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Research Article

Stability Analysis of Standing Matter Wave Dark Solitons in a Coupled Bose-Einstein Condensate

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Abstract: We study the existence and stability of standing matter waves in two quasi one-dimensional parallel coupled Bose-Einstein condensates in the presence of a magnetic trap. The system is modeled by linearly coupled Gross-Pitaevskii equations. In particular, the coupled dark soliton solutions are considered. The effect of changing strength of the magnetic trap on the stability of dark solitons is studied. It is found that the presence of a magnetic trap does not affect the stability of coupled dark solitons.

Keywords: Bose-Einstein condensate, magnetic trap, Gross-Pitaevskii equation, Josephson tunneling, stability

1. INTRODUCTION

A soliton is a nonlinear wave which does not alter its shape or speed during its motion. A Scottish engineer, John Scott Russell, was the first who observed a solitary wave on the Union Canal in Edinburgh in 1834. He named this wave as “the wave of translation”. Later on, in 1965, Norman Zabusky and Martin Kruskal named this wave as a “soliton” while studying solitary waves in Korteweg-de Vries (KdV) equation [1].

It is a fact that there are several phenomena in physics, engineering and biology which can be explained by the physical and mathematical theory of solitons. For example, solitons are exactly suitable for fibre optics communications networks where in one second millions and billions of solitons carry information down fiber circuits for telephones, computers and televisions [2, 3]. The importance of solitons is apparent as they emerge as a solution of more than one hundred nonlinear partial differential equations [2].

The dynamics of solitons can be modeled by the nonlinear Schrödinger (NLS) equation. The NLS equation arises in the description of various physical phenomena such as Bose-Einstein Condensates (BEC) [4], non-linear optics [4, 5], non-linear water waves, non-linear acoustics [6], plasma waves [7] and so on. From a mathematical view point, on the Euclidean space, this equation has been studied since the seventies [8]. Although the NLS equation has infinitely many solutions, the mostly studied solutions are plane wave solution, bright soliton, dark soliton, etc.

Dark solitons are the basic indispensable envelope excitations sustained in non-linear dispersive media and comprise of a swift dip in the intensity of a wide pulse or a continuous waveguide background

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with a jump in its phase at its intensity minimum. Not only theoretically but many experimental observations and results on dark solitons exist, such as pulses in optical fibers (from which the name dark was conceived) [9], thin magnetic films [10], movement of static kink in a shallow liquid driven parametrically [11], standing waves in mechanical systems [12], and many more.

The study on matter wave dark solitons continue to be an interesting subject. In one dimension, the criterion for the dynamical stability of dark soliton was given in [13]. The snake instability was crushed and the dark soliton was stabilized by strongly confining the radial motion and by keeping the radial frequency greater than the mean field interaction of the particles. The experimental and theoretical studies of vortices in Bose-Einstein condensates were presented in [14].

The idea of electron tunneling also called Josephson tunneling between two superconductors linked with each other by a very thin insulator [15] was presented by Josephson in 1962. Such tunneling in BEC was predicted by Smerzi et al. [16, 17]. Josephson tunneling for a single and an arrangement of short Bose-Josephson junction [18] was realized experimentally. The idea of Bose-Josephson junction was extended to long Bose-Josephson junction in [19]. This junction was similar to long superconducting Josephson junction. It was proposed in [19] that atomic vortices could be seen in coupled BEC that are weakly linked with each other and that these vortices are analogues to Josephson fluxons in superconducting long Josephson junction [20]. Moreover, it was shown that the atomic Josephson vortices can be transformed to a matter wave dark soliton and vice versa at a critical value of coupling strength. Josephson tunneling of matter wave dark solitons in a double-well potential was investigated in [21].

In this work, we consider the existence and stability of matter wave dark solitons in two coupled cigar-shaped BEC with a magnetic trap. In particular, we investigate the effects of change in the strength of a magnetic trap on the stability of dark solitons in BEC.

