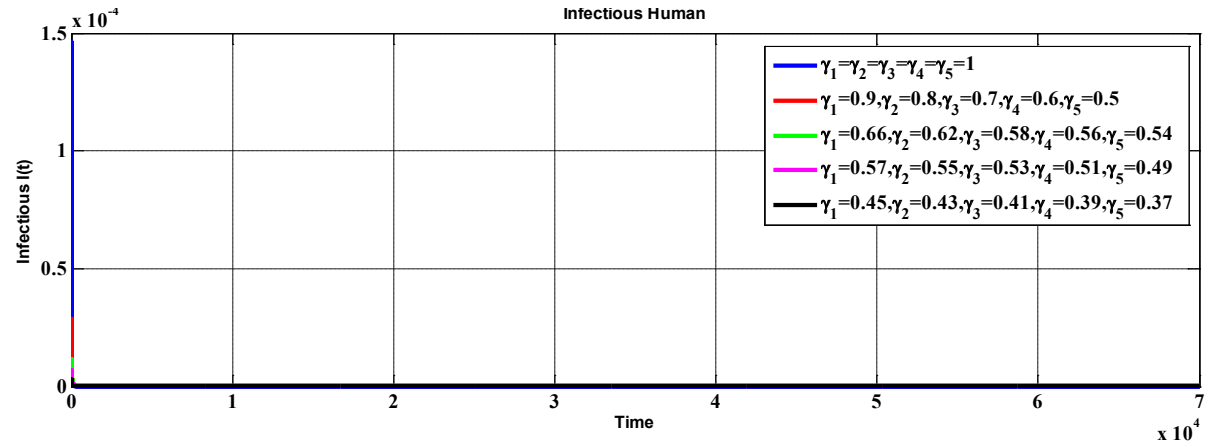

Fig. 14. The infectious humans for different values of γ *'s* with $h = 2.4$.

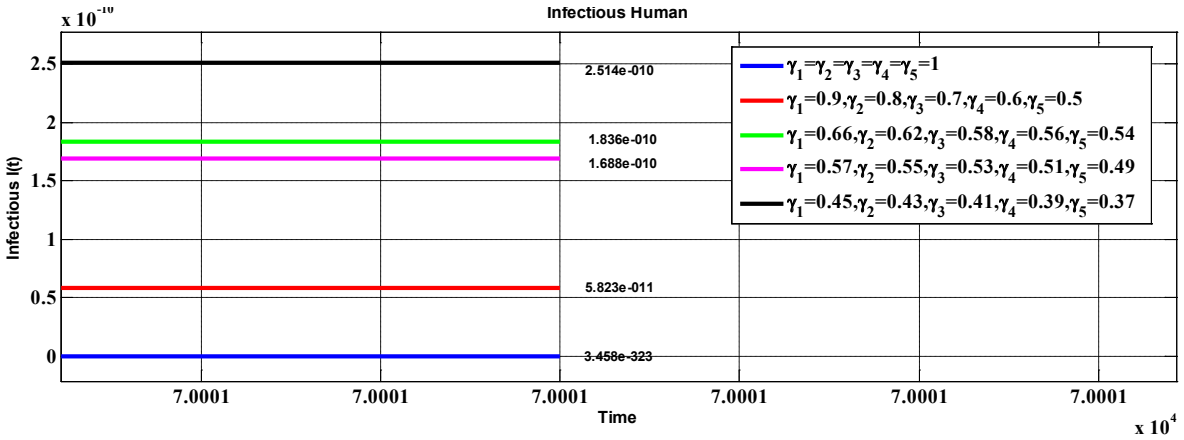

Fig. 15. The infectious humans, in zoom.

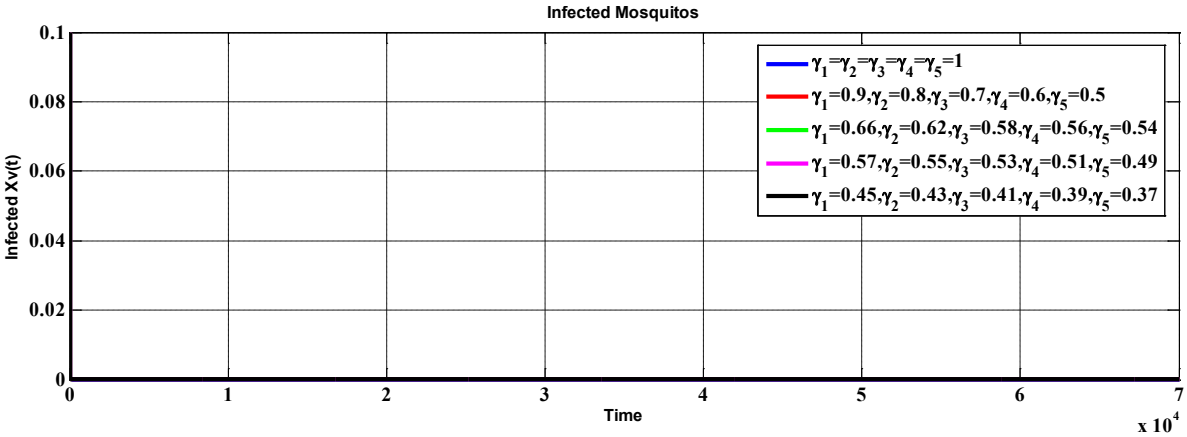

Fig. 16. The infected mosquitoes for different values of γ *'s* with $h = 2.4$.

