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Research Article

Single-Tile Semigroups of Shift Operators

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Abstract: We introduce some new subsemigroups of the finite full transformation semigroups T_n. We consider various irregular boards of different shapes and sizes to generate subsemigroups of T_n using the four idempotent operators *L, R, U, D*. These operators shift tiles on a board in four different directions (left, right, up and down) in their respective rows and columns. In this way each operator $O \in G = \{L, R, U, D\}$ defines a member of T_n on the base set $X = \{1, 2, \ldots, n\}$ and the semigroups of various properties are given as S [redacted] T_X.

Keywords and phrases: Finite semigroups, shift operators, transformation semigroups, $\mathcal{D}$-classes

1. INTRODUCTION AND PRELIMINARIES

We will consider different shaped boards consisting of tiles. We assume that the tiles of the board can be dragged in four different directions (left, right, up and down) by introducing the four idempotent operators *L, R, U, D* on the board. We denote the operator *L(T) = L*, as that which moves the tile *T* from co-ordinates (i, j) to (i', j) that can be reached by sliding *T* to the left until it encounters a barrier. (It is assumed that each row and column has barriers at the edges of the boards and perhaps elsewhere as well.) In a similar way we define the right operator *R*, while the up operator *U* and the down operator *D* acts on the second co-ordinate in an entirely analogous fashion.

In this way each operator $O \in G = \{L, R, U, D\}$ defines a member of the full transformation Semigroup "Ganyushkin and Mazorchuk[5]" on the base set $X = \{1, 2, \ldots, n\}$ and put S [redacted] T_X. We wish to study this finite semigroup, which is generatedby a pair of disjoint two-element [redacted]-zero subsemigroups $R_h = \{L, R\}$ and $R_v = \{U, D\}$. (The subscript h, v stands respectively for horizontal and vertical.) A similar finite subsemigroup of T_n on rectangular $m \times n$ bi-coloured board B with $2 \le m \le n$ with m rows and n columns has been discussed recently in "Ahmad [1, 6]". We will apply the operators from right to left so the operator *UD = UD(B)* will mean, first operate *D* followed by *U*.Before going into details of the semigroups, we will define some basic notions. For undefined semigroup terminology we will refer to "Peter [2], Howie [3], Ganyushkin and Mazorchuk [5]".

An element a of a semigroup S will be called idempotent if $a^2 = a$. A non-emptysubset A of S is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$ and an (two-sided) ideal if it is both a left and right ideal. If a is an element of a semigroup S, the smallest left ideal of S containing a is $aSa \cup \{a\}$, denoted by S^1a and will be called the principal left ideal generated by a. For any set X, the full transformation Semigroup $(T_X, \circ)$ is the semigroup of all mappings from X into X under the operation of composition of mappings. An element*a*of a semigroup S is called regular if there exists an element x

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in S such that $a = axa$; S is called regular if all its elements are regular. The set of all regular elements of S is denoted by Reg(S). A semigroup S is calleda rectangular band if $aba = a$ for all a, b in S. The term band is used in general for a semigroup consisting of idempotents. A commutative band is called a semilattice. Any band is a semilattice of rectangular bands "Howie[3,Theorem 4.4.1]". For $a \in S$, we say that a is an inverse of a if $aa\,a = a$ and $a\,aa = a$. The set of all inverses of an element a in S is denoted by $V(a)$. Obviously every idempotent e is regular $(e = eee)$ and every regular element a has an inverse.

The equivalence $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if and only if a and b generate the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly the equivalence $\mathcal{R}$ is defined by the rule that $a\mathcal{R}b$ if and only if $aS^1 = bS^1$. The $\mathcal{H}$ relation is defined to be the intersection of the $\mathcal{L}$ and $\mathcal{R}$ relations.The join $\mathcal{L} \vee \mathcal{R}$ of $\mathcal{L}$ and $\mathcal{R}$ will be called the $\mathcal{D}$-relation. Similarly, since the principal two-sided ideal of S generated by a is S^1aS^1, we can definethe equivalence $\mathcal{J}$ by the rule that $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$. It is immediate that $\mathcal{L} \subseteq \mathcal{J}$ and $R \subseteq J$. Hence, since $\mathcal{D}$ is the smallest equivalence containing $\mathcal{L}$ and $\mathcal{R}$, we must have $\mathcal{D} \subseteq \mathcal{J}$. For finite semigroups and for any T_X, we have the equality $\mathcal{D} = \mathcal{L}$ as appears in "Howie [3, Proposition 2.1.4]".