2. MATHEMATICAL MODEL AND DESCRIPTION

Let us consider a system of two coupled cigar-shaped BEC under the influence of an external magnetic trap. The intra atomic interaction is assumed to be repulsive. The system can be described by two coupled one-dimensional nonlinear Schrodinger equations which can be written as

$$i\,\frac{\partial u_1}{\partial t} = -\,\frac{1}{2}\,\frac{\partial^2 u_1}{\partial x^2} + \alpha|u_1|^2 u_1 + \beta u_1 - \gamma u_2 + v_{ext}\; u_1 \tag{1}$$

$$i\,\frac{\partial u_2}{\partial t} = -\,\frac{1}{2}\,\frac{\partial^2 u_2}{\partial x^2} + \alpha|u_2|^2 u_2 + \beta u_2 - \gamma u_1 + v_{ext}\; u_2\,, \tag{2}$$

where x and t represent the spatial and temporal variables respectively. u_1 and u_2 denote the wave functions of atoms of BEC in the two wells of a magnetic trap. The parameters $\alpha,\ \beta$ and γ represent respectively the nonlinearity coefficient, the chemical potential and the coupling strength between the two condensates. v_{ext} is the external magnetic trap and is given as

$$v_{ext} = \frac{1}{2}\,\Omega^2 x^2, \tag{3}$$

where Ω denotes the strength of external magnetic trap and i represents iota.

When $\gamma = 0$, i.e. the two condensates are uncoupled, the dynamics of matter wave dark solitons in BEC with an external potential was studied theoretically [22, 23] and experimentally [24, 25]. Intriguing phenomena on the joint behavior of a quantum degenerate boson gas, for instance, oscillations of solitons [24] and shifts in frequency due to collisions were observed.

In this study, we consider the case when $\gamma \neq 0$. For the steady state solution, putting $\frac{\partial u_1}{\partial t} = 0 = \frac{\partial u_2}{\partial t}$, we obtain

$$\frac{1}{2}\frac{\partial^2 u_1}{\partial x^2} - \alpha |u_1|^2 u_1 - \beta u_1 + \gamma u_2 - v_{ext}\ u_1 = 0, \quad (3)$$

$$\frac{1}{2}\frac{\partial^2 u_2}{\partial x^2} - \alpha |u_2|^2 u_2 - \beta u_2 + \gamma u_1 - v_{ext}\ u_2 = 0. \quad (4)$$

Discretizing eq.(3) and eq.(4) using central difference approximations, we obtain

$$\frac{u_{1,m+1} - 2u_{1,m} + u_{1,m-1}}{2\Delta x^2} - \alpha |u_{1,m}|^2 u_{1,m} - \beta u_{1,m} + \gamma u_{2,m} - v_{ext}\ u_{1,m} = 0, \quad (5)$$

$$\frac{u_{2,m+1} - 2u_{2,m} + u_{2,m-1}}{2\Delta x^2} - \alpha |u_{2,m}|^2 u_{2,m} - \beta u_{2,m} + \gamma u_{1,m} - v_{ext}\ u_{2,m} = 0, \quad (6)$$

where $m = 1,2,\ldots,M$.

Eq. (5) and eq. (6) represent a coupled system of nonlinear algebraic equations. We solve this system numerically using Newton's method with the Neumann boundary conditions $u_{n,0} = u_{n,1}$ and $u_{n,M} = u_{n,M+1}$, $n = 1,2$ and obtain linearly coupled dark soliton solution as shown in Fig. 1.

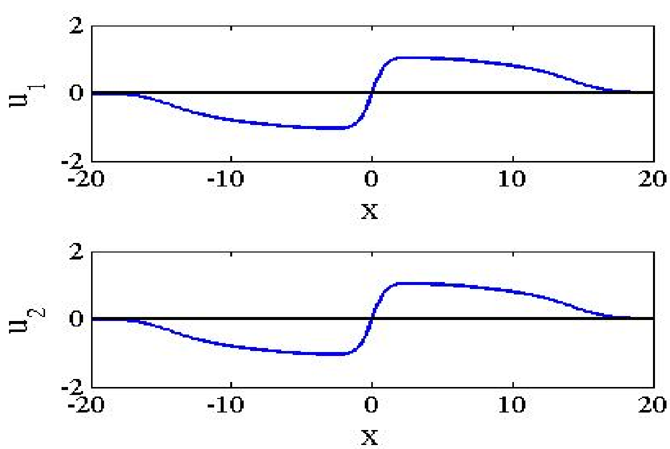

Fig. 1. Numerically obtained coupled dark soliton solution for the parameter values $\alpha = 1$, $\beta = 1$, $\gamma = 0.15$ and $\Omega = 0.1$. The real and imaginary parts of solution are represented by blue curves and black lines, respectively.