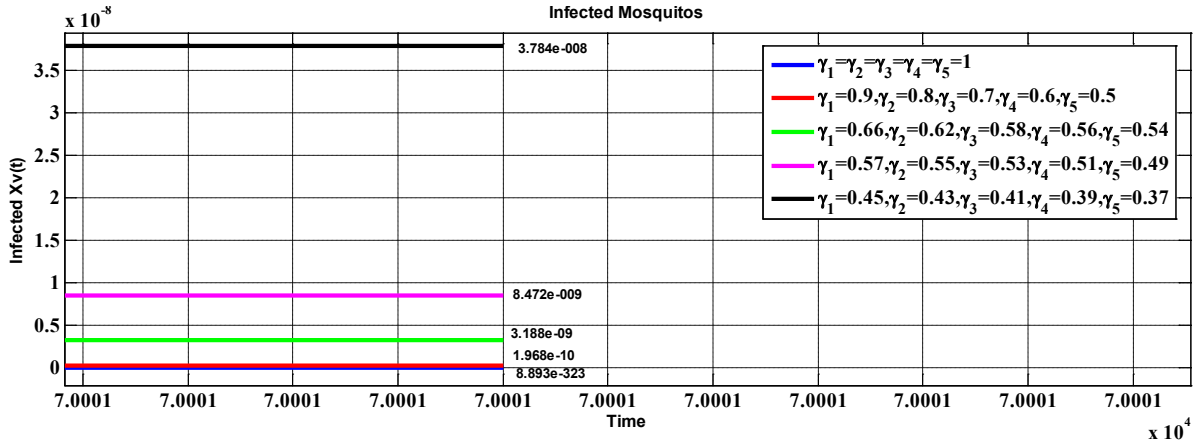

Fig. 17. The infected mosquitoes, in zoom.

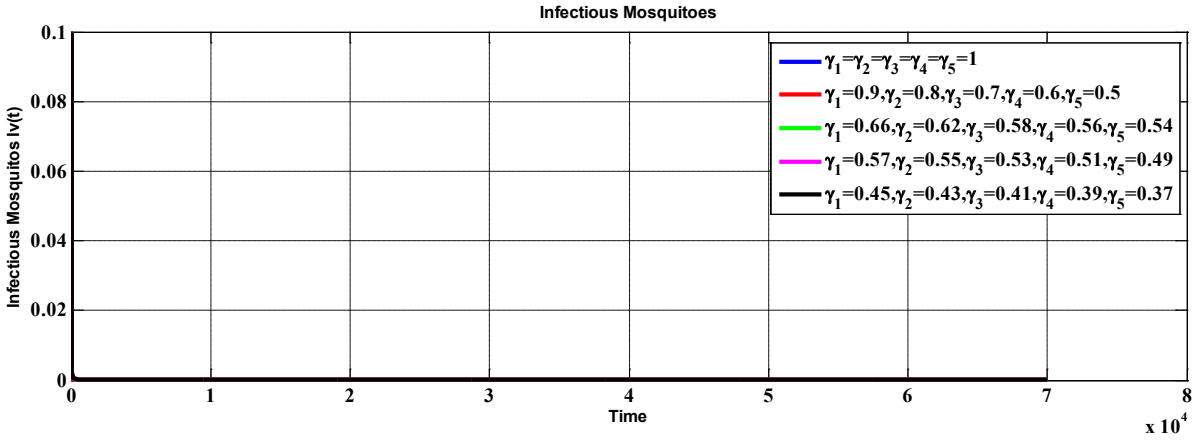

Fig. 18. The infectious mosquitoes for different values of γ 's with $h = 2.4$.

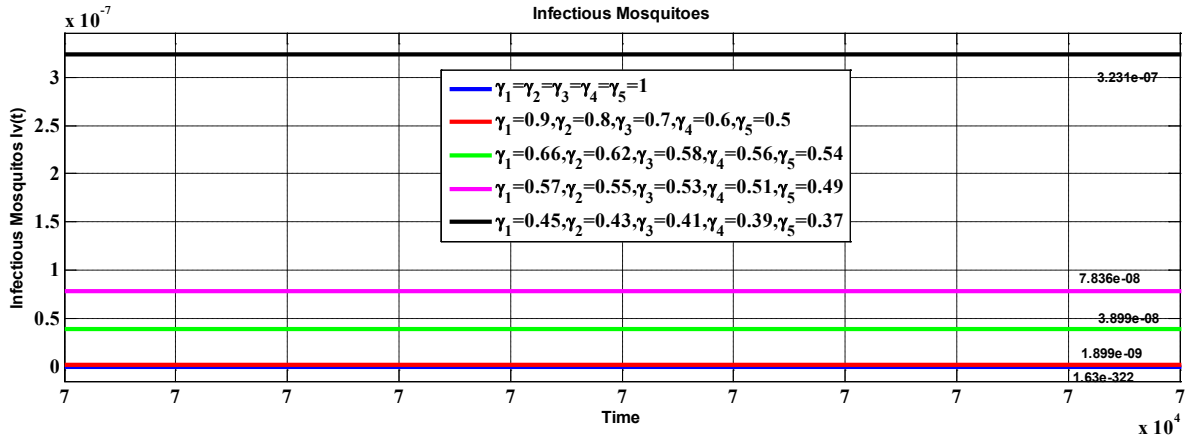

Fig. 19. The infectious mosquitoes, in zoom.

6. CONCLUSIONS

Describing the reality through a mathematical model, usually a system of differential equations, is hard task that has an inherent compromise between simplicity and accuracy. In this article we studied the fractional-order dengue model. It turns out that, in general, this classical model does not provide enough good results. In order to get better results, that fit the reality, fractional order system is needed. It allows us to model memory effects, and result in a more powerful approach to epidemiological models. Our investigation show that even a simple fractional model may give surprisingly good results. From the obtained results from the presented figures, it turns out that the results are non-negative because susceptible, infected and recovered can never be less than zero. Then the local stability analysis of the model in fractional order is presented. The results obtained are in agreement with the numerical simulations attained from the graph.

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