2. MATERIALS AND METHODS

We will use different types of boards, like L-shape, T and Y- shape boards consisting of n tiles to introduce various finite subsemigroups of the finite semigroups T_n.

3. RESULTS AND DISCUSSION

One-Tile Semigroup that is Regular but not a Band

Here, we will consider an L-shaped one-tile board of four cells that will be labelled {1, 2, 3, 4}. We will assume that this board has one moving cell that will be called its tile.

Definition 1. A board B will be called convex if we can travel between any two cells by a horizontal and then a vertical movement (in some order).

Let B be the L-shaped board as in the following Fig. 1.

The board B in the Fig. 1 has four cells with base set {1, 2, 3, 4}. The *L, R, U, D* operators, as defined in the previous section act on the moveabletile $T \in \{1, 2, 3, 4\}$ as follows:

3.1. Calculations of Presentation for the L-semigroup.

For this type of calculations, we follow the method as outlined by "Pin [4, Section 3.1]" whichguarantees to produce a presentation of a semigroup S (but the presentation maycontain redundant relations).

The Actions of Operators of Length 1

The actions of operators *L, R, U,* and *D* on the L-shaped board are given as follows:

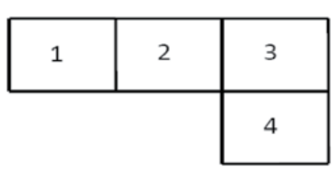

Fig. 1. L-shaped board with a single movable tile.

Operators	Tiles			
	1	**2**	**3**	**4**
L	1	1	1	4
R	3	3	3	4
U	1	2	3	3
D	1	2	4	4

Note that the operator *L* moves the tiles 1, 2, 3, 4 of the board B respectivelytowards 1, 1, 1, 4.

The Actions of Operators of Length 2

We calculate successively (1). *LL* to (16).*DD* and list the relatedrelations as in the following table.

Operatros	Tiles				Relations	
	1	**2**	**3**	**4**		
L	1	1	1	4	*(1) LL = L*	*(15) DU = D*
R	3	3	3	4	*(2) LR = L*	*(16) DD = D*
U	1	2	3	3	*(5) RL = R*	*(21) LUL = LU*
D	1	2	4	4	*(6) RR = R*	*(22) LDR = DR*
(3) LU	1	1	1	1	*(10) UR = RU*	*(28) RDR = DR*
(4) LD	1	1	4	4	*(11) UU = U*	*(29) ULU = LU*
(7) RU	3	3	3	3	*(12) UD = U*	*(38) DRD = DR*
(8) RD	3	3	4	4	*(13) DL = L*	
(9) UL	1	1	1	3		
(14) DR	4	4	4	4		
(30)ULD	1	1	3	3		

The Actions of Operators of Length 3

We calculate successively the actions of operators of length 3from (17) to (40) as (17) *LLU*, (18) *LLD*, (19) *LRU*, ... (39) *DU L*, (40) *DDR*.

The relations already known enable us to avoid the calculation of (17), . . . , (20),(22), . . . , (28), (32), . . . , (37), (39), (40), (42), (43), (44) since *LLU* = *LU,LLD* = *LD, LRU* = *LU* and so on.