3. STABILITY OF COUPLED DARK SOLITONS

In order to discuss the stability of dark soliton, we first assume that $u_1^{(0)}$ and $u_2^{(0)}$ are the static solutions of system of eq. (1) and eq. (2). Let us perturb these solutions $u_1^{(0)}$ and $u_2^{(0)}$ by adding small perturbations $\eta_1(x,t)$ and $\eta_2(x,t)$ in them respectively, i.e.

$$u_1(x,t) = u_1^{(0)}(x) + \eta_1(x,t), \quad (7)$$

$$u_2(x,t) = u_2^{(0)}(x) + \eta_2(x,t), \tag{8}$$

Substituting these values of $u_1(x,t)$ and $u_2(x,t)$ from eq. (7) and eq. (8) into eq. (1) and eq. (2), we get

$$i\,\frac{\partial \eta_1}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2 u_1^{(0)}}{\partial x^2} + \frac{\partial^2 \eta_1}{\partial x^2}\right) + \alpha\left(u_1^{(0)} + \eta_1\right)^2 \overline{\left(u_1^{(0)} + \eta_1\right)} - \beta\left(u_1^{(0)} + \eta_1\right) + \mathcal{v}_{ext}\left(u_1^{(0)} + \eta_1\right) - \gamma\left(u_2^{(0)} + \eta_2\right), \tag{9}$$

$$i\,\frac{\partial \eta_2}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2 u_2^{(0)}}{\partial x^2} + \frac{\partial^2 \eta_2}{\partial x^2}\right) + \alpha\left(u_2^{(0)} + \eta_2\right)^2 \overline{\left(u_2^{(0)} + \eta_2\right)} - \beta\left(u_2^{(0)} + \eta_2\right) + \mathcal{v}_{ext}\left(u_2^{(0)} + \eta_2\right) - \gamma\left(u_1^{(0)} + \eta_1\right), \tag{10}$$

where bar represents the complex conjugate. Keeping in view that $u_1^{(0)}$ and $u_2^{(0)}$ are the solutions of system of eq. (3) and eq. (4) and assuming that the perturbations η_1 and η_2 are so small that their squares and higher power terms can be neglected, the above eq. (9) and eq. (10) can be written as

$$i\,\frac{\partial \eta_1}{\partial t} = -\frac{1}{2}\frac{\partial^2 \eta_1}{\partial x^2} + 2\alpha|u_1^{(0)}|^2\eta_1 + \alpha\left(u_1^{(0)}\right)^2\overline{\eta_1} - \beta\eta_1 + \mathcal{v}_{ext}\,\eta_1 - \gamma\eta_2, \tag{11}$$

$$i\,\frac{\partial \eta_2}{\partial t} = -\frac{1}{2}\frac{\partial^2 \eta_2}{\partial x^2} + 2\alpha|u_2^{(0)}|^2\eta_2 + \alpha\left(u_2^{(0)}\right)^2\overline{\eta_2} - \beta\eta_2 + \mathcal{v}_{ext}\,\eta_2 - \gamma\eta_1. \tag{12}$$

Taking the complex conjugate of these equations, we obtain

$$-i\,\frac{\partial \overline{\eta_1}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \overline{\eta_1}}{\partial x^2} + 2\alpha|u_1^{(0)}|^2\overline{\eta_1} + \alpha\left(\overline{u_1^{(0)}}\right)^2\eta_1 - \beta\overline{\eta_1} + \mathcal{v}_{ext}\,\overline{\eta_1} - \gamma\overline{\eta_2}, \tag{13}$$

$$-i\,\frac{\partial \overline{\eta_2}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \overline{\eta_2}}{\partial x^2} + 2\alpha|u_2^{(0)}|^2\overline{\eta_2} + \alpha\left(\overline{u_2^{(0)}}\right)^2\eta_2 - \beta\overline{\eta_2} + \mathcal{v}_{ext}\,\overline{\eta_2} - \gamma\overline{\eta_1}. \tag{14}$$