We now have the following representation for the semigroup generated by the idempotents *L, R, U, D*.

$$S_1 = \langle L, U, R, D: L^2 = L, R^2 = R, U^2 = U, D^2 = D, LR = L, RL = R, UR = RU, DL = L,$$

$$DU = D, U D = U, U LU = LU, LU L = LU, LDR = DR, DRD = DR, RD R = DR\rangle$$

Removing the redundancies we have,

$$S_1 = \langle L, U, R, D : LR = L, RL = R, UR = RU, DL = L, DU = D,$$

$$U D = U, U LU = LU, LU L = LU, LDR = DR, DRD = DR, R DR = DR\rangle .$$

As in the case of previous boards, we can show that $|S_1| = |S| = 11$ and so $S_1 = S$. Now we have Table 1.

Table 1.

	L	*R*	*U*	*D*	*UL*	*RU*	*DR*	*LU*	*LD*	*RD*	*ULD*
L	*L*	*L*	*LU*	*LD*	*LU*	*LU*	*DR*	*LU*	*LD*	*LD*	*LU*
R	*R*	*R*	*RU*	*RD*	*RU*	*RU*	*DR*	*RU*	*RD*	*RD*	*RU*
U	*UL*	*RU*	*U*	*U*	*UL*	*RU*	*RU*	*LU*	*ULD*	*RU*	*ULD*
D	*L*	*DR*	*D*	*D*	*L*	*DR*	*DR*	*LU*	*LD*	*DR*	*LD*
UL	*UL*	*UL*	*LU*	*ULD*	*LU*	*LU*	*RU*	*LU*	*ULD*	*ULD*	*LU*
RU	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*
DR	*DR*	*DR*	*DR*	*DR*	*DR*	*DR*	*DR*	*DR*	*DR*	*DR*	*DR*
LU	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*
LD	*L*	*DR*	*LD*	*LD*	*L*	*DR*	*DR*	*LU*	*LD*	*DR*	*LD*
RD	*R*	*DR*	*RD*	*RD*	*R*	*DR*	*DR*	*RU*	*RD*	*DR*	*DR*
ULD	*UL*	*RU*	*ULD*	*ULD*	*UL*	*RU*	*RU*	*LU*	*ULD*	*RU*	*ULD*

Note that each element is idempotent except *UL&RD*, where $(UL)^2 = LU$ and $(RD)^2 = DR$, hence S is not an orthodox. However, it can be checked that *UL* and *RD* are still regular, since $(UL)D(UL) = UL$ and $(RD)L(RD) = RD$.

3.2. D-classes for the L-semigroup

The D-classes are given as follows:

$\mathcal{D}_1$:

U
D

$\mathcal{D}_2$:

L	*LD*
R	*RD*
UL	*ULD*

$\mathcal{D}_3$:

RU
DR
LU

4. OPERATIONS ON A T-SHAPED BOARD

Now we consider a T-shaped 4-cell board with one moveable cell with the base set {1, 2, 3, 4} as in the following Fig. 2.

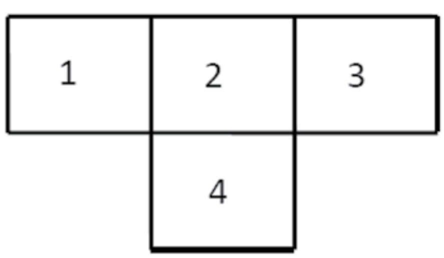

Fig. 2. T-shaped board with a single movable tile.

The *L, R, U, D* operators then act as follows and generate an idempotent-generated semigroup that is not a band but still is regular and furthermore it is conventional. The operators are given as follows:

4.1. Calculations of Presentation for the T-semigroup

The Actions of Operators of Length 1: The actions of the operators *L, R, U*, and *D* are given as

OPERATORS	TILES			
	1	**2**	**3**	**4**
L	1	1	1	4
R	3	3	3	4
U	1	2	3	2
D	1	4	3	4

The Actions of Operators of Length 2: We calculate successively *(1). LL, (2). LR, up to (16).DD* as follows.