For simplicity we replace η_i by χ_i and $\overline{\eta_i}$ by ξ_i, where $i = 1,2$ in equations (11), (12), (13) and (14) to obtain

$$i\,\frac{\partial \chi_1}{\partial t} = -\frac{1}{2}\frac{\partial^2 \chi_1}{\partial x^2} + 2\alpha|u_1^{(0)}|^2\chi_1 + \alpha\left(u_1^{(0)}\right)^2\xi_1 - \beta\chi_1 + \mathcal{v}_{ext}\,\chi_1 - \gamma\chi_2 = \lambda\chi_1, \tag{15}$$

$$i\,\frac{\partial \chi_2}{\partial t} = -\frac{1}{2}\frac{\partial^2 \chi_2}{\partial x^2} + 2\alpha|u_2^{(0)}|^2\chi_2 + \alpha\left(u_2^{(0)}\right)^2\xi_2 - \beta\chi_2 + \mathcal{v}_{ext}\,\chi_2 - \gamma\chi_1 = \lambda\chi_2, \tag{16}$$

$$i\,\frac{\partial \xi_1}{\partial t} = \frac{1}{2}\frac{\partial^2 \xi_1}{\partial x^2} - 2\alpha|u_1^{(0)}|^2\xi_1 - \alpha\left(\overline{u_1^{(0)}}\right)^2\chi_1 + \beta\xi_1 - \mathcal{v}_{ext}\,\xi_1 + \gamma\xi_2 = \lambda\xi_1, \tag{17}$$

$$i\,\frac{\partial \xi_2}{\partial t} = \frac{1}{2}\frac{\partial^2 \xi_2}{\partial x^2} - 2\alpha|u_1^{(0)}|^2\xi_2 - \alpha\left(\overline{u_2^{(0)}}\right)^2\chi_2 + \beta\xi_2 - \mathcal{v}_{ext}\,\xi_2 + \gamma\xi_1 = \lambda\xi_2, \tag{18}$$

where λ is a scalar representing the eigenvalues.

Discretizing equations (15), (16), (17) and (18) gives

$$-\frac{\chi_{1,m+1}-2\chi_{1,m}+\chi_{1,m-1}}{2\Delta x^2}+2\alpha|u_{1,m}^{(0)}|^2\chi_{1,m}+\alpha\left(u_{1,m}^{(0)}\right)^2\xi_{1,m}-\beta\chi_{1,m}+v_{ext}\,\chi_{1,m}-\gamma\chi_{2,m}$$
$$=\lambda\chi_{1,m}, \tag{19}$$

$$-\frac{\chi_{2,m+1}-2\chi_{2,m}+\chi_{2,m-1}}{2\Delta x^2}+2\alpha|u_{2,m}^{(0)}|^2\chi_{2,m}+\alpha\left(u_{2,m}^{(0)}\right)^2\xi_{2,m}-\beta\chi_{2,m}+v_{ext}\,\chi_{2,m}-\gamma\chi_{1,m}$$
$$=\lambda\chi_{2,m}, \tag{20}$$

$$\frac{\xi_{1,m+1}-2\xi_{1,m}+\xi_{1,m-1}}{2\Delta x^2}-2\alpha|u_{1,m}^{(0)}|^2\xi_{1,m}-\alpha\left(\overline{u_{1,m}^{(0)}}\right)^2\chi_{1,m}+\beta\xi_{1,m}-v_{ext}\,\xi_{1,m}+\gamma\xi_{2,m}$$
$$=\lambda\xi_{1,m}, \tag{21}$$

$$\frac{\xi_{2,m+1}-2\xi_{2,m}+\xi_{2,m-1}}{2\Delta x^2}-2\alpha\left|u_{2,m}^{(0)}\right|^2\xi_{2,m}-\alpha\left(\overline{u_{2,m}^{(0)}}\right)^2\chi_{2,m}+\beta\xi_{2,m}-v_{ext}\,\xi_{2,m}+\gamma\xi_{1,m}$$
$$=\lambda\xi_{2,m}, \tag{22}$$

where $m=1,2,\dots,M$. Using the Neumann boundary conditions $\chi_{n,0}=\chi_{n,1}$ and $\xi_{n,M}=\xi_{n,M+1}$, $n=1,2$, the above system of equations (19), (20), (21) and (22) can be expressed as an eigenvalue problem