Operator	Tiles				Relations
	1	**2**	**3**	**4**	
L	1	1	1	4	***(1) LL = L***
R	3	3	3	4	*(2) LR = L*
U	1	2	3	2	*(5) RL = R*
D	1	4	3	4	*(6) RR = R*
(3) LU	1	1	1	1	*(11) U U = U*
(4) LD	1	4	1	4	*(12) U D = U*
(7) RU	3	3	3	3	*(13) DL = L*
(8) RD	3	4	3	4	*(14) DR=R*
(9) UL	1	1	1	2	*(15) DU = D*
(10) UR	3	3	3	2	*(16) DD = D*

The Actions of Operators of Length 3: We continue to calculate successively from (17) to (40) as (17) *LLU*, (18) *LLD*, (19) *LRU*, ..., (39*) DU L*, (40) *DU R*.

The known relations enable us to avoid the calculations of (17), . . . , (20),(23), . . . (26), (33), . . . , (40) since *U LL = U L, DLL = DL, URL = U R* andso on. The continuation of the calculations, then gives

Calculaitons of Length 3

OPERATORS	TILES				RELATIONS	
	1	2	3	4		
L	1	1	1	4	*(1) LL = L*	*(15) DU = D*
R	3	3	3	4	*(2) LR = L*	*(16) DD = D*
U	1	2	3	2	*(5) RL = R*	*(21) LUL = LU*
D	1	4	3	4	*(6) RR = R*	*(22) LUR = LU*
(3) LU	1	1	1	1	*(11) UU = U*	*(27) RUL = RU*
(4) LD	1	4	1	4	*(12) UD = U*	*(28) RU R = RU*
(7) RU	3	3	3	3	*(13) DL = L*	*(29) ULU = LU*
(8) RD	3	4	3	4	*(14) DR = R*	*(31) URU = RU*
(9) UL	1	1	1	2		
(10) UR	3	3	3	2		
(30) ULD	1	2	1	2		
(32) URD	3	2	3	2		

The Actions of Operators of Length 4: No more new calculations for length 4 are possible as $LU\,L = LU = LU\,R$ and $RU\,L = RU = RU\,R$, etc.is avoiding the new entries.

We now have the following representation for the semigroup of the **T**-shaped board.

$$S_1 = \langle L, U, R, D : L^2 = L, R^2 = R, U^2 = U, D^2 = D, LR = L, RL = R, DR = R, DU = D, DL = L, UD = U, RU\,L = RU, ULU = LU, LU\,L = LU, LU\,R = LU, RU\,R = RU, U\,RU = RU \rangle$$

Again, on removing the redundancies we have

$$S_1 = \langle L, U, R, D : LR = L, RL = R, DR = R, DU = D, U\,D = U, DL = L, RU\,L = RU, ULU = LU, LU\,L = LU, LU\,R = LU, RU\,R = RU, URU = RU \rangle$$

Using these relations and generators we have Table 2 for the semigroup S of the **T**-shaped board having one moveable tile under the shift operators

Table 2.

	L	*R*	*U*	*D*	*UL*	*UR*	*LU*	*RU*	*LD*	*RD*	*ULD*	*URD*
L	*L*	*L*	*LU*	*LD*	*LU*	*LU*	*LU*	*LU*	*LD*	*LD*	*LU*	*LU*
R	*R*	*R*	*RU*	*RD*	*RU*	*RU*	*RU*	*RU*	*RD*	*RD*	*RU*	*RU*
U	*UL*	*UR*	*U*	*U*	*UL*	*UR*	*LU*	*RU*	*ULD*	*URD*	*ULD*	*URD*
D	*L*	*R*	*D*	*D*	*L*	*R*	*LU*	*RU*	*LD*	*RD*	*LD*	*RD*
UL	*UL*	*UL*	*LU*	*ULD*	*LU*	*LU*	*LU*	*LU*	*ULD*	*ULD*	*LU*	*LU*
UR	*UR*	*UR*	*RU*	*URD*	*RU*	*RU*	*RU*	*RU*	*URD*	*URD*	*RU*	*RU*
LU	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*	*LU*
RU	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*	*RU*
LD	*L*	*L*	*LD*	*LD*	*L*	*L*	*LU*	*LU*	*LD*	*LD*	*LD*	*LD*
RD	*R*	*R*	*RD*	*RD*	*R*	*R*	*RU*	*RU*	*RD*	*RD*	*RD*	*RD*
ULD	*UL*	*UL*	*ULD*	*ULD*	*UL*	*UL*	*LU*	*LU*	*ULD*	*ULD*	*ULD*	*ULD*
URD	*UR*	*UR*	*URD*	*URD*	*UR*	*UR*	*RU*	*UR*	*URD*	*URD*	*URD*	*URD*

Clearly the table shows that the elements UL and UR are not idempotents but are regular as

$$U\,L = U\,L(LD)U\,L$$

$$U\,R = U\,R(LD)U\,R$$

4.2. $\mathcal{D}$-classes for the T-semigroup.

The $\mathcal{D}$-classes for the T-shaped board semigroup are given as follows:

$\mathcal{D}_1$:

U
D

$\mathcal{D}_2$:

L	*LD*
R	*RD*
UL	*ULD*
UR	*URD*

$\mathcal{D}_3$:

LU
RU

The **L** and **T**-shaped boards semigroups are not orthodox. However, the **T**-semigroup is an example of a finite conventional semigroup that is notorthodox as follows by the following result.

Proposition 2. The **T**-semigroup associated with the **T**-shaped board is conventional.

Proof. To show the single tile semigroup T is conventional, we need to show that aea is always idempotent (as we know that T is regular but not orthodox). However, there are only two non-idempotent elements, $U L$ and $U R$, and since the tile has left-right symmetry, we only need to check that there is no factorization of UL of the form aea .

Suppose that $UL = aea$. The Table 2 shows this is possible only if a is a member of the set $X = \{U, U\ L, U\ LD\}$ and a is a member of the set $Y = \{L, R, U\ L, U\ R\}$. On the other hand, since $V(U) = U$, $V(U L) = \{LD, RD\}$ and $V(U LD) = U LD$. Clearly the inverses of each element of the set X has an empty intersection with Y. This shows that no factorization of the from, $UL = ata$ (for any $t \in S_1$) is possible. Hence T is conventional Semigroup that is not orthodox.

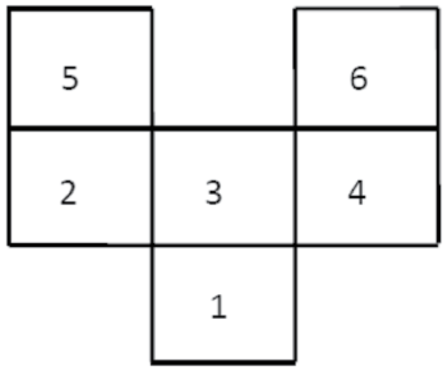

Fig. 3. Y-shaped board with a single movable tile.

Definition 3. A semigroup is called E-solid if the idempotents e, f, g satisfy, $e\mathcal{R}f$ and $f\mathcal{L}g$, then there is an idempotent h such that $e\mathcal{L}h$ and $h\mathcal{R}g$.

T. E. Hall, recently asked (personal communication) if conventional semigroups are E-solid. The answer is indeed 'no' as T is not E-solid: R, RD, $ULD \in E(S)$ but $UL \notin E(S)$.

5. IRREGULAR ONE-TILE SEMIGROUP

We will include here another special kind of one-tile board of the shape **Y** consisting of six cells on the base set {1, 2, 3, 4, 5, 6}. We will assume again that the board has a single movable tile.

The L, R, U, D operators on the **Y**-shaped board are defined as follows:

$$L(B) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 2 & 5 & 6 \end{pmatrix} \qquad R(B) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 4 & 4 & 5 & 6 \end{pmatrix}$$

$$U(B) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 3 & 6 & 5 & 6 \end{pmatrix} \qquad D(B) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 4 & 2 & 4 \end{pmatrix}$$

$$LU(B) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 2 & 6 & 5 & 6 \end{pmatrix}$$

Proposition 4. The semigroup S generated by the operators on the **Y**- shaped board is irregular.