$$AX=\lambda X,$$

where

$$\mathrm{A}=\begin{bmatrix} A_1 & G & B_1 & 0\\ G & A_2 & 0 & B_2\\ -\overline{B_1} & 0 & -A_1 & -G\\ 0 & -\overline{B_2} & -G & -A_2\end{bmatrix},$$

$$A_1=\begin{bmatrix} \frac{1}{2\Delta x^2}+2\alpha|u_{1,1}^{(0)}|^2-\beta+v_{ext} & \frac{-1}{2\Delta x^2} & 0 & 0 & \dots & 0\\ \frac{-1}{2\Delta x^2} & \frac{1}{\Delta x^2}+2\alpha|u_{1,2}^{(0)}|^2-\beta+v_{ext} & \frac{-1}{2\Delta x^2} & 0 & \dots & 0\\ 0 & \frac{-1}{2\Delta x^2} & \frac{1}{\Delta x^2}+2\alpha|u_{1,3}^{(0)}|^2-\beta+v_{ext} & \frac{-1}{2\Delta x^2} & \dots & 0\\ & \ddots & \ddots & \ddots & & \\ 0 & 0\dots & \frac{-1}{2\Delta x^2} & & & \frac{1}{2\Delta x^2}+2\alpha|u_{1,M}^{(0)}|^2-\beta+v_{ext}\end{bmatrix},$$

$$A_2=\begin{bmatrix} \frac{1}{2\Delta x^2}+2\alpha|u_{2,1}^{(0)}|^2-\beta+v_{ext} & \frac{-1}{2\Delta x^2} & 0 & 0 & \dots & 0\\ \frac{-1}{2\Delta x^2} & \frac{1}{\Delta x^2}+2\alpha|u_{2,2}^{(0)}|^2-\beta+v_{ext} & \frac{-1}{2\Delta x^2} & 0 & \dots & 0\\ 0 & \frac{-1}{2\Delta x^2} & \frac{1}{\Delta x^2}+2\alpha|u_{2,3}^{(0)}|^2-\beta+v_{ext} & \frac{-1}{2\Delta x^2} & \dots & 0\\ & \ddots & \ddots & \ddots & & \\ 0 & 0\dots & \frac{-1}{2\Delta x^2} & & & \frac{1}{2\Delta x^2}+2\alpha|u_{2,M}^{(0)}|^2-\beta+v_{ext}\end{bmatrix},$$

$$B_1 = \begin{bmatrix} \alpha\left(u_{1,1}^{(0)}\right)^2 & 0 & 0 & \dots & 0 \\ 0 & \alpha\left(u_{1,2}^{(0)}\right)^2 & 0 & \dots & 0 \\ 0 & 0 & \alpha\left(u_{1,3}^{(0)}\right)^2 & \dots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \alpha\left(u_{1,M}^{(0)}\right)^2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \alpha\left(u_{2,1}^{(0)}\right)^2 & 0 & 0 & \dots & 0 \\ 0 & \alpha\left(u_{2,2}^{(0)}\right)^2 & 0 & \dots & 0 \\ 0 & 0 & \alpha\left(u_{2,3}^{(0)}\right)^2 & \dots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \alpha\left(u_{2,M}^{(0)}\right)^2 \end{bmatrix},$$

$$G = \begin{bmatrix} -\gamma & 0 & 0 & \dots & 0 \\ 0 & -\gamma & 0 & \dots & 0 \\ 0 & 0 & -\gamma & \dots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & -\gamma \end{bmatrix}.$$

The solution will be unstable if the imaginary part of at least one of the eigenvalues is positive.

We find the eigenvalues of the stability matrix A and are shown in Fig. 2. One can see that all eigenvalues are lying on the horizontal axis except a pair of eigenvalues which is lying on the vertical axis and depicting the instability of the coupled dark soliton solution. In order to confirm the results found, we solve the system of eq. (1) and eq. (2) numerically by perturbing the solution shown in Fig. 1 and use the 4th order Runge-Kutta method. The time evolution of the coupled dark soliton is shown in Fig. 3. The radiation are emerging at nearly $t = 40$ and reveals that the solution is unstable which justifies the results

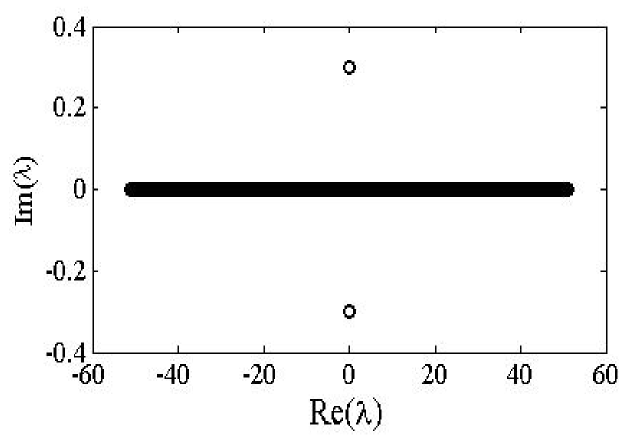

Fig. 2. The eigenvalues structure for the solution shown in Fig. 1. The two eigenvalues are lying on the vertical axis and depicts that the solution is unstable.

already obtained. We then investigate the stability of the coupled dark soliton for different values of Ω and noticed that the critical value of the coupling parameter γ is γ_c which in our case is nearly 0.334. This value remains unchanged with Ω as shown in Fig. 4. This means that the magnetic trap does not affect the critical value of the stability of the solution. Thus the coupled dark soliton solution exists and is unstable for $\gamma < 0.334$ but becomes stable for $\gamma > 0.334$.

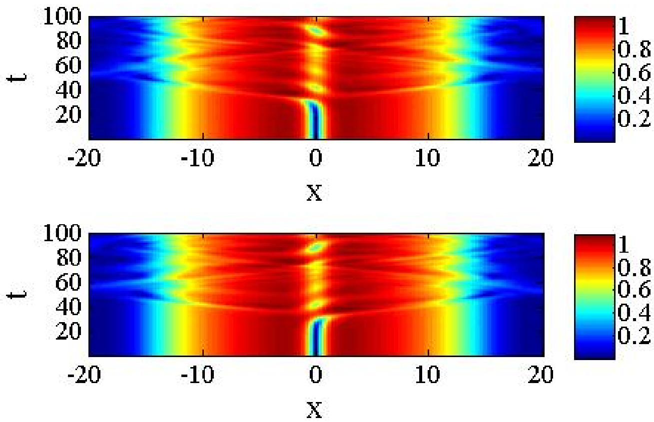

Fig. 3. The evolution of time for the solution shown in Fig. 1. The upper panel corresponds to $|u_1|$ and lower panel to $|u_2|$. The emergence of radiation in both panels shows the instability of the solution.

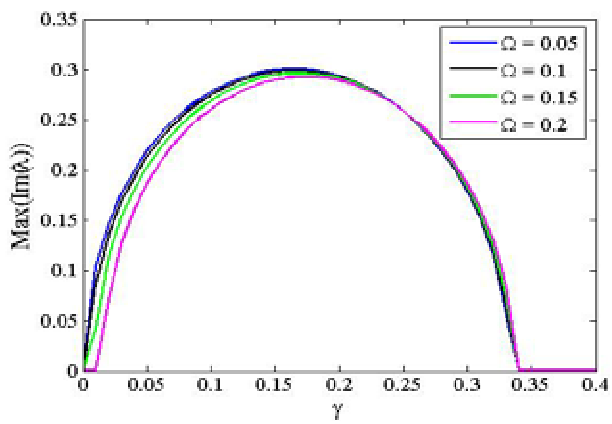

Fig. 4. Stability curves for the coupled dark soliton solution for distinct values of trapping strength **Ω**. The critical value remains unchanged for each value of **Ω** showing that the magnetic trap does not affect the stability of coupled dark soliton solution.

4. CONCLUSIONS

In this paper, we have considered the existence and stability of standing matter wave dark solitons in two quasi one-dimensional parallel coupled BEC in a magnetic trap. Especially, the stability of coupled dark soliton solution has been investigated while varying the strength of the magnetic trap. It has been found that a critical value γ_c of the coupling parameter exists which remains unchanged with the strength of the magnetic trap. The coupled dark soliton solution exists and remains unstable for $\gamma < \gamma_c$ and is stable for $\gamma > \gamma_c$.

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