Proof. $LU\ (1) = L(2) = 2$. Let $W \in S^1$. Then $W(2) \in \{2, 4, 5, 6\}$. Hence

$(LU)\,W(LU)(1) = LUW(2) \subseteq LU\ \{2, 4, 5, 6\} = L\{5, 6\} = \{5, 6\}$. Since $2 \notin \{5, 6\}$ it follows that $(LU)\,W(LU) = LU, \forall W \in S^1$. Hence $LU \notin Reg(S)$.

Remark 5. We observe that the semigroups associated with the convex boards, for example, any **T, L**-shaped boards are all regular while the semigroup of **Y**-shaped board is irregular. Hence we conjecture that if the board is convex then the associated semigroup is regular.

Definition 6. A semigroup S is called aperiodic semigroup if all subgroups of S are trivial.

Conjecture. The one-tile semigroup on a board with no internal barrier is aperiodic.

We include an example of a non-aperiodic semigroup when internal barriers are allowed. In the Fig. 4 hard lines represent the barriers.

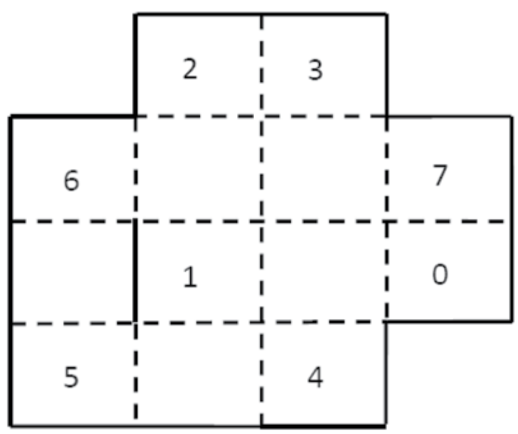

Fig. 4. A board with an internal barrier.

Since an internal barrier to the left of square 1 exists. We have

$$DRUL(0) = DRU(1) = DR(2) = D(3) = 4;$$
$$DRU\,L(4) = DRU(5) = DR(6) = D(7) = 0.$$

Hence, $DRU\,L(0) = 4$, $DRU\,L(4) = 0$ and so, $S = DRU\,L$ contains a nontrivial subgroup. Whence, the semigroup of this board is not aperiodic.

6. ONE-TILE SEMIGROUPS CORRESPONDING TO RECTANGLE OF ANY SIZE ARE ISOMORPHIC

Theorem 7. One-tile semigroups corresponding to any $m \times n,\ (2 \leq m, n)$ rectangle are isomorphic

Proof. Let $2 \leq m, n$. Consider a one-tile $m \times n$ rectangular board B as follows;

$B = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$.

We list the action of the four operators *L, R, U, D* on the board B as under;

$L(i, j) = (i, 1) = LL(i, j), i \leq m; \quad R(i, j) = (i, n) = RR(i, j), i \leq m;$

$U\ (i, j) = (1, j) = U\ U\ (i, j), j \leq n; \quad D(i, j) = (m, j) = DD(i, j), j \leq n,$

Similarly,

$LU\ (i, j) = (1, 1) = U\ L(i, j); \quad RU\ (i, j) = (1, n) = U\ R(i, j);$

$LD(i, j) = (m, 1) = DL(i, j); \quad RD(i, j) = (m, n) = DR(i, j).$

Since *L, R* commute with *U, D* we can write any word in the form $w = uv$ where $u \in \{L, R\}$ and $v \in \{U, D\}$. Further, since $\{L, R\}$ and $\{U, D\}$ are left-zero semigroups *w* equals one of the 8 listed elements in the one-tile semigroup *S*. Hence the semigroup *S* on a one-tile rectangular board has at most 8 elements *L, R, U, D, LU, RU, LD* and *RD*. Hence one-tile semigroups corresponding to any $m \times n, (2 \leq m, n)$ rectangle are isomorphic.

7. CONCLUSIONS

Finite semigroups are very rare in the literature of Semigroup theory. This work provides a variety of subsemigroups of the finite transformation semigorups in an amazing way while playing with tiles of irregular boards. We provide various several finite subsemigroups of different types.

8. ACKNOWLEDGEMENT